Contents

Chapter 0

Sample Chapter

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0.1 Some recomendations

sec:1:recomendaciones

- Please use the environments `theorem`, `corollary`, `proposition` etc. and `proof`.
- In definitions, use `define{}` for what you want to define.
- For labels, please use the following conventions:
  - `\label{chapter:1}` for chapters
  - `\label{sec:1}` for sections
  - `\label{thm:number-of-chapter:something}` for theorems,
  - `\label{lem:number-of-chapter:something}` for lemmas,
  - `\label{prop:number-of-chapter:something}` for propositions
  - `\label{cor:number-of-chapter:something}` for corollaries
  - `\label{eq:number-of-chapter:something}` for equations
- If you refer to a formula, use `eqref{}`. If you refer to something in an enumerated list, use `enumiref{}`.
- Do not use `eqnarray`. Use `equation`, `align gather`, etc. (see, e.g., amsldoc.pdf).
- Use `\index` when appropriate.
- `\commentbox{}` may be useful. It takes as an optional argument colors.
- In proofs, you may want to use `\proofenumerate{}` instead of `\enumerate{}`.
- Use `\I` for the imaginary unit and `\e` for e.

Important comment!
Comment in teal!
Comment in cyan!
Comment in violet!
Comment in purple!
Comment in red!

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- Use `\rd` to produce d in integrals:
  `\int_{0}^{1}f(tz+\left(1-t\right)z_{1})\,\rd t` gives
  $\int_0^1 f(tz+(1-t)\,z_1)\,\mathrm{d}t$.
- Never ever use i as index in sums etc.
- Use punctuation in formulas.

0.2 Dictionary

sec:dictionary

	[AI89]	[Bog74]
V vector space	$\mathcal{F}$	$\mathfrak{E}$
linear subspace	lineal	
closed linear subspace	subspace	
$[\,,]$ indefinite inner product	$[\,,]$	$[\,,]$
$\langle\,,\rangle$ Hilbert space inner product	$(\,,)$	$(\,,)$
x is positive, $\langle x\,,x\rangle > 0$	strictly positive	positive
x is nonnegative, $\langle x\,,x\rangle \geq 0$	positive	nonnegative
x is negative, $\langle x\,,x\rangle < 0$	strictly negative	nonpositive
x is nonpositive, $\langle x\,,x\rangle \leq 0$	negative	nonpositive
$\mathcal{P}^{++} = \{x \in V : \langle x\,,x\rangle > 0\}$	$\mathcal{P}^{++}$	$\mathfrak{P}^{++}$
$\mathcal{P}^{+} = \{x \in V : \langle x\,,x\rangle \geq 0\}$	$\mathcal{P}^{+}$	$\mathfrak{P}^{+}$
$\mathcal{P}^{--} = \{x \in V : \langle x\,,x\rangle < 0\}$	$\mathcal{P}^{--}$	$\mathfrak{P}^{--}$
$\mathcal{P}^{-} = \{x \in V : \langle x\,,x\rangle \leq 0\}$	$\mathcal{P}^{-}$	$\mathfrak{P}^{-}$
$\mathcal{P}^{00} = \{x \in V : \langle x\,,x\rangle = 0\}$	$\mathcal{P}^{00}$	$\mathfrak{P}^{00}$
$\mathcal{P}^{0} = \{x \in V : \langle x\,,x\rangle \leq 0\}$	$\mathcal{P}^{0}$	$\mathfrak{P}^{0}$
$\perp$ orthogonal in Hilbert space		
$[\perp]$ Q-orthogonal in inner product space		
$+$ algebraic sum: $U+V=\{u+v : u\in U,\, v\in V\}$		
$\dot{+}$ direct sum `\dplus`: $U\dot{+}V = V+V$ if $U\cap V=\varnothing$		
$\oplus$ Hilbert space orthogonal sum: $U\oplus V = U+V$ if $U\perp V$		
$[+]$ inner product space orthogonal sum `\operp`: $U[+]V = U+V$ if $U[\perp]V$		
$\boxplus$ inner product space orthogonal sum `\dperp`: $U\boxplus V = U+V$ if $U[\perp]V$ and $U\cap V=\varnothing$		

Chapter 1

chapter:01

Linear spaces with indefinite metric

Usually we denote vector spaces without any additional structure by V, and we assume all vector spaces to be complex vector spaces.

1.1 Bounded Hermitian forms

Definition 1.1. Let V be a complex vector space. A mapping $Q : V \times V \to \mathbb{C}$ is a *hermitian sesquilinear form* on V if for all $x, y, z \in V$, $\lambda \in \mathbb{C}$

(i) $Q(\alpha x + y,\, z) = \alpha Q(x,\, z) + Q(y,\, z)$ (linearity in the first component),

(ii) $Q(x,\, z) = \overline{Q(z,\, x)}$ (symmetry)

We often write $[x\,, y]$ instead of $Q(x, y)$ if it is clear which form Q is considered.

Clearly, a hermitian sesquilinear form satisfies

$$Q(x,\, \alpha y + z) = \overline{\alpha} Q(x,\, y) + Q(x,\, z), \qquad x, y, z \in V,\ \alpha \in \mathbb{C}.$$

An *inner product space* $(V, [\,,])$ is a vector space V with a sesquilinear form $[\,,]$. Clearly, if V is vector space with inner product $[\,,]$ and U is a subspace of V, then $(U, [\,,]|_{U\times U})$ is again an inner product space, usually denoted by $(U, [\,,])$ or simply U.

enumi:1:basic1 ex:1:basic **Example 1.2.** (i) Let $V = \mathbb{C}^2$ and define the inner product

$$V \times V \to \mathbb{C}, \qquad \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] = x_1\overline{y_1} - x_2\overline{y_2}.$$

enumi:1:basic2 (ii) Let $V = \mathbb{C}^2$ and define the inner product

$$V \times V \to \mathbb{C}, \qquad \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] = x_1\overline{y_2} + x_2\overline{y_1}.$$

Example 1.3. Examples 1.1 and 1.2. from [AI89].

Definition 1.4. Let $[\,,]$ be a sesquilinear form on a vector space V and let $x \in V$.

- x is called *positive* if $[x\,,x] > 0$,
- x is called *non-negative* if $[x\,,x] \geq 0$,
- x is called *negative* if $[x\,,x] < 0$,
- x is called *non-positive* if $[x\,,x] \leq 0$,
- x is called *neutral* if $[x\,,x] = 0$.

Observe that $[x\,,x] = 0$ if $x = 0$, but the reverse is in general not true.

Definition 1.5. Let $(V, [\,,])$ be an inner product space. Then $(V, -[\,,])$ is called its *antispace*.

Definition 1.6. Let V be a vector space with hermitian sesquilinear form $[\,,]$. We define the following sets:

$$\begin{aligned}
\mathcal{P}^{++}(V) &:= \{x \in V : [x\,,x] > 0\},\\
\mathcal{P}^{+}(V) &:= \{x \in V : [x\,,x] \geq 0\},\\
\mathcal{P}^{--}(V) &:= \{x \in V : [x\,,x] < 0\},\\
\mathcal{P}^{-}(V) &:= \{x \in V : [x\,,x] \leq 0\},\\
\mathcal{P}^{00}(V) &:= \{x \in V : x \neq 0,\ [x\,,x] = 0\},\\
\mathcal{P}^{0}(V) &:= \{x \in V : [x\,,x] = 0\}.
\end{aligned}$$

If the underlying space V is clear, we sometimes write $\mathcal{P}^{++}, \mathcal{P}^{+}, \mathcal{P}^{--}, \mathcal{P}^{-}, \mathcal{P}^{00}$ and $\mathcal{P}^{0}$ instead of $\mathcal{P}^{++}(V), \mathcal{P}^{+}(V), \mathcal{P}^{--}(V), \mathcal{P}^{-}(V), \mathcal{P}^{00}(V)$ and $\mathcal{P}^{0}(V)$.

Clearly $\mathcal{P}^{+} = \mathcal{P}^{++} \cup \mathcal{P}^{0}$, $\mathcal{P}^{-} = \mathcal{P}^{--} \cup \mathcal{P}^{0}$ and $\mathcal{P}^{0} = \mathcal{P}^{+} \cap \mathcal{P}^{-}$. Moreover, if $x \in \mathcal{P}^{++}$, then $\alpha x \in \mathcal{P}^{++}$ for all $\alpha \in \mathbb{C} \setminus \{0\}$. The same is true for the other sets defined above.

Example 1.7.

- Example 1.2(i) :

$$\begin{aligned}
\mathcal{P}^{++} &= \{(x_1,\, x_2)^t : |x_1| > |x_2|\}, & \mathcal{P}^{+} &= \{(x_1,\, x_2)^t : |x_1| \geq |x_2|\},\\
\mathcal{P}^{--} &= \{(x_1,\, x_2)^t : |x_1| < |x_2|\}, & \mathcal{P}^{-} &= \{(x_1,\, x_2)^t : |x_1| \leq |x_2|\},\\
\mathcal{P}^{0} &= \{(x_1,\, x_2)^t : |x_1| = |x_2|\}.
\end{aligned}$$

- Example 1.2(ii) :

$$\begin{aligned}
\mathcal{P}^{++} &= \{(x_1,\, x_2)^t : \operatorname{Re}(x_1\overline{x_2}) > 0\}, & \mathcal{P}^{+} &= \{(x_1,\, x_2)^t : \operatorname{Re}(x_1\overline{x_2}) \geq 0\},\\
\mathcal{P}^{--} &= \{(x_1,\, x_2)^t : \operatorname{Re}(x_1\overline{x_2}) < 0\}, & \mathcal{P}^{-} &= \{(x_1,\, x_2)^t : \operatorname{Re}(x_1\overline{x_2}) \leq 0\},\\
\mathcal{P}^{0} &= \{(x_1,\, x_2)^t : \operatorname{Re}(x_1\overline{x_2}) = 0\}.
\end{aligned}$$

def:mP **Definition 1.8.** . Let V be a vector space with hermitian sesquilinear form $[\,,]$ and let $U \subseteq V$ be a subspace.

- U is *positive* if $U \subseteq \mathcal{P}^{++} \cup \{0\}$,
- U is *non-negative* if $U \subseteq \mathcal{P}^{+}$,
- U is *negative* if $U \subseteq \mathcal{P}^{--} \cup \{0\}$,
- U is *non-positive* if $U \subseteq \mathcal{P}^{-}$,
- U is *semidefinite* if $U \subseteq \mathcal{P}^{+}$ or $U \subseteq \mathcal{P}^{-}$,
- U is *definite* if $U \subseteq \mathcal{P}^{++} \cup \{0\}$ or $U \subseteq \mathcal{P}^{--} \cup \{0\}$,
- U is *neutral* if $U \subseteq \mathcal{P}^{0}$,

The sesquilinear form $[\,,]$ is *positive, non-negative, etc.* if V is *positive, non-negative, etc.*. The sesquilinear form $[\,,]$ is called *indefinite* it is not semidefinite.

Clearly, a neutral subspace is semidefinite. It is both non-negative and non-positive.

Note that $\mathcal{P}^{++}$ and $\mathcal{P}^{--}$ are not vector spaces (they do not contain 0). Next we show that if $\mathcal{P}^{++} \neq \varnothing$, then the span of $\mathcal{P}^{++} \cup \{0\}$ is all of V, in particular, in general the sets $\mathcal{P}^{+}$ and $\mathcal{P}^{-}$ are not vector spaces too. Note however that $\mathcal{P}^{0}$ is a vector space, see Corollary 1.12.

Proposition 1.9. *Let $(V, [\,,])$ be an inner product space and let $x \in V$ with $[x\,,x] > 0$. Then every element of V is the sum of two positive vectors.*

Clearly a similar statement is true if there exists $x \in V$ with $[x\,,x] < 0$.

Proof. Let $x \in V$ such that $[x\,,x] > 0$ and fix $z \in V$. Then we can choose $\alpha \in \mathbb{R}$ large enough such that

$$[z+\alpha x\,, z+\alpha x] = [z\,,z] + 2\alpha\,\mathrm{Re}[z\,,x] + \alpha^2[x\,,x] > 0.$$

Then $z = (z+\alpha x) + (-\alpha x)$ with $z+\alpha x, -\alpha x \in \mathcal{P}^{++}$. □

Proposition 1.10. *Let $(V, [\,,])$ be an inner product space. If $[\,,]$ is indefinite, then there exists $x \neq 0$ such that $[x\,,x] = 0$.*

Proof. Let $y, z \in H$ such that $[y\,,y] > 0$ and $[z\,,z] < 0$. Clearly, y and z are linearly independent. Consider the continuous map $f : [0,1] \to \mathbb{R}$, $f(t) = [(1-t)y + tz\,, (1-t)y + tz]$. Since $f(0) > 0$ and $f(1) < 1$, there exists $t_0 \in (0,1)$ such that $f(t_0) = 0$ and $x = (1-t_0)y + t_0 z \neq 0$ does the job. □

Proposition 1.11 (Cauchy-Schwarz-Bunyakovski inequality). *Let $(V, \langle\,,\rangle)$ be a semidefinite inner product space. Then for all $x, y \in V$*

$$|[x\,,y]|^2 \leq |[x\,,x]|\,|[y\,,y]|. \tag{1.1}$$ eq:CSB

Proof. Without restriction we may assume that $[\,,]$ is non-negative. Observe that for all $\alpha \in \mathbb{C}$

$$0 \le [x - \alpha y\,, x - \alpha y] = [x\,, x] - 2\,\mathrm{Re}\,\big(\alpha[x\,, y]\big) + \alpha^2[y\,, y].$$

If $[x\,, x] = [y\,, y] = 0$, then clearly $[x\,, y] = 0$, otherwise the choice $\alpha = 2[x\,, y]^{-1}$ would lead to a contradiction. Now assume that at least one of the vectors x, y is not neutral, without restriction let $[y\,, y] > 0$. Choose $\alpha = \frac{[x\,,y]}{[y\,,y]}$. Then the above inequality gives

$$0 \le [x\,, x] - 2\frac{|[x\,, y]|^2}{[y\,, y]} + \frac{|[x\,, y]|^2}{[y\,, y]} = [x\,, x] - \frac{|[x\,, y]|^2}{[y\,, y]}.$$

Multiplication by $[y\,, y]$ completes the proof. □

cor:1:P0 **Corollary 1.12.** *$\mathcal{P}^0(V)$ is a subspace of V.*

1.2 Orthogonality and isotropic subspaces

Definition 1.13. Let $(V, [\,,])$ be an inner product space. We write $x[\perp]y$ if and only if $[x\,, y] = 0$. In this case x an y are called Q-orthogonal.
Let $M \subset V$ be a set. Then $x[\perp]M$ if and only if $x[\perp]y$ for all $y \in M$ and

$$M^{[\perp]} := \{x \in V : x[\perp]y \text{ for all } \ y \in M\}$$

is called the *orthogonal complement of M*. We set $M \subset M^{[\perp][\perp]} := \big(M \subset M^{[\perp]}\big)^{[\perp]}$.

Remark 1.14. Let $x, y \in V$ and let $M, N \subset V$.

(i) $x[\perp]y \iff y[\perp]x$.

(ii) Then $M^{[\perp]}$ is a linear subspace of V.

(iii) $M \subset M^{[\perp][\perp]}$.

(iv) If $N \subset M$, then $M^{[\perp]} \subset N^{[\perp]}$.

(v) $M^{[\perp]} \cap N^{[\perp]} \subset (M \cup N)^{[\perp]}$.

(vi) $M^{[\perp]} + N^{[\perp]} \subset (M \cap N)^{[\perp]}$.

(vii) If U, W are linear subspaces of V, then $U^{[\perp]} + NW[\perp] = (U \cap W)^{[\perp]}$.

Remark 1.15. Let $U \subset V$ be a subspace of V. In general $V + V^{[\perp]} \neq V$, even if V is closed, and $U \cap U^{[\perp]}$ may occur. For instance, let V as in example 1.2 i and let $U = \{(1, 1)^t\} \subset V$. Then $U^{[\perp]} = U$.

Definition 1.16. Let $(V, [\,,])$ be an inner product space. A vector $x \in V, x \neq 0$, is called *isotropic* if $x[\perp]V$. If $U \subset V$, then a vector $x \in U, x \neq 0$, is called *isotropic for* U if $x[\perp]U$.

Clearly, linear combinations of isotropic vectors are again isotropic and a vector x is isotropic for a subspace U if and only if $x \in U \cap U^{[\perp]} \setminus \{0\}$.

Definition 1.17. Let $(V, [\,,])$ be an inner product space and $U \subset V$ a linear subspace. Then $U^0 := U \cap U^{[\perp]}$ is called the *isotropic part of* U. The subspace U is called *degenerate* if $U^0 = \{0\}$, otherwise it is called *non-degenerate*.

Observe that by the Cauchy-Schwarz-Bunyakovski inequality $V^0 = \mathcal{P}^0(V)$ if the inner product is semidefinite.

Definition 1.18. Let $(V, [\,,])$ be an indefinite inner product space. A subspace U is called *maximal positive* if it is a positive subspace of V and for every positive subspace $W \subset V$ with UW it follows that $U = W$. The definition of *maximal non-negative*, *maximal negative*, *maximal non-positive*, *maximal neutral*, *maximal non-degenerate subspace* is analogous.

Theorem 1.19. *Let $(V, [\,,])$ be an indefinite inner product space and $U \subset V$ a positive subspace. Then U is contained in a maximal positive subspace.*
Analogous statements are true for non-negative, negative, non-positive, neutral, non-degenerate subspace subspaces.

Proof. Zorn. See [AI89, Theorem 1.19]. □

1.3 Decomposition of subspaces

Definition 1.20. Let $(V, [\,,])$ be an inner product space. We write $x[\perp]y$ if and only if $[x\,, y] = 0$. In this case x an y are called Q-orthogonal.
Let $M \subset V$ be a set. Then $x[\perp]M$ if and only if $x[\perp]y$ for all $y \in M$ and

$$M^{[\perp]} := \{x \in V : x[\perp]y \text{ for all } \ y \in M\}$$

is called the *orthogonal complement of* M. We set $M \subset M^{[\perp][\perp]} := \big(M \subset M^{[\perp]}\big)^{[\perp]}$.

Remark 1.21. Let $x, y \in V$ and let $M, N \subset V$.

(i) $x[\perp]y \iff y[\perp]x$.

(ii) Then $M^{[\perp]}$ is a linear subspace of V.

(iii) $M \subset M^{[\perp][\perp]}$.

(iv) If $N \subset M$, then $M^{[\perp]} \subset N^{[\perp]}$.

(v) $M^{[\perp]} \cap N^{[\perp]} \subset (M \cup N)^{[\perp]}$.

(vi) $M^{[\perp]} + N^{[\perp]} \subset (M \cap N)^{[\perp]}$.

(vii) If U, W are linear subspaces of V, then $U^{[\perp]} + W^{[\perp]} = (U \cap W)^{[\perp]}$.

Remark 1.22. Let $U \subset V$ be a subspace of V. In general $V + V^{[\perp]} \neq V$, even if V is closed, and $U \cap U^{[\perp]}$ may occur. For instance, let V as in example 1.2 i and let $U = \{(1,1)^t\} \subset V$. Then $U^{[\perp]} = U$.

Definition 1.23. Let $(V, [\,,])$ be an inner product space. A vector $x \in V, x \neq 0$, is called *isotropic* if $x[\perp]V$. If $U \subset V$, then a vector $x \in U, x \neq 0$, is called *isotropic for* U if $x[\perp]U$.

Clearly, linear combinations of isotropic vectors are again isotropic and a vector x is isotropic for a subspace U if and only if $x \in U \cap U^{[\perp]} \setminus \{0\}$.

Definition 1.24. Let $(V, [\,,])$ be an inner product space and $U \subset V$ a linear subspace. Then $U^0 := U \cap U^{[\perp]}$ is called the *isotropic part of* U. The subspace U is called *degenerate* if $U^0 = \{0\}$, otherwise it is called *non-degenerate*.

Observe that by the Cauchy-Schwarz-Bunyakovski inequality $V^0 = \mathcal{P}^0(V)$ if the inner product is semidefinite.

Definition 1.25. Let $(V, [\,,])$ be an indefinite inner product space. A subspace U is called *maximal positive* if it is a positive subspace of V and for every positive subspace $W \subset V$ with UW it follows that $U = W$. The definition of *maximal non-negative*, *maximal negative*, *maximal non-positive*, *maximal neutral*, *maximal non-degenerate subspace* is analogous.

Theorem 1.26. *Let $(V, [\,,])$ be an indefinite inner product space and $U \subset V$ a positive subspace. Then U is contained in a maximal positive subspace.*
Analogous statements are true for non-negative, negative, non-positive, neutral, non-degenerate subspace subspaces.

Proof. Zorn. See [AI89, Theorem 1.19]. □

1.4 Quotient spaces

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