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These lecture notes are work in progress. They may be abandoned or changed radically at any moment. If you find mistakes or have suggestions how to improve them, please let me know.

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Chapter 1

Preliminaries

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In this chapter we collect some well-known facts from functional analysis.

Definition 1.1. Let X be a vector space over $\mathbb{K}$. $(X, \|\cdot\|)$ is called a *normed space* with *norm* $\|\cdot\|$ if

$$\|\cdot\| : X \to \mathbb{R}$$

is a map such that for all $x, y \in X$, $\alpha \in \mathbb{K}$

(i) $\|x\| = 0 \iff x = 0$,
(ii) $\|\alpha x\| = |\alpha|\, \|x\|$,
(iii) $\|x + y\| \le \|x\| + \|y\|$.

If $\|\cdot\|$ satisfies only (ii) and (iii), it is called a *seminorm*.

Note that $\|x\| \ge 0$ for all $x \in X$ because $0 = \|x - x\| \le 2\|x\|$. The last inequality follows from the triangle inequality (iii) and (ii) with $\alpha = -1$.

Definition 1.2. A normed space $(X, \|\cdot\|)$ is a *Banach space* if it is complete with respect to the topology induced by $\|\cdot\|$.

Definition 1.3. Let X be a $\mathbb{K}$-vector space. A map

$$\langle \cdot\,, \cdot \rangle : X \times X \to \mathbb{K}$$

is a *sesquilinear form* on X if for all $x, y, z \in X$, $\lambda \in \mathbb{K}$

(i) $\langle \lambda x + y\,, z \rangle = \lambda \langle x\,, z \rangle + \langle y\,, z \rangle$,
(ii) $\langle x\,, \lambda y + z \rangle = \overline{\lambda} \langle x\,, y \rangle + \langle x\,, z \rangle$.

The inner product is called

- *hermitian* $\iff \langle x\,, y \rangle = \overline{\langle y\,, x \rangle}, \qquad x, z \in X$,

- *positive semidefinite* $\iff \langle x\,, x \rangle \ge 0, \qquad x \in X$,
- *positive (definite)* $\iff \langle x\,, x \rangle > 0, \qquad x \in X \setminus \{0\}$.

Definition 1.4. A positive definite hermitian sesquilinear form on a $\mathbb{K}$-vector X is called an *inner product* on X and $(X, \langle \cdot\,, \cdot \rangle)$ is called an *inner product space* (or *pre-Hilbert space*).

Note that for a hermitian sesquilinear form $\langle x\,, x \rangle \in \mathbb{R}$ for every $x \in X$ because $\langle x\,, x \rangle = \overline{\langle x\,, x \rangle}$.

Lemma 1.5 (Cauchy-Schwarz inequality). *Let X be a $\mathbb{K}$-vector space with inner product $\langle \cdot\,, \cdot \rangle$. Then for all $x, y \in X$*

$$|\langle x\,, y \rangle|^2 \le |\langle x\,, x \rangle|\, |\langle y\,, y \rangle|, \tag{1.1}$$

with equality if and only if x and y are linearly dependent.

Proof. For $x = 0$ or $y = 0$ there is nothing to show. Now assume that $y \ne 0$. For all $\lambda \in \mathbb{K}$

$$0 \le \langle x + \lambda y\,, x + \lambda y \rangle = \langle x\,, x \rangle + \lambda \langle y\,, x \rangle + \overline{\lambda} \langle x\,, y \rangle + |\lambda|^2 \langle y\,, y \rangle.$$

In particular, when we choose $\lambda = -\frac{\langle x\,, y \rangle}{\langle y\,, y \rangle}$ we obtain

$$\begin{aligned} 0 \le \langle x + \lambda y\,, x + \lambda y \rangle &= \langle x\,, x \rangle - \frac{|\langle y\,, x \rangle|^2}{\langle y\,, y \rangle} - \frac{|\langle x\,, y \rangle|^2}{\langle y\,, y \rangle} + \frac{|\langle x\,, y \rangle|^2}{\langle y\,, y \rangle} \\ &= \langle x\,, x \rangle - \frac{|\langle x\,, y \rangle|^2}{\langle y\,, y \rangle} \end{aligned}$$

which proves (1.1). If there exist $\alpha, \beta \in K$ such that $\alpha x + \beta y = 0$, then obviously equality holds in (1.1). On the other hand, if equality holds, then $\langle x + \lambda y\,, x + \lambda y \rangle = 0$ with λ chosen as above, so x and y are linearly dependent. $\square$

Note that (1.1) is true also in a space X with a semidefinite hermitian sesquilinear form but equality in (1.1) does not imply that x and y are linearly dependent.

Lemma 1.6. *An inner product space $(X, \langle \cdot\,, \cdot \rangle)$ becomes a normed space by setting $\|x\| := \langle x\,, x \rangle^{\frac{1}{2}}$, $x \in X$.*

Definition 1.7. A complete inner product space is called a *Hilbert space*.

Definition 1.8. Let X, Y be normed spaces. A map $T : X \to Y$ is called a *linear operator* from X to Y if

$$T(\alpha x + y) = \alpha T x + T y, \qquad \alpha \in \mathbb{K},\ x, y \in X.$$

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A linear operator T from X to Y is called *bounded* with *norm* $\|T\|$ if

$$\|T\| := \sup\{\|Tx\| : x \in X,\ \|x\| = 1\} < \infty.$$

If T is not bounded it is called *unbounded*. The set of all bounded linear operators from X to Y is denoted by $L(X, Y)$.

It is easy to check that

$$\begin{aligned} \|T\| &= \sup\{\|Tx\| : x \in X,\ \|x\| = 1\} \\ &= \sup\{\|Tx\| : x \in X,\ \|x\| \le 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in X,\ x \neq 0\right\} \\ &= \inf\{M \in \mathbb{R} : \forall\, x \in X\ \|Tx\| \le M\|x\|\}. \end{aligned}$$

and that the following is equivalent:

(i) T is continuous.
(ii) T is continuous in 0.
(iii) T is bounded.
(iv) T is uniformly continuous.

Theorem 1.9. *Let X, Y be normed spaces. Then $(L(X, Y), \|\cdot\|)$ is a normed space. If Y is a Banach space, then $L(X, Y)$ is a Banach space.*

Remark. Sometimes T is defined only on a (not necessarily closed) subspace $\mathcal{D} \subset X$. Then we write

$$T : X \supseteq \mathcal{D}(T) \to Y$$

if $T|_{\mathcal{D}} : \mathcal{D} \to Y$ is a linear operator in the sense above. When the domain is not mentioned explicitely, we sometimes write $T(X, Y)$ or $T(X \to Y)$.

In general, linear operators which are not defined on all of X will be unbounded.

Example 1.10. Let $X = (C[0, 1], \|\cdot\|_\infty)$ be the space of the continuous functions on $[0, 1]$ together with the supremum norm $\|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$ and let $\mathcal{D} := C^1[0, 1]$ the space of the once continuously differentiable functions. Then the differential operator

$$T : X \supseteq \mathcal{D} \to X, \quad Tf = f'$$

is an unbounded linear operator.

Proof. Well-definedness and linearity is clear. For $n \in \mathbb{N}_0$ define $f_n \in C[0, 1]$ by $f_n(t) = t^n$. Obviously $\|f_n\|_\infty = 1$ and $\|Tf_n\|_\infty = n\|f_{n-1}\|_\infty = n$ for all $n \in \mathbb{N}$. Hence T is unbounded. □

The bounded linear maps from a normed space to $\mathbb{K}$ play a very important role.

Definition 1.11. Let X be a normed space over $\mathbb{K}$. A bounded linear map $X \to \mathbb{K}$ is called a definebounded linear functional on X. The *dual space* X' of X is the set all bounded bounded linear functionals on X, i. e., $X' = L(X, \mathbb{K})$.

Note that by Theorem 1.9 the dual space is Banach space.
That the dual space of a Hilbert space is isomorphic to itself and that every Hilbert space is reflexive follows from the following theorem.

Theorem 1.12 (Fréchet-Riesz representation theorem). *Let H be a Hilbert space. Then the map*

$$\Phi : H \to H', \quad y \mapsto \langle \cdot\, , y\rangle$$

is an isometric antilinear bijection.

We have the natural injection $X \to X''$, $x \mapsto \hat{x}$ where $\hat{x}(x') = x'(x)$ for all $x \in X$. This map is an isometry. If it is even a bijection, then X is called *reflexive.* Note that there are normed spaces which are not reflexive but nevertheless isomorphic to their bidual.

Theorem 1.13 (Hahn-Banach). *Let X be a normed space and $p : X \to \mathbb{R}$ a seminorm (a sublinear functional). Let Y be a subspace of X and $\varphi_0 \in Y'$ such that $|\varphi_0(y)| \le p(y)$ for all $y \in Y$. Then there exists an extension $\varphi \in X'$ of φ_0 with $\|\varphi\| = \|\varphi_0\|$ and $|\varphi(x)| \le p(x)$ for all $x \in X$.*

Example 1.14. Examples for dual spaces: Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$(\ell_p(\mathbb{N}))' = \ell_q(\mathbb{N})), \qquad (L_p(\Omega))' = L_q(\Omega)$$

where (Ω, Σ, μ) is a σ-finite measure space. Note that $(\ell_\infty(\mathbb{N}))' \neq \ell_1(N)$ and $(L_\infty(Omega))' = L_1(\Omega)$.
Denote by $c_0(\mathbb{N})$ the space of all sequences $(x_n)_{n\in\mathbb{N}}$ which converge to 0. Then $(c_0(\mathbb{N}))' = \ell_1(\mathbb{N})$.
The analogon for function spaces is given by the following theorem.

Theorem 1.15 (Riesz representation theorem). *Let K be a compact metric space and $M(K)$ the set of regular Borel measures of finite variation on K. Then $(C(K))' = M(K)$.*

An important role plays the *uniform boundedness principle.*

Theorem 1.16 (Uniform boundedness principle). *Let X be a complete metric space, Y a normed space and $\mathcal{F} \subseteq C(X, Y)$ a family of continuous func-*

tions which is pointwise bounded, i. e.,

$$\forall x \in X \quad \exists C_x \geq 0 \quad \forall f \in \mathcal{F} \quad \|f(x)\| < C_x.$$

Then there exists an $M \in \mathbb{R}$, $x_0 \in X$ *and* $r > 0$ *such that*

$$\forall x \in B_r(x_0) \quad \forall f \in \mathcal{F} \quad \|f(x)\| < M. \tag{1.2}$$

The following is an immediate corollary of the uniform boundedness principle.

Theorem 1.17 (Banach-Steinhaus theorem). *Let* X *be a Banach space,* Y *a normed space and* $\mathcal{F} \subseteq L(X,Y)$ *a family of continuous linear functions which is pointwise bounded, i. e.,*

$$\forall x \in X \quad \exists C_x \geq 0 \quad \forall f \in \mathcal{F} \quad \|f(x)\| < C_x.$$

Then there exists an $M \in \mathbb{R}$ *such that*

$$\|f\| < M, \qquad f \in \mathcal{F}.$$

Linear operators

Definition 1.18. Let X, Y be Banach spaces. A linear map $T \in L(X,Y)$ is called *open* if $T(U)$ is open in Y for every open subset U of X.

Theorem 1.19 (Open mapping theorem). *Let* X, Y *be Banach spaces and* $T \in L(X,Y)$. *Then* T *is open if and only if it is surjective.*

The open mapping theorem has the following important corollary.

Corollary 1.20 (Inverse mapping theorem). *Let* X, Y *be Banach spaces and* $T \in L(X,Y)$ *a bijection. Then* T^{-1} *exists and is continuous.*

For the definition of a closed operator we introduce the graph of a linear operator.
Let X, Y be Banach spaces. Then we can introduce a norm on $X \times Y$ by $\|(x,y)\|_{X\times Y} = \|x\| + \|y\|$ or $\|(x,y)\|_{X\times Y} = \sqrt{\|x\|^2 + \|y\|^2}$. The topolopies generated by either of these norms coincide.

Definition 1.21. Let X, Y be Banach spaces, $\mathcal{D} \subseteq X$ a subspace of X and $T : X \supseteq X \to Y$ a linear operator. The *graph* $\mathrm{G}(T)$ is

$$\mathrm{G}(T) := \big\{(x, Tx) : x \in \mathcal{D}\big\} \subseteq X \times Y.$$

The linear operator T is called *closed* if its graph is closed. It is called *closable* if the closure of its graph is the graph of a linear operator. If $\overline{\mathrm{G}(T)} = \mathrm{G}(\overline{T})$ then $\overline{T}$ is called the *closure* of T.

Obviously, the closure of a closable linear operator T is unique and the smallest closed extension if T. The following characterisation of closed and closable operators is often useful.

Lemma 1.22. *Let* X, Y *normed space,* $\mathcal{D} \subseteq X$ *a subspace and* $T : X \supseteq \mathcal{D} \to Y$.

(i) *T is closed if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ the following is true:*

$$\begin{aligned} &(x_n)_{n\in\mathbb{N}} \text{ and } (Tx_n)_{n\in\mathbb{N}} \text{ converge} \\ &\qquad \implies x_0 := \lim_{n\to\infty} x_n \in \mathcal{D} \text{ and } \lim_{n\to\infty} Tx_n = Tx_0. \end{aligned} \tag{1.3}$$

(ii) *T is closable if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ the following is true:*

$$\lim_{n\to\infty} x_n = 0 \text{ and } (Tx_n)_{n\in\mathbb{N}} \text{ converges} \implies \lim_{n\to\infty} Tx_n = 0. \tag{1.4}$$

The closure $\overline{T}$ of T is given by

$$\begin{aligned} \mathcal{D}(\overline{T}) &= \{x \in X : \exists\, (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D} \text{ with } \lim_{n\to\infty} x_n = x \text{ and } (Tx_n)_{n\in\mathbb{N}} \text{ converges }\}, \\ \overline{T}x &= \lim_{n\to\infty} (Tx_n) \qquad \text{for } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D} \text{ with } \lim_{n\to\infty} x_n = x. \end{aligned} \tag{1.5}$$

Theorem 1.23 (Closed graph theorem). *Let* X, Y *be Banach spaces and* $T : X \to Y$ *be a closed linear operator. Then* T *is bounded.*

The following corollary shows how closedness and continuity are related.

Lemma 1.24. *Let* X, Y *be Banach spaces,* $\mathcal{D} \subseteq X$ *a subspace and* $T : \mathcal{D} \to Y$ *linear. Then the following are equivalent:*

(i) *T is closed and $\mathcal{D}(T)$ is closed.*
(ii) *T is closed and T is continuous.*
(iii) *$\mathcal{D}(T)$ is closed and T is continuous.*

Definition 1.25. Let X, Y be Banach spaces, $\mathcal{D} \subseteq X$ a subspace and $T : X \supseteq \mathcal{D} \to Y$ a linear operator. Then

$$\|\cdot\|_T : \mathcal{D} \to \mathbb{R}, \quad \|x\|_T = \sqrt{\|x\|^2 + \|Tx\|^2}$$

is called the *graph norm* of T.

It is easy to see that $\|\cdot\|_T$ is a norm on $\mathcal{D}$. Moreover, the norm defined above is equivalent to the norm $\|x\|'_T = \sqrt{\|x\|^2 + \|Tx\|^2}$ on $\mathcal{D}$.
Note that the operator

$$\widetilde{T} : (\mathcal{D}(T), \|\cdot\|_\infty) \to Y, \quad \widetilde{T}x = Tx$$

is continuous. In general we write T instead of $\widetilde{T}$.

Linear operators

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Definition 1.26. Let X, Y be Banach spaces and $\mathcal{D}(T) \subseteq X$ a dense subspace. For a linear map $T : X \supseteq \mathcal{D}(T) \to Y$ we define

$$\mathcal{D}(T') := \{\varphi \in Y' \ : \ x \mapsto \varphi(Tx) \ \text{ is a bounded linear functional on } \ \mathcal{D}(T)\},$$

Since $\mathcal{D}(T)$ is dense in X, the map $\mathcal{D}(T) \to \mathbb{K}$, $x \mapsto \varphi(Tx)$ has a unique continuous extension $T'\varphi \in X'$ for $\varphi \in \mathcal{D}(T')$. Hence the *Banach space adjoint* T'

$$T' : Y' \supseteq \mathcal{D}(T') \to X', \qquad (T'\varphi)(x) = \varphi(Tx), \quad x \in \mathcal{D}(T),\ \varphi \in \mathcal{D}(T').$$

is well-defined.

If a linear operator acts between Hilbert spaces then its adjoint can be defined as above. However, we can also use the canonical identification of a Hilbert space with its dual to define its adjoint.

Definition 1.27. Let H_1, H_2 be Hilbert spaces and $\mathcal{D}(T) \subseteq H_1$ a dense subspace. For a linear map $T : H_1 \supseteq \mathcal{D}(T) \to H_2$ its *Hilbert space adjoint* T^* is defined by

$$\mathcal{D}(T^*) := \{y \in H_2 \ : \ x \mapsto \langle Tx\,, y\rangle \ \text{ is a bounded on } \ \mathcal{D}(T)\},$$
$$T^* : H_2 \supseteq \mathcal{D}(T^*) \to H_1, \qquad T^*y = y^*,$$

where $y^* \in H_1$ such that $\langle Tx\,, y\rangle = \langle x\,, y^*\rangle$ for all $x \in \mathcal{D}(T)$.
Note that for $y \in \mathcal{D}(T^*)$ the map $x \mapsto \langle Tx\,, y\rangle$ is continuous and densely defined and can therefore be extended uniquely to an element $\varphi_y \in H_1'$. By the Fréchet-Riesz representation theorem (Theorem 1.12) there exists exactly one $y^* \in H_1$ as desired.

Remark 1.28. Note that the application $T \mapsto T'$ is linear wheras $T \mapsto T^*$ is antilinear (that is, $(\alpha T)^* = \overline{\alpha} T^*$ for $\alpha \in \mathbb{K}$).
If Φ_1 and Φ_2 are the maps of the Fréchet-Riesz representation theorem (Theorem1.12) corresponding to H_1 and H_2 respectively, then $T^* = \Phi_1^{-1} T' \Phi_2$.

Note that T is bounded if and only if its adjoint is bounded. In this case $\|T\| = \|T^*\|$.
The following two theorems are true for Banach or Hilbert spaces.

Theorem 1.29. *Let X, Y, Z be Banach spaces and $R(X \to Y)$, $S(X \to Y)$, $T(Y \to Z)$ densely defined linear operators. Then*

(i) $(R + S)' \subseteq R' + S'$ *if* $\mathcal{D}(R + S) = \mathcal{D}(R) \cap \mathcal{D}(S)$ *is dense in* X.

(ii) $(TS)' \subseteq S'T'$ *if* $\mathcal{D}(TS) = \{x \in \mathcal{D}(S) : Sx \in \mathcal{D}(TS)\}$ *is dense in* X.

Theorem 1.30. *Let X, Y be Banach spaces and $T(X \to Y)$ a densely defined linear operator. Then T' is closed.*

Now we consider linear operators between Hilbert spaces.

Theorem 1.31. *Let H_1, H_2 be Hilbert spaces and $T(H_1 \to H_2)$ a densely defined linear operator. Then the following is true.*

(i) *T^* is closed.*

(ii) *If T^* is densely defined, then $T \subseteq T^{**}$.*

(iii) *If T^* is densely defined and S is a closed extension of T, then $T^{**} \subseteq S$, in particular T is closable and $\overline{T} = T^{**}$.*

(iv) *If T is closable then T^* is densely defined and $\overline{T} = T^{**}$.*

Definition 1.32. Let H be a Hilbert space and $T(H \to H)$ a densely defined linear operator.

(i) T is *symmetric* $\iff$ $T \subseteq T^*$.

(ii) T is *selfadjoint* $\iff$ $T = T^*$.

(iii) T is *essentially selfadjoint* $\iff$ $\overline{T}$ is selfadjoint.

Proposition 1.33. (i) *T symmetric $\implies$ $T \subseteq T^{**} \subseteq T^* = T^{***}$.*

(ii) *T closed and symmetric $\iff$ $T = T^{**} \subseteq T^*$.*

(iii) *T selfadjoint $\iff$ $T = T^{**} = T^*$.*

(iv) *T essentially selfadjoint $\iff$ $T \subseteq T^{**} = T^*$.*

Theorem 1.34. *Let H_1, H_2 be Hilbert spaces and $T(H_1 \to H_2)$ a densely defined linear operator.*

(i) $\operatorname{rg}(T)^\perp = \ker(T^*)$.

(ii) $\overline{\operatorname{rg}(T)} = \ker(T^*)^\perp$.

(iii) $\operatorname{rg}(T^*)^\perp = \ker(T*)$.

(iv) $\overline{\operatorname{rg}(T^*)} = \ker(T*)^\perp$.

Theorem 1.35 (Hellinger-Toeplitz). *Let H be a Hilbert space, $T : H \to H$ a linear operator such that $\langle Tx\,, y\rangle = \langle x\,, Ty\rangle$ for all $x, y \in H$ (that is, T is formally symmetric). Then T is bounded.*

Spectrum of linear operapors

Definition 1.36. Let X be a Banach space and $T(X \to X)$ a densely defined linear operator.

$$\begin{aligned} \rho(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is bijective}\} && \textit{resolvent set} \text{ of } T, \\ \sigma(T) &:= \mathbb{C} \setminus \rho(T) && \textit{spectrum} \text{ of } T. \end{aligned}$$

The spectrum of T is further divided in *point spectrum* $\sigma_{\mathrm{p}}(T)$, *continuous spectrum* $\sigma_{\mathrm{c}}(T)$ and *residual spectrum* $\sigma_{\mathrm{r}}(T)$:

$$\begin{aligned} \sigma_{\mathrm{p}}(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is not injective}\}, \\ \sigma_{\mathrm{c}}(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is injective, } \operatorname{rg}(T - \lambda \operatorname{id}) \neq X,\ \overline{\operatorname{rg}(T - \lambda \operatorname{id})} = X\}, \\ \sigma_{\mathrm{r}}(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is injective, } \overline{\operatorname{rg}(T - \lambda \operatorname{id})} \neq X\}. \end{aligned}$$

It follows immediately from the definition that

$$\sigma(T) = \sigma_{\mathrm{p}}(T) \,\dot\cup\, \sigma_{\mathrm{c}}(T) \,\dot\cup\, \sigma_{\mathrm{r}}(T).$$

In the following, we often write $\lambda - T$ instead of $\lambda \operatorname{id} - T$.

Remark 1.37. If T is closed, then $(T - \lambda)^{-1}$ is closed if it exists. Therefore, by the closed graph theorem,

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is injective and } (T - \lambda)^{-1} \in L(X)\}.$$

Often the resolvent set of a linear operator is defined slightly different: Let $T(X \to X)$ is a densely defined linear operator. Then $\lambda \in \rho(T)$ if and only if $\lambda - T$ is bijective and $(\lambda - T) \in L(X)$. With this definition it follows that $\rho(T) = \emptyset$ for every non-closed $T(X \to X)$ because one of the following cases holds:

(i) $\lambda - T$ is not bijective $\implies \lambda \notin \rho(T)$;

(ii) $\lambda - T$ is bijective, then $(\lambda - T)^{-1}$ is defined everywhere and closed, so by the closed graph theorem it cannot be bounded, which implies $\lambda \notin \rho(T)$.

Remark 1.38. (i) If T is bounded, then $\sigma(T) \neq \emptyset$ and $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$.

(ii) If T is unbounded, then $\sigma(T) = \emptyset$ is possible.

Lemma 1.39. *Let X be a Banach space and $T(X \to X)$ a closed linear operator. Then the resolvent set $\rho(T)$ is open and the* resolvent map

$$\rho(T) \to L(H), \qquad \lambda \mapsto R(\lambda, T) := (\lambda - T)^{-1}$$

is analytic. Moreover

(i) $\|R(\lambda_0, T)\| \geq \dfrac{1}{\operatorname{dist}(\lambda_0, \sigma(T))}$ *for all* $\lambda_0 \in \rho(T)$.

(ii) *For* $\lambda_0 \in \rho(T)$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda - \lambda_0| < \|R(\lambda_0, T)\|^{-1}$

$$R(\lambda, T) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (R(\lambda_0, T))^{n+1}.$$

Let X be a Banach space and $T \in L(X)$. Then the *spectral radius* of T is defined by $r(T) := \limsup_{n\to\infty} \|T^n\|^{1/n}$. The spectral radius gives an estimate for the spectrum of T.

Theorem 1.40. *For a Banach space X and $T \in L(X)$ the following holds:*

(i) $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$*, in particular* $r(T) \leq \|T\|$.

(ii) $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}$.

(iii) *If X is a complex Banach space, then* $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$.

(iv) *If X is a Hilbert space, then* $r(T) = \|T\|$.

It can be shown that a linear operator T on a *complex* Hilbert space H is symmetric if and only if $\langle Tx\,, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{D}(T)$ and that $\sigma(T) \subseteq \mathbb{R}$. The next theorems show how the spectrum of a symmetric operator T is related to selfadjointness of

Projections

Definition 1.41. Let X be Banach space. An operator $P : X \to X$ is called *projection* if and only of $P^2 = P$.

Remark 1.42. (i) If P is an projection then also $\operatorname{id} - P$ is an projection.

(ii) If $P \in L(X)$ is an projection then either $\|P\| = 0$ or $\|P\| \geq 1$.

Definition 1.43. Let H be Hilbert space. A projection $P \in L(H)$ is called *orthogonal projection* if there exists a closed subspace $U \subseteq H$ such that $\operatorname{rg} P = U$ and $\ker U = (\operatorname{rg} P)^\perp$.
In this case, $\|P\| = 0$ or $\|P\| = 1$.

Note that every $x \in H$ can be written as $x = Px + (1 - P)x$. If P is an orthogonal projection on U, then Px is the unique element in U such that $\|x - Px\| = \operatorname{dist}(x, U)$.
In the following, we collect some useful results on orthogonal projections.

Theorem 1.44. *Let H be a Hilbert space, $P \in L(H)$ a projection with $P \neq 0$. The the following are equivalent.*

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(i) *P is an orthogonal projection.*

(ii) $\|P\| = 1$.

(iii) *$\|P\|$ is selfadjoint.*

(iv) *$\|P\|$ is normal (i. e. $PP^* = P^*P$).*

(v) *$\langle Px\,, x\rangle \geq 0$ for all $x \in H$.*

Theorem 1.45. *Let H be a Hilbert space, $P, Q \in L(H)$ orthogonal projections.*

(i) *The the following are equivalent:*

  (a) *PQ is an orthogonal projection.*

  (b) *QP is an orthogonal projection.*

  (c) *$PQ = QP$ is an orthogonal projection.*

  In this case $\operatorname{rg}(PQ) = \operatorname{rg}(QP) = \operatorname{rg}(P) \cap \operatorname{rg}(Q)$.

(ii) *The the following are equivalent:*

  (a) *$P + Q$ is an orthogonal projection.*

  (b) *$PQ = QP = 0$.*

  (c) $\operatorname{rg}(P) \perp \operatorname{rg}(Q)$.

(iii) *The the following are equivalent:*

  (a) *$P - Q$ is an orthogonal projection.*

  (b) *$PQ = QP = Q$.*

  (c) $\operatorname{rg}(Q) \subseteq \operatorname{rg}(P)$.

  (d) *$\|Qx\| \leq \|Px\|$ for all $x \in H$.*

  (e) *$\langle Qx\,, x\rangle \leq \langle Px\,, x\rangle$ for all $x \in H$.*

Theorem 1.46. *Every monotonic sequence of orthogonal projections $(P_n)_{n\in\mathbb{N}}$ converges strongly to an orthogonal projection.*
If the sequence is increasing, then the strong limit is the orthotgonal projection on $\overline{\bigcup_{n\in\mathbb{N}} \operatorname{rg} P_n}$.
If the sequence is decreasing, then the strong limit is the orthotgonal projection on $\bigcap_{n\in\mathbb{N}} \operatorname{rg} P_n$.

Compact linear operators

Definition 1.47. Let X, Y be normed spaces. An operator $T \in L(X, Y)$ is called *compact* if for every bounded set $A \subseteq X$ the set $T(A)$ is relatively compact. The set of all compact operators from X to Y is denoted by $K(X, Y)$. Sometimes compact operators are called *completely continuous*.

Remarks 1.48. (i) Every compact linear operator is bounded.

(ii) $T \in L(X, Y)$ is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a convergent subsequence.

(iii) $T \in L(X, Y)$ is compact if and only if $T(B_X(0, 1))$ is relatively compact.

(iv) Let $T \in L(X, Y)$ with finite dimensional $\operatorname{rg}(T)$. The T is compact.

(v) The identity map $\mathrm{id} \in L(X)$ is compact if and only if X is finite-dimensional.

(vi) $K(X)$ is a two-sided closed ideal in $L(X)$.

Theorem 1.49 (Schauder). *Let X, Y be Banach space and $T \in L(X, Y)$. Then T is compact if and only if T' is compact.*

Let X be a vector space and $T : X \to X$ a linear operator. Note that for $\lambda \in \mathbb{C} \setminus \{0\}$ the ascent $\alpha(\lambda - T)$ and the descent $\delta(\lambda - T)$ are finite and equal where

$$\alpha(\lambda - T) := \begin{cases} \min\{k \in \mathbb{N}_0 : \ker(\lambda - T)^k = \ker(\lambda - T)^{k+1}\}, & \text{if the minimum exists,} \\ \infty & \text{else} \end{cases}$$

$$\delta(\lambda - T) := \begin{cases} \min\{k \in \mathbb{N}_0 : \operatorname{rg}(\lambda - T)^k = \operatorname{rg}(\lambda - T)^{k+1}\}, & \text{if the minimum exists,} \\ \infty & \text{else.} \end{cases}$$

The number $p := \alpha(\lambda - T) = \delta(\lambda - T)$ is called the *Riesz index* of $\lambda - T$.

Theorem 1.50 (Spectrum of a compact operator). *Let X be a Banach space. For a compact operator $T \in L(X)$ the following holds.*

(i) *If $\lambda \in \mathbb{C} \setminus \{0\}$, then λ either belongs to $\rho(T)$ or it is an eigenvalue of T, that is $\mathbb{C} \setminus \{0\} \subseteq \rho(T) \cup \sigma_p(T)$.*

(ii) *The spectrum of T is at most countable and 0 is the only possible accumulation point.*

(iii) *If $\lambda \in \sigma(T) \setminus \{0\}$, then the dimension of the algebraic eigenspace $\mathcal{A}_\lambda(T)$ is finite and $\mathcal{A}_\lambda(T) = \ker(\lambda - T)^p$ where p is the Riesz index of $\lambda - T$.*

(iv) *$X = \ker(\lambda - T)^p \oplus \operatorname{rg}(\lambda - T)^p$ for $\lambda \in \sigma(T) \setminus \{0\}$ where p is the Riesz index of $\lambda - T$ and $\ker(\lambda - T)^p$ and $\operatorname{rg}(\lambda - T)^p$ are T-invariant.*

(v) *$\sigma_p(T) \setminus \{0\} = \sigma_p(T') \setminus \{0\}$ and $\sigma(T) = \sigma(T')$. If H is a Hilbert space then $\sigma_p(T) \setminus \{0\} = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma_p(T^*)\} \setminus \{0\} = \overline{\sigma_p(T^*)} \setminus \{0\}$, where the bar denotes complex conjugation, and $\sigma(T) = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma(T^*)\} = \overline{\sigma(T^*)}$.*

Theorem 1.51 (Spectral theorem for compact selfadjoint operators). *Let H be a Hilbert space and $T \in L(H)$ a compact selfadjoint operator.*

(i) *There exists an orthonormal system* $(\mathrm{e}_n)_{n=1}^N$ *of eigenvectors of* T *with eigenvalues* $(\lambda_n)_{n=1}^N$ *where* $N \in \mathbb{N} \cup \{\infty\}$ *such that*

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x \,, \mathrm{e}_n \rangle \, \mathrm{e}_n, \qquad x \in H. \tag{1.6}$$

The λ_n *can be chosen such that* $|\lambda_1| \geq |\lambda_2| \geq \cdots > 0$. *The only possible accumulation point of the sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *is* 0.

(ii) *If* P_0 *is the orthogonal projection on* $\ker T$, *then*

$$x = P_0 x + \sum_{n=1}^{N} \langle x \,, \mathrm{e}_n \rangle \, \mathrm{e}_n, \qquad x \in H. \tag{1.7}$$

(iii) *If* $\lambda \in \rho(T)$, $\lambda \neq 0$

$$(\lambda - T)^{-1} x = \lambda^{-1} P_0 x + \sum_{n=1}^{N} \frac{\langle x \,, \mathrm{e}_n \rangle}{\lambda_n - \lambda} \, \mathrm{e}_n, \qquad x \in H.$$

Note that the representation in (1.6) is not unique. A unique represention is obtained if we define orthogonal projections P_j on the eigenspaces corrsponding to μ_j where the μ_j are the pairwise distinct non-zero eigenvalues of T. Then for all $x \in H$

$$Tx = \sum_{n=1}^{N} \mu_n P_n x, \qquad x = P_0 x + \sum_{n=1}^{N} P_n x. \tag{1.8}$$

Note also that $T = \sum_{n=1}^{N} \mu_n P_n$ in the operator norm.

Interpretation/Application of the spectral theorem

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Diagonalisation of T.

From finite dimensional linear algebra it is known that for every hermitian linear operator T there exists an orthoganal basis with respect to which the matrix representation of T has diagonal form. Writing T as an infinite matrix with respect to the orthogonal system introduced in Theorem 1.51 (i) we obtain

$$Tx = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}$$

where $x = \sum_{n=1}^{N} x_n \, \mathrm{e}_n = (x_1, \, x_2, \, x_3, \, \dots)^t$. Note that $x_n = \langle x \,, e_n \rangle$.

T is unitarily equivalent to a multiplication operator on an L_2-space.

Assume that $\ker T = \{0\}$. Then from the above representation it is clear that

$$T = U M_T U^{-1}$$

where

$$U : H = \overline{\mathrm{rg}(T)} \to \ell(\mathbb{N}), \qquad U\Big(\sum_{n=1}^{\infty} \alpha_n \, \mathrm{e}_n \Big) = (\alpha_n)_{n\in\mathbb{N}}$$

and

$$M_T : \ell(\mathbb{N}) \to \ell(\mathbb{N}), \qquad M_T x = (\lambda x_n)_{n\in\mathbb{N}} \quad \text{for } \ x = (x_n)_{n\in\mathbb{N}}.$$

If T has only finitely many eigenvalues then the space $\ell(\mathbb{N})$ has to be replaced by $\ell(\{1, 2, \dots, N\})$ and the operator U has to be modified accordingly.

T as an integral.

Assume that all eigenvalues of T are positive: $\mu_1 < \mu_2 < \cdots < 0$ and let P_j be the orthogonal projection on the eigenspace corresponding to μ_j. Define $E_\lambda = \sum_{\mu_j < \lambda} P_j$. Then $P_n = E_{\lambda_n} - E_{\lambda_{n-1}} =: \Delta E_n$ and therefore

$$T = \sum_{n=1}^{N} \mu_n P_n = \sum_{n=1}^{N} \mu_n \big(E_{\lambda_n} - E_{\lambda_{n-1}} \big) = \sum_{n=1}^{N} \mu_n \Delta E_{\lambda_n}.$$

Functional calculus for T.

If f is a bounded function defined on $\sigma(T)$ then we can define $f(T)$ by

$$f(T) = \sum_{n=1}^{N} f(\mu_n) P_n.$$

When f is polynomial, this definition coincides with the usual definition of the polynomial of a bounded linear operator. Also for $f(x) = (\lambda_0 - x)^{-1}$ where $\lambda \in \rho(T)$ the definition above and the usual definition coincide. Note that for an eigenvector x of T with eigenvalue μ we have that $f(T)x = f(\mu)x$.

In the next chapter we will see how the above can be extended to selfadjoint linear operators that are not necessarily compact.

DRAFT

Chapter 2

The spectral theorem

2.1 The Riemann-Stieltjes integral

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Definition 2.1. The *total variation* of a function $\alpha : [a, b] \to \mathbb{K}$ is defined by

$$\operatorname{var} \alpha := \sup \Big\{ \sum_{j=1}^{n} |\alpha(t_j) - \alpha(t_{j-1})| \ : \ a = t_0 < t_1 < \cdots < t_n = b \Big\}.$$

α is said to be of *bounded variation* (or *finite variation*) if $\operatorname{var} \alpha < \infty$.
The set of all functions of bounded variation on $[a, b]$ is denoted by $\mathrm{BV}[a, b]$.

Remark 2.2.

- $\mathrm{BV}[a, b]$ with $\|\alpha\| = \alpha(a) + \operatorname{var} \alpha$, $\alpha \in \mathrm{BV}[a, b]$ is a non-separable normed space.
- Every $\alpha \in \mathrm{BV}[a, b]$ can be written as difference of two monotonic functions (Jordan decomposition).

Definition 2.3. Let $a \le t_0 < t_1 < \ldots < t_n \le b$. We say $f = \binom{t_0, t_1, \ldots, t_n}{c_1, \ldots, c_n}$ is a *step function* if $f : [a, b] \to \mathbb{K}$ and $f(t) = c_j$ if and only if $t \in [t_{j-1}, t_j]$.
The set of all step functions on $[a, b]$ is denoted by $T[a, b]$.

Remark 2.4. $(T[a, b], \|\cdot\|)$ is a normed space. It is a subspace of $(B[a, b], \|\cdot\|)$ where $B[a, b]$ is the set of all bounded functions on $[a, b]$ and

$$\|f\|_\infty := \sup \big\{ |f(t)| : t \in [a, b] \big\}, \qquad f \in B[a, b].$$

Definition 2.5. The closure of $T[a, b]$ in $B[a, b]$ is denoted by $I[a, b]$.

Remark 2.6. The following can be shown:

- $C[a, b] \subseteq I[a, b]$.
- If $f \in I[a, b]$, then $f(x + 0)$ exists for $x \in [a, b)$ and $f(x - 0)$ exists for $x \in (a, b]$, where as usual $f(x \pm 0) := f(x\pm) := \lim_{\varepsilon \searrow 0} f(x \pm \varepsilon)$.

Integration with respect to $\alpha \in \mathrm{BV}[a, b]$

Definition 2.7. Fix $\alpha \in \mathrm{BV}[a, b]$. For $f = \binom{t_0, t_1, \ldots, t_n}{c_1, \ldots, c_n} \in T[a, b]$ define

$$i_\alpha(f) := \int f \, \mathrm{d}\alpha := \sum_{j=1}^{n} c_j (\alpha(t_j) - \alpha(t_{j-1})).$$

Observe that $i_\alpha(f)$ is independent of the representation of f, hence it is well defined. Obviously, i_α is linear in f and

$$|i_\alpha(f)| \le \sum_{j=1}^{n} |c_j| |\alpha(t_j) - \alpha(t_{j-1})| \le \|f\|_\infty \operatorname{var} \alpha, \qquad f \in T[a, b].$$

Proposition 2.8. *The function $i_\alpha : (T[a, b], \| \, \|_{infty}) \to \mathbb{K}$ is a bounded linear operator with $\|i_\alpha\| \le \operatorname{var} \alpha$. It can be extended to a continuous linear operator $\hat{i}_\alpha : I[a, b] \to \mathbb{K}$. The extension is unique and $\|\hat{i}_\alpha\| = \|i_\alpha\|$.*

For $f \in I[a, b]$, we write

$$\int f \, \mathrm{d}\alpha := \hat{i}_\alpha(f).$$

Note that for $f \in I[a, b]$

$$\Big\| \int f d\alpha \Big\| = \|\hat{i}_\alpha(f)\| \le \|\hat{i}_\alpha\| \|f\|_\infty = \|i_\alpha\| \|f\|_\infty = \operatorname{var} \alpha \|f\|_\infty.$$

If $\alpha \in \mathrm{BV}[a, b]$ and $[a', b'] \subseteq [a, b]$, then it is easy to see that $\alpha|_{[a', b']} \in \mathrm{BV}[a', b']$.

Proposition 2.9. *For $\alpha \in \mathrm{BV}[a, b]$, $f \in I[a, b]$ and $x \in [a, b]$ let*

$$K : [a, b] \to \mathbb{K}, \quad K(x) := \int_a^x f \, \mathrm{d}\alpha, \quad \text{if } x \in (a, b] \quad \text{and} \quad K(a) = 0.$$

Then we have:

(i) *$K \in \mathrm{BV}[a, b]$ and $K(a) = 0$.*
(ii) *If f is right-continuous then K is right continuous.*
(iii) *For all $g \in I[a, b]$ we have $\int g \, \mathrm{d}K = \int gf \, \mathrm{d}\omega$.*

Proof. Exercise **??**. □

Proposition 2.9 shows that $\mathrm{BV}[a, b] \subseteq (C[a, b])'$. The reverse inclusion is shown in the following theorem.

Theorem 2.10 (F. Riesz). *ω is right-continuous in (a, b); For $\varphi \in (C[a, b], \mathbb{R})'$ there exists a unique real valued $\omega \in \mathrm{BV}[a, b]$ satisfying*

(i) ω *is right-continuous in* (a, b)*;*

(ii) $\omega(a) = 0$*;*

(iii) $\varphi(f) = \int f \, \mathrm{d}\omega$ *for all* $f \in C[a, b]$*;*

(iv) $\operatorname{var} \omega = \|\varphi\|$.

Proof. A proof can be found for instance in [Tay58, §4.32] (A. Taylor, Introduction to Functional Analysis). □

Remark 2.11. Without conditions (i) and (ii) the representation of φ as a function $\omega \in \mathrm{BV}[a, b]$ is *not* unique.

2.2 Spectral families

Definition 2.12. Let H be a Hilbert space. $(E_\lambda)_{\lambda \in \mathbb{R}} \subseteq L(H)$ is called a *spectral family* (or *spectral resolution of the identity*) if and only in for all $x \in H$ we have:

(i) E_λ is an orthogonal projection for all $\lambda \in \mathbb{R}$.

(ii) $E_\lambda E_\mu = E_\mu E_\lambda = E_\mu$ for $\mu \leq \lambda$.

(iii) $E_\mu x \to E_\lambda x$ if $\mu \searrow \lambda$ (strong-right continuity).

(iv) $E_\mu x \to x$ for $\mu \to \infty$.

(v) $E_\mu x \to 0$ for $\mu \to -\infty$.

Remark 2.13. Let $(E_\lambda)_{\lambda \in \mathbb{R}}$ be a spectral family.

(i) If $\mu < \lambda$, then $E_\mu < E_\lambda$ by (i) and (ii) and Theorem 1.45 (iii).

(ii) Since $(E_\lambda)_\lambda$ is increasing, then, by Theorem 1.46, the strong left limit exists and is an orthogonal projection (that is, for all $\mu \in \mathbb{R}$ and $x \in H$ the limit $\lim_{\lambda \nearrow \mu} E_\lambda x$ exists). Note, however, that in general $E(\lambda) \neq s\text{-}\lim_{\lambda \nearrow \mu} E_\lambda$.

Notation 2.14. Let $(E_\lambda)_{\lambda \in \mathbb{R}}$ be a spectral family.

- Instead of E_λ we also write $E(\lambda)$.
- Let $-\infty \leq a < b \leq \infty$. Then

$$\begin{aligned} E((a, b]) &:= E(b) - E(a), & E([a, b)) &:= E(b-) - E(a), \\ E((a, b)) &:= E(b-) - E(a-), & E([a, b]) &:= E(b) - E(a-), \\ E(\{b\}) &:= E(b) - E(b-) \end{aligned}$$

where $E(-\infty) := 0$ and $E(\infty) := \mathrm{id}$.

Example 2.15. Let $T \in L(H)$ be a compact self-adjoint operator with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \ldots > 0$, $\lambda_j \neq \lambda_h$ for $j \neq h$, and let P_j be the projection on the eigenspace corresponding to λ_j.
For $\lambda \in \mathbb{R}$ and $x \in H$ define

$$E_\lambda x := \begin{cases} \sum\limits_{\lambda_j \leq \lambda} P_j x, & \text{if } \lambda < 0 \\ x - \sum\limits_{\lambda_j > \lambda} P_j x & \text{if } \lambda \geq 0. \end{cases}$$

Then $(E_\lambda)_\lambda$ is a spectral family (the spectral resolution of T).

Proof. Exercise **??**. □

Remark 2.16. Let $\mathcal{B}(\mathbb{R})$ be the set of all Borel sets on $\mathbb{R}$. A map

$$P : (\mathbb{R}) \to L(H)$$

is called a *projection valued measure* if it is additive, that is, if for pairwise disjoint $U_j \in \mathcal{B}(\mathbb{R})$ and all $x \in H$

$$P(\dot{\bigcup_{j=1}^{\infty}} U_j)x = \sum_{j=1}^{\infty} P(U_j)x.$$

Observe that the sum on the right hand side does not converge in the operator in general.

Lemma 2.17 (Properties of spectral families). *Every spectral family* $(E_\lambda)_{\lambda \in \mathbb{R}}$ *satisfies the following:*

(i) $E_\lambda - E_\mu$ *is an orthogonal projection if* $\mu \leq \lambda$.

(ii) *If* $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$,

$$(E_{\lambda_2} - E_{\lambda_1})(E_{\lambda_4} - E_{\lambda_3}) = (E_{\lambda_4} - E_{\lambda_3})(E_{\lambda_2} - E_{\lambda_1}) = 0.$$

(iii) *If* $\lambda_1 < \lambda_2 < \lambda_3$ *and* $x \in H$,

$$\|(E_{\lambda_3} - E_{\lambda_1})x\|^2 = \|(E_{\lambda_3} - E_{\lambda_2})x\|^2 + \|(E_{\lambda_2} - E_{\lambda_1})x\|^2 = \langle (E_{\lambda_3} - E_{\lambda_1})x \,, x \rangle.$$

(iv) *For fixed* $x \in H$ *the function* $\lambda \longmapsto \langle E_\lambda x \,, x \rangle$ *is monotonically increasing and bounded by* $\|x\|^2$.

(v) *The function* $\lambda \longmapsto E_\lambda$ *is strongly right-continuous. For every* $\lambda \in \mathbb{R}$ *the strong left limit exists and is an orthogonal projection but in general* $E_{\lambda-} \neq E_\lambda = E_{\lambda+}$.

(vi) *For all* $x, y \in H$ *the function* $\omega_{xy} : \lambda \longmapsto \langle E_\lambda x \,, y \rangle$ *belongs to* $\mathrm{BV}[a, b]$ *for every* $[a, b] \subseteq \mathbb{R}$ *and* $\operatorname{var} \omega_{xy}|_{[a,b]} \leq \|x\| \|y\|$.

Proof. (i) follows from properties of orthogonal projections (Theorem 1.45).

(ii) is verified by straightforward calculation.

(iii) Since $(E_{\lambda_3} - E_{\lambda_1})$ is a projection we obtain

$$\begin{aligned}\|(E_{\lambda_3} - E_{\lambda_1})x\|^2 &= \langle (E_{\lambda_3} - E_{\lambda_1})^2 x\,, x\rangle \\ &= \langle (E_{\lambda_3} - E_{\lambda_1}) x\,, x\rangle \\ &= \langle (E_{\lambda_3} - E_{\lambda_2}) x\,, x\rangle + \langle (E_{\lambda_2} - E_{\lambda_1}) x\,, x\rangle \\ &= \|(E_{\lambda_3} - E_{\lambda_2})x\|^2 + \|(E_{\lambda_2} - E_{\lambda_1})x\|^2.\end{aligned}$$

(iv) follows from properties of orthogonal projections (Theorem 1.45) and the fact that $\langle E_\lambda x\,, x\rangle \le \|E_\lambda\|\|x\|^2 \le \|x\|^2$.

(v) follows from Theorem 1.46.

(vi) Fix $x, y \in H$ and $[a,b] \subseteq \mathbb{R}$. For every partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$

$$\begin{aligned}\sum_{j=1}^n |\omega_{xy}(t_j) - \omega_{xy}(t_{j-1})| &= \sum_{j=1}^n |\langle (E_{t_j} - E_{t_{j-1}})x\,, y\rangle| \\ &= \sum_{j=1}^n |\langle (E_{t_j} - E_{t_{j-1}})x\,, (E_{t_j} - E_{t_{j-1}})y\rangle| \\ &\le \sum_{j=1}^n \|(E_{t_j} - E_{t_{j-1}})x\|\|(E_{t_j} - E_{t_{j-1}})y\| \\ &\le \Big(\sum_{j=1}^n \|(E_{t_j} - E_{t_{j-1}})x\|^2\Big)^{\frac12}\Big(\sum_{j=1}^n \|(E_{t_j} - E_{t_{j-1}})y\|^2\Big)^{\frac12} \qquad (2.1) \\ &= \|(E_b - E_a)x\|\|(E_b - E_a)y\| \qquad (2.2) \\ &\le \|x\|\|y\|\end{aligned}$$

where in (2.1) we used the Cauchy-Schwarz inequality and in (2.2) we used (iii). □

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Definition 2.18 (Integration with respect to a spectral family). Let H be a Hilbert space and $(E_\lambda)_{\lambda\in\mathbb{R}}$ a spectral family. For a step function $f = \binom{t_0, t_1, \ldots, t_n}{c_1, \ldots, c_n} \in T[a,b]$ we define in analogy to definition 2.7 the integral with respect to $(E_\lambda)_{\lambda\in\mathbb{R}}$ by

$$\int_a^b f \,\mathrm{d}E_\lambda = \sum_{j=1}^n c_j (E_{t_j} - E_{t_{j-1}}).$$

Observe that the integral does not depend on the representation of f.

Theorem 2.19. $(T[a,b], \|\cdot\|_\infty) \to L(H)$, $f \mapsto \displaystyle\int_a^b f \,\mathrm{d}E_\lambda$ *is a bounded linear map with bound* ≤ 1.

Proof. The linearity of the map is clear. To prove that is bounded by one, we calculate for $f = \binom{t_0, t_1, \ldots, t_n}{c_1, \ldots, c_n} \in T[a,b]$ and $x \in H$ we obtain, using Lemma 2.17 (iii)

$$\begin{aligned}\Big\|\int_a^b f \,\mathrm{d}E_\lambda\, x\Big\|^2 &= \Big\|\sum_{j=1}^n c_j (E_{t_j} - E_{t_j})x\Big\|^2 = \sum_{j=1}^n |c_j|\,\big\|(E_{t_j} - E_{t_j})x\,\big\|^2 \\ &\le \max\{|c_j|^2 : j = 1, \ldots, n\} \sum_{j=1}^n \big\|(E_{t_j} - E_{t_j})x\,\big\|^2 \\ &= \|f\|_\infty \,\|(E_{t_n} - E_{t_0})x\,\|^2 \le \|f\|_\infty \|x\|^2.\end{aligned}$$ □

Definition 2.20. By the theorem above there exists exactly one continuous extension of $\int_a^b \cdot\, \mathrm{d}E_\lambda$ from the space $T[a,b]$ to $I[a,b] = \overline{T[a,b]}$. This extension will again be denoted by

$$\int_a^b f \,\mathrm{d}E_\lambda \qquad \text{for} \quad f \in I[a,b].$$

is a bounded linear map with bound ≤ 1.

Note that the extension has norm ≤ 1.

Lemma 2.21 (Properties of the integral). *Let* $(E_\lambda)_{\lambda\in\mathbb{R}}$ *be a spectral resolution on a Hilbert space* H *and* $f, g \in I[a,b]$. *Then the following holds:*

(i) $\displaystyle\left\langle \left(\int_a^b f(\lambda)\,\mathrm{d}E_\lambda\right) x\,, y \right\rangle = \int_a^b f(\lambda)\,\mathrm{d}\langle E_\lambda x\,, y\rangle, \qquad x, y \in H.$

(ii) $\displaystyle E_\mu \int_a^b f(\lambda)\,\mathrm{d}E_\lambda = \int_a^\mu f(\lambda)\,\mathrm{d}E_\lambda, \qquad a \le \mu \le b.$

(iii) $\displaystyle\left(\int_a^b f(\lambda)\,\mathrm{d}E_\lambda\right)\left(\int_a^b g(\lambda)\,\mathrm{d}E_\lambda\right) = \left(\int_a^b (fg)(\lambda)\,\mathrm{d}E_\lambda\right).$

(iv) $\displaystyle\left(\int_a^b f(\lambda)\,\mathrm{d}E_\lambda\right)^* = \int_a^b \overline{f}(\lambda)\,\mathrm{d}E_\lambda.$

(v) $\displaystyle\left\|\int_a^b f(\lambda)\,\mathrm{d}E_\lambda x\right\|^2 = \int_a^b |f(\lambda)|^2\,\mathrm{d}\langle E_\lambda x\,, x\rangle.$

Proof. Exercise 2.5. □

Corollary 2.22. *Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a spectral resolution on a Hilbert space H and $[a,b]\subseteq\mathbb{R}$. Then*

$$A := \int_a^b \lambda \, \mathrm{d}E_\lambda$$

is a bounded selfadjoint linear operator with bound $\|A\| = \max\{|a|, |b|\}$.

2.3 The spectral theorem for bounded selfadjoint operators

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For the proof of the spectral theorem we will construct certain sesquilinear forms. The following theorem shows to associate a bounded linear operator to a bounded sesquilinear form. Recall that a sesquilinear form $\mathfrak{t} : H\times H\to\mathbb{C}$ is called *bounded* if there exists a $M > 0$ such that

$$|\mathfrak{t}[x,y]| \le M\|x\|\,\|y\|, \qquad x,y\in H. \tag{2.3}$$

The form is called *symmetric* if $\mathfrak{t}(x,y) = \overline{\mathfrak{t}[y,x]}$ for all $x,y\in H$.

Theorem 2.23 (Lax-Milgram). *Let H be a Hilbert space and $\mathfrak{t} : H\times H\to\mathbb{K}$ a bounded sesquilinear form. Then there exists exactly one bounded operator $T\in L(H)$ such that*

$$\langle Tx\,, y\rangle = \mathfrak{t}[x,y], \qquad x,y\in H.$$

If $\mathfrak{t}$ is symmetric, then T is selfadjoint.

An extension of the theorem to symmetric unbounded sesquilinear forms will be proved in **??**.

Proof of Theorem 2.23. Let M as in (2.3). For every $y\in H$ we define the map

$$\varphi_y : H\to\mathbb{K}, \quad \varphi_y(x) := \mathfrak{t}[x,y].$$

Obviously, φ_y is linear and bounded by $M\|y\|$. Hence, by the Riesz-Frechét theorem, there exists a $Sy\in H$ such that $\mathfrak{t}[x,y] = \langle x\,, Sy\rangle$ for every $x\in H$. In order to prove that $y\mapsto Sy$ is linear, we fix $y_1, y_1, x\in H$ and $c\in\mathbb{K}$. It follows that

$$\begin{aligned}\langle x\,, S(cy_1+y_2) - cSy_1 - Sy_2\rangle &= \mathfrak{t}[x,\, S(cy_1+y_2) - cSy_1 - Sy_2]\\ &= \mathfrak{t}[x,\, S(cy_1+y_2)] - \overline{c}\,\mathfrak{t}[x,\, Sy_1 - Sy_2] - \mathfrak{t}[x,\, Sy_2]\\ &= \langle x\,, S(cy_1+y_2)\rangle - \overline{c}\,\langle x\,, Sy_1 - Sy_2\rangle - \langle x\,, Sy_2\rangle = 0.\end{aligned}$$

Since this is true for every $x\in H$, it follows that $S(cy_1+y_2) = cSy_1 + Sy_2$.

Next we show that S is bounded by M. This is clear from

$$\|Sy\| = \|\mathfrak{t}[\,\cdot\,,y]\| \le M\|y\|, \qquad y\in H.$$

Now $T := S^*$ has all the desired properties.

Now if $\mathfrak{t}$ is symmetric, T is symmetric because

$$\langle Tx\,, y\rangle = \mathfrak{t}[x,y] = \overline{\mathfrak{t}[y,x]} = \overline{\langle Ty\,, x\rangle} = \langle x\,, Ty\rangle = \langle T^*x\,, y\rangle, \qquad x,y\in H. \quad \square$$

Theorem 2.24 (Spectral mapping theorem for polynomials). *Let H be a complex Hilbert space, $T\in L(H)$ and P a polynomial. Then*

$$\sigma(P(T)) = P(\sigma(T)) := \{P(\lambda) : \lambda\in\sigma(T)\}.$$

Proof. The assertion is clear if $\deg P = 0$. Now assume $\deg P \ge 1$.
For $\mu\in\sigma(P(T))$ consider the factorisation $P(X) - \mu = a\prod_{j=1}^n (X - \lambda_j)$. By assumption, $P(T) - \mu$ is not invertible, so at least one of the factors in $P(T) - \mu = a\prod_{j=1}^n (T-\lambda_j)$ cannot be invertible. This implies that there is at least one $k\in\{1,\dots,n\}$ such that $\lambda_k\in\sigma(T)$ man it follows that $\mu = P(\lambda_k)\in P(\sigma(T))$.

Now let $\mu\in P(\sigma(T))$ and $\lambda\in\sigma(T)$ such that $\mu = P(\lambda)$. Then there exists a polynomial Q such that $P(X) - \mu = (X-\lambda)Q(X)$, hence $P(T) - \mu = (T-\lambda)Q(T) = Q(T)(T-\lambda)$. Since by assumption $T - \lambda$ is not bijective, $P(T) - \mu$ cannot be bijective, so $\mu\in\sigma(P(T))$. $\square$

Theorem 2.24 will be extended to continuous functions in Exercise 2.11.

Corollary 2.25. *Let H be a complex Hilbert space, $A\in L(H)$ a selfadjoint operator and $[a,b]\subseteq\mathbb{R}$ such that $\sigma(A)\subseteq[a,b]$. Then for every polynomial $p : [a,b]\to\mathbb{C}$*

(i) $(p(A))^* = \overline{p}(A)$,

(ii) $\|p(A)\| \le \|p\|_\infty$.

Proof. The first assertion is clear because A is selfadjoint. In particular, $\overline{p}p(A) = |p|^2(A)$ is a nonnegative selfadjoint operator.
Recall that for every selfadjoint operator T on H (see Theorem 1.40)

$$\|T\| = \sup\{|\langle Tx\,, x\rangle| : x\in H,\ \|x\| = 1\} = r(T) = \max\{|\lambda| : \lambda\in\sigma(T)\}.$$

Thus we obtain

$$\begin{aligned}\|p(a)\|^2 &= \sup\{\|p(A)x\|^2 \,:\, x\in H,\ \|x\| = 1\}\\ &= \sup\{\langle p(A)x\,, p(A)x\rangle \,:\, x\in H,\ \|x\| = 1\}\\ &= \sup\{\langle \overline{p}p(A)x\,, x\rangle \,:\, x\in H,\ \|x\| = 1\}\\ &= \|\overline{p}p(A)\| = \max\{\lambda \,:\, \lambda\in\sigma(\overline{p}p(A))\}\\ &= \max\{\overline{p}p(\lambda) \,:\, \lambda\in\sigma(A)\},\end{aligned}$$

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where we used $\sigma(\overline{p}p(A)) \geq 0$ and, in the last step, the spectral mapping theorem for polynomials. □

Theorem 2.26 (Spectral theorem for bounded selfadjoint linear operators). *Let H be a complex Hilbert space and $T \in L(H)$ a bounded selfadjoint linear operator. We set*

$$m := m(T) := \inf\{\langle Tx\,, x\rangle : x \in H, \|x\| = 1\},$$
$$M := M(T) := \sup\{\langle Tx\,, x\rangle : x \in H, \|x\| = 1\}.$$

Then there exists exactly one spectral resolution $(E_\lambda)_{\lambda\in\mathbb{R}}$ such that

(i) $E_\lambda = 0,\ \lambda < m,$
$E_\lambda = \mathrm{id},\ \lambda \geq M,$

(ii) $TE_\lambda = E_\lambda T,\ \lambda \in]\mathbb{R},$

(iii) $P(T) = \displaystyle\int_{m-}^{M} P(\lambda)\,\mathrm{d}E_\lambda$ *for every polynomial P. In particular*

$$A = \int_{m-}^{M} \lambda\,\mathrm{d}E_\lambda.$$

Note that the convergence of the integral in the last point as limit of linear operators is convergence in norm, see Theorem 2.19.

Definition 2.27. The resolution of the identity in the spectral theorem is called the *spectral resolution of A*.

Proof of Theorem 2.26. Let us denote the set of all polynomials $[m, M] \to \mathbb{R}$ with real coefficients by P_r and define the polynomials

$$f_n : [m, M] \to \mathbb{R}, \quad f_n(t) := t^n.$$

In the proof, we will use several times the following consequence of the Weierstrass approximation theorem (which says that $\{f_n : n \in \mathbb{N}\}$ is a total subset of $C([m, M], \mathbb{R})$) and the uniqueness assertion in the Riesz representation theorem (Theorem 2.10)

$$\left.\begin{array}{l} \alpha, \beta \in \mathrm{BV}[a, b] \text{ real valued, right-continuous in } (a, b),\\ \alpha(a) = \beta(a) = 0,\\ \displaystyle\int_a^b f_n\,\mathrm{d}\alpha = \int_a^b f_n\,\mathrm{d}\beta \text{ for all } n \in \mathbb{N} \end{array}\right\} \implies \alpha = \beta. \qquad (*)$$

We divide the proof in several steps.

Step 1. Definition of $\alpha_{xy} \in \mathrm{BV}[m, M]$.
Let $x, y \in H$ and define

$$\varphi_{xy} : P_r \to \mathbb{C}, \qquad \varphi_{xy}(p) := \langle p(A)x\,, y\rangle.$$

This map is obviously linear and it is bounded by $\|x\|\,\|y\|$ by Corollary 2.25 because

$$|\varphi_{xy}(p)| = |\langle p(A)x\,, y\rangle| \leq \|p(A)\|\,\|x\|\,\|y\| \leq \|p\|_\infty \|x\|\,\|y\|.$$

Now we define the real and imaginary part of φ_{xy} by $\varphi^r_{xy}(p) := \mathrm{Re}(\varphi_{xy}(p))$ and $\varphi^i_{xy}(p) := \mathrm{Im}(\varphi_{xy}(p))$. Then $\varphi^{r/i}_{xy} : P_r \to \mathbb{R}$ are obviously $\mathbb{R}$-linear and bounded by $\|x\|\,\|y\|$. By the Weierstraß theorem, P_r is dense in $C([m, M], \mathbb{R})$, the set of all continuous functions on $[m, M]$ with values in R. Hence there exist unique continuous extensions of $\varphi^{r/i}_{xy}$ to $C([m, M], \mathbb{R})$. We denote these extensions again by $\varphi^{r/i}_{xy}$. By the Riesz representation theorem (Theorem 2.10) there exist uniquely determined functions $\alpha^{r/i}_{xy} \in \mathrm{BV}[m, M]$ which are right-continuous in (m, M) and satisfy $\alpha_{xy}(m) = 0$ and

$$\int_m^M f(t)\,\mathrm{d}\alpha^{r/i}_{xy}(t) = \varphi^{r/i}_{xy}(f), \qquad f \in C([m, M], \mathbb{R}).$$

Moreover, $\mathrm{var}\ \alpha^{r/i}_{xy} = \|\varphi^{r/i}_{xy}\| \leq \|x\|\,\|y\|$. Hence the function $\alpha_{xy} := \alpha_{xy} + \mathrm{i}\alpha_{xy}$ belongs to $\mathrm{BV}[m, M]$, is right-continuous in (m, M) and satisfies

$$\begin{aligned}\int_m^M f(t)\,\mathrm{d}\alpha_{xy}(t) &= \int_m^M f(t)\,\mathrm{d}\alpha^r_{xy}(t) + \mathrm{i}\int_m^M f(t)\,\mathrm{d}\alpha^i_{xy}(t)\\ &= \varphi^r_{xy}(f) + \mathrm{i}\varphi^i_{xy}(f) = \varphi_{xy}(f), \qquad f \in C([m, M], \mathbb{R}).\end{aligned}$$

and $\mathrm{var}\ \alpha_{xy} \leq \mathrm{var}\ \alpha^r_{xy} + \mathrm{var}\ \alpha^i_{xy} \leq 2\|x\|\,\|y\|$.
For later use, observe:

- $\displaystyle\int_m^M p(t)\,\mathrm{d}\alpha_{xy}(t) = \varphi_{xy}(p) = \langle p(A)x\,, y\rangle$ for polynomials $p : [m, M] \to \mathbb{R}$.
- $\alpha_{xy}(m) = 0$ for all $x, y \in H$.
- $\alpha_{xy}(M) = \langle x\,, y\rangle$ for all $x, y \in H$ because

$$\alpha_{xy}(M) = \alpha_{xy}(M) - \alpha_{xy}(m) = \int_m^M 1\,\mathrm{d}\alpha_{xy}(t) = \langle A^0x\,, y\rangle = \langle x\,, y\rangle.$$

Step 2. Definition of $(F_\lambda)_{\lambda\in[m,M]}$.
We will show that for every fixed $\lambda \in [m, M]$, the map

$$H \times H \to \mathbb{C}, \quad (x, y) \mapsto \alpha_{xy}(\lambda) \tag{2.4}$$

is a bounded symmetric sesquilinear form. Then, by the Lax-Milgram theorem (Theorem 2.23), there exists a unique bounded selfadjoint $F_\lambda \in L(H)$ such that $\langle F_\lambda x\,, y\rangle = \alpha_{xy}(\lambda)$ for all $x, y \in H$.

First we prove the linearity in the first variable of (2.4), that is, $\alpha_{cx_1+x_2,y} = c\alpha_{x_1,y} + \alpha_{x_2,y}$ for all $x_1, x_2, y \in H$, $c \in \mathbb{C}$. Note that $\alpha_{cx_1+x_2,y}$ and $c\alpha_{x_1,y} + \alpha_{x_2,y}$ belong to $\mathrm{BV}[m, M]$, are right-continuous in (m, M) and are 0 in m. Hence, by $(*)$ applied to their real and imaginary parts, they are equal if $\int_m^M p \,\mathrm{d}\alpha_{cx_1+x_2,y} = \int_m^M p \,\mathrm{d}\big[c\alpha_{x_1,y} + \alpha_{x_2,y}\big]$ for all polynomials $p \in P_r$. This equality follow from

$$\begin{aligned}\int_m^M p \,\mathrm{d}\alpha_{cx_1+x_2,y} &= \varphi_{cx_1+x_2,y}(p) = \langle p(A)(cx_1 + x_2)\,, y\rangle \\ &= c\langle p(A)x_1\,, y\rangle + \langle p(A)x_2\,, y\rangle = c\varphi_{x_1y}(p) + \varphi_{x_2y}(p) \\ &= c\int_m^M p \,\mathrm{d}\alpha_{x_1,y} + \int_m^M p \,\mathrm{d}\alpha_{x_2,y} = \int_m^M p \,\mathrm{d}[c\alpha_{x_1,y} + \alpha_{x_2,y}].\end{aligned}$$

For the symmetry of the form in (2.4) we have to show $\alpha_{xy}(\lambda) = \overline{\alpha_{yx}(\lambda)}$ for all $x, y \in H$. As before, this follows from $(*)$ (after separation in real and imaginary part) and

$$\begin{aligned}\int_m^M p \,\mathrm{d}\alpha_{xy} &= \varphi_{xy}(p) = \langle p(A)x\,, y\rangle = \overline{\langle y\,, p(A)x\rangle} = \overline{\langle p(A)y\,, x\rangle} = \overline{\varphi_{yx}(p)} \\ &= \overline{\int_m^M p \,\mathrm{d}\alpha_{yx}} = \int_m^M p \,\mathrm{d}\overline{\alpha_{yx}}.\end{aligned}$$

“Antilinearity” in the second argument is now a consequence of symmetry and linearity in the first argument in (2.4). It remains to show boundedness of the sesquilinear form $(x, y) \mapsto \alpha_{xy}(\lambda)$. This follows from

$$|\alpha_{xy}(\lambda)| = |\alpha_{xy}(\lambda) - \alpha_{xy}(m)| \le \mathrm{var}\ \alpha_{xy} \le 2\|x\|\,\|y\|, \qquad x, y \in H.$$

Step 3. For all $\lambda \in [m, M]$, F_λ is an orthogonal projection and $F_\mu F_\lambda = F_\lambda F_\mu = F_\mu$ for $m \le \mu \le \lambda \le M$.

Observe that

$$F_\mu F_\lambda = F_\lambda F_\mu = F_\mu \qquad \text{for } \mu \in [m, \lambda] \tag{2.5}$$

is equivalent to $\langle F_\mu F_\lambda x\,, y\rangle = \langle F_\lambda F_\mu x\,, y\rangle = \langle F_\mu x\,, y\rangle$ for all $x, y \in H$ by the Riesz-Frechét theorem. By $(*)$, the latter is equivalent to

$$\int_m^\lambda f_n(\mu)\,\mathrm{d}\langle F_\mu F_\lambda x\,, y\rangle = \int_m^\lambda f_n(\mu)\,\mathrm{d}\langle F_\lambda F_\mu x\,, y\rangle = \int_m^\lambda f_n(\mu)\,\mathrm{d}\langle F_\mu x\,, y\rangle, \qquad x, y \in H, n \in \mathbb{N}_0. \tag{2.6}$$

because for every $x, y \in H$ the functions

$$\begin{aligned}&\mu \mapsto \langle F_\mu F_\lambda x\,, y\rangle = \alpha_{F_\lambda x,y}(\mu), \\ &\mu \mapsto \langle F_\lambda F_\mu x\,, y\rangle = \langle F_\mu x\,, F_\lambda y\rangle = \alpha_{x,F_\lambda y}(\mu), \\ &\mu \mapsto \langle F_\mu x\,, y\rangle = \alpha_{x,y}(\mu)\end{aligned}$$

belong to $\mathrm{BV}[m, \lambda]$, are right-continuous in (m, λ) and take the value 0 in m. For the proof of (2.6) we will use

$$k_n(\lambda) := \int_m^\lambda f_n(\mu)\ \mathrm{d}\langle F_\mu x\,, y\rangle = \langle F_\lambda x\,, A^n y\rangle, \qquad n \in \mathbb{N}_0,\ \lambda \in [m, M]. \tag{2.7}$$

Note that $k_n \in \mathrm{BV}[m, M]$, it is right-continuous in (m, M) and $k_n(m) = 0$ (see Exercise 2.1) and that $\lambda \mapsto \langle F_\lambda x\,, A^n y\rangle = \alpha_{x,A^n y}(\mu)$ has the same properties. So, again by $(*)$, (2.7) is true because for all $\ell \in \mathbb{N}_0$

$$\begin{aligned}\int_m^M f_\ell(t)\ \mathrm{d}k_n(t) &= \int_m^M f_\ell(t) f_n(t)\ \mathrm{d}\langle F_t x\,, y\rangle = \int_m^M f_{\ell+n}(t)\ \mathrm{d}\langle F_t x\,, y\rangle \\ &= \langle A^{\ell+n}x\,, y\rangle = \langle A^\ell x\,, A^n y\rangle = \int_m^M f_\ell(t)\ \mathrm{d}\langle F_t x\,, A^n y\rangle.\end{aligned}$$

In the first step we used Exercise 2.1.

Now we are ready to prove (2.6). Fix $\lambda \in [m, M]$. Then

$$\begin{aligned}\int_m^M f_n(\mu)\ \mathrm{d}\langle F_\mu F_\lambda x\,, y\rangle &= \langle A^n F_\lambda x\,, y\rangle = \langle F_\lambda x\,, A^n y\rangle \overset{(2.7)}{=} \int_m^\lambda f_n(\mu)\ \mathrm{d}\langle F_\mu x\,, y\rangle \\ &= \int_m^M f_n(\mu)\ \mathrm{d}\widetilde{\alpha}_{xy}(\mu),\end{aligned}$$

where

$$\widetilde{\alpha}_{xy} : [m, M] \to \mathbb{C}, \quad \widetilde{\alpha}_{xy}(\mu) = \begin{cases}\alpha_{xy}(\mu) = \langle F_\mu x\,, y\rangle, & m \le \mu \le \lambda \le M, \\ \alpha_{xy}(\lambda) = \langle F_\lambda x\,, y\rangle, & m \le \lambda \le \mu \le M.\end{cases}$$

By construction, $\widetilde{\alpha}_{xy}$ belongs to $\mathrm{BV}[m, M]$, is right-continuous in (m, M) and $\widetilde{\alpha}_{xy}(m) = 0$. Hence, again by $(*)$, it follows that

$$\langle F_\mu F_\lambda x\,, y\rangle = \widetilde{\alpha}_{xy}(\mu) = \begin{cases}\langle F_\mu x\,, y\rangle, & m \le \mu \le \lambda \le M, \\ \langle F_\lambda x\,, y\rangle, & m \le \lambda \le \mu \le M.\end{cases}$$

This shows that $F_\lambda F_\mu = F_{\min\{\mu,\lambda\}} = F_\mu F_\lambda$, so (2.6) is proved. If we choose $\mu = \lambda$, we obtain $F_\lambda^2 = F_\lambda$, so F_λ is an orthogonal projection.

Step 4. $\lambda \mapsto F_\lambda$ is strongly right continuous in (m, M) (but not necessarily in m).

Let $x \in H$. Since F_λ is increasing in λ, the strong limit $\lim_{t\searrow\lambda} F_\lambda x$ exists (Theorem 1.46) and must be equal to its weak limit. For all $y \in H$ and $t > \lambda$

$$\langle F_t x\,, y\rangle - \langle F_\lambda x\,, y\rangle = \alpha_{xy}(t) - \alpha_{xy}(\lambda) \longrightarrow 0, \qquad t \searrow \lambda,$$

by right-continuity of α_{xy} in (m, M). Therefore $\lim_{t\searrow\lambda} F_t x = w\text{-}\lim_{t\searrow\lambda} F_t x = F_\lambda x$.

Step 5. Redefine F_m and extend $(F_\lambda)_{\lambda\in[m,M]}$ to a resolution of identity $(E_\lambda)_{\lambda\in\mathbb{R}}$.

For all $x, y \in H$ define

$$\beta_{xy} : \mathbb{R} \to \mathbb{R}, \quad \beta_{xy}(t) = \begin{cases} 0, & t < m, \\ \alpha_{xy}(m+0), & t = m, \\ \alpha_{xy}(t), & t \in (m, M], \\ \alpha_{xy}(M), & t > M. \end{cases}$$

Note that by construction β is right-continuous and belongs to $\mathrm{BV}[a, b]$ for every compact interval $[a, b] \subseteq \mathbb{R}$. For every $\lambda \in \mathbb{R}$, $H \times H \to \mathbb{C}, (x, y) \mapsto \beta_{xy}(\lambda)$ is a bounded symmetric bilinear. This is immediately clear for $\lambda \in \mathbb{R} \setminus \{m\}$. For $\lambda = m$ this follows from

$$\begin{aligned} \beta_{cx_1+x_2,y}(m) &= \lim_{t\searrow m} \alpha_{cx_1+x_2,y}(t) = \lim_{t\searrow m} \big(c\alpha_{x_1,y}(t) + \alpha_{x_2,y}(t)\big) = c\beta_{x_1,y}(m) + \beta_{x_2,y}(m), \\ \beta_{xy}(m) &= \lim_{t\searrow m} \alpha_{xy}(t) = \lim_{t\searrow m} \overline{\alpha_{yx}(t)} = \overline{\beta_{yx}(m)}, \\ |\beta_{xy}(m)| &= |\lim_{t\searrow m} \alpha_{xy}(t)| = \lim_{t\searrow m} |\alpha_{xy}(t)| \le 2\|x\|\,\|y\|. \end{aligned}$$

Now for every $\lambda \in \mathbb{R}$ let $E_\lambda \in L(H)$ be the unique selfadjoint linear operator such that $\langle E_\lambda x\,, y\rangle = \beta_{xy}(\lambda)$ for all $x, y \in H$. Observe that by construction

$$\left.\begin{array}{ll} E_t = 0, & t < m, \\ E_t = s\text{-}\lim\limits_{\lambda\searrow m} F_t, & t = m, \\ E_t = F_t, & t \in (m, M], \\ E_t = F_M = \mathrm{id}, & t > M \end{array}\right\} \begin{array}{l}\text{because for} \\ \text{all } x, y \in H\end{array} \left\{\begin{array}{l} \langle E_t x\,, y\rangle = \beta_{xy}(t) = 0, \\ \langle E_t x\,, y\rangle = \beta_{xy}(t) = \alpha_{xy}(t) = \langle F_t x\,, y\rangle, \\ \langle E_t x\,, y\rangle = \beta_{xy}(t) = \alpha_{xy}(t) = \langle F_t x\,, y\rangle, \\ \langle E_t x\,, y\rangle = \beta_{xy}(M) = \alpha_{xy}(M) = \langle F_M x\,, y\rangle = \langle x\,, y\rangle. \end{array}\right.$$

Now it is easy to verify that $(E_\lambda)_{\lambda\in\mathbb{R}}$ is a resolution of the identity and that by construction

$$\int_{m-0}^{M} f \,\mathrm{d}\langle E_\lambda x\,, y\rangle = \int_{m-0}^{M} f \,\mathrm{d}\langle F_\lambda x\,, y\rangle$$

for all continuous functions f.

Step 6. $(E_\lambda)_{\lambda\in\mathbb{R}}$ commutes with A.

$E_\lambda A = AE_\lambda$ is clear for $\lambda < m$ or $\lambda \ge M$. Now let $\lambda \in [m, M]$ and $x, y \in H$. Then

$$\begin{aligned} \langle AE_\lambda x\,, y\rangle = \varphi_{xy}(f_1) &= \int_{m-0}^{M} f_1(t)\,\mathrm{d}\langle E_t E_\lambda x\,, y\rangle = \int_{m-0}^{M} f_1(t)\,\mathrm{d}\langle E_\lambda E_t x\,, y\rangle \\ &= \int_{m-0}^{M} f_1(t)\,\mathrm{d}\langle E_t x\,, E_\lambda y\rangle = \varphi_{x,E_\lambda y}(f_1) = \langle Ax\,, E_\lambda y\rangle = \langle E_\lambda A x\,, y\rangle \end{aligned}$$

Since this is true for all $x, y \in H$, we obtain $AE_\lambda = E_\lambda A$.

Step 7. Representation of $p(A)$ as an integral.
Recall that for all $x, y \in H$ and $n \in \mathbb{N}_0$

$$\langle A^n x\,, y\rangle = \varphi_{xy}(f_n) = \int_{m-0}^{M} f_n \,\mathrm{d}\alpha_{xy}(t) = \int_{m-0}^{M} f_n \,\mathrm{d}\beta_{xy}(t),$$

hence by linearity of the integral and Lemma 2.21 (i)

$$\langle p(A)x\,, y\rangle = \int_{m-0}^{M} p(t)\,\mathrm{d}\beta_{xy}(t) = \int_{m-0}^{M} p(t)\,\mathrm{d}\langle E_t x\,, y\rangle = \left\langle \int_{m-0}^{M} p(t)\,\mathrm{d}E_t x\,, y \right\rangle$$

for all $x, y \in H$, so again, by Riesz-Frechét, $p(A) = \displaystyle\int_{m-0}^{M} p(t)\,\mathrm{d}E_t$.

Step 8. Uniqueness of $(E_\lambda)_{\lambda\in\mathbb{R}}$.
Suppose that $(\widetilde{E}_\lambda)_{\lambda\in\mathbb{R}}$ is a resolution of the identity such that (i), (ii) and (iii) of the theorem hold. Then, for all $x, y \in H$

$$\int_{m-0}^{M} f_n(t)\,\mathrm{d}\langle E_t x\,, y\rangle = \langle A^n x\,, y\rangle = \int_{m-0}^{M} f_n(t)\,\mathrm{d}\langle \widetilde{E}_t x\,, y\rangle,$$

hence, by the Riesz representation theorem, $\langle E_t x\,, y\rangle = \langle \widetilde{E}_t x\,, y\rangle$ for all $x, y \in H$, which implies $E_t = \widetilde{E}_t$ for all $t \in [m, M]$. Since $E_t = \widetilde{E}_t = 0$ for $t < m$ and $E_t = \widetilde{E}_t = \mathrm{id}$ for $t \ge M$, uniqueness is proved. □

Next we use the spectral resolution of a bounded selfadjoint operator to define $f(T)$ where f is a continuous function on $\sigma(T)$, that is, $f \in C(\sigma(T))$.

Theorem 2.28 (Continuous functional calculus). *Let H be a complex Hilbert space and $T \in L(H)$ a bounded selfadjoint linear operator with spectral resolution $(E_\lambda)_{\lambda\in\mathbb{R}}$. As before let*

$$\begin{aligned} m &:= m(T) := \inf\{\langle Tx\,, x\rangle : x \in H, \|x\| = 1\}, \\ M &:= M(T) := \sup\{\langle Tx\,, x\rangle : x \in H, \|x\| = 1\}. \end{aligned}$$

For $f \in C(\sigma(T))$ we define

$$f(T) := \int_{m-}^{M} f(\lambda)\,\mathrm{d}E_\lambda.$$

The map

$$C([m,M]) \to L(H), \quad f \mapsto f(T) \tag{2.8}$$

is a continuous homomorphism of Banach algebras and has norm 1.
More precisely:

(i) *Let* $f_n(t) := t^n$. *Then* $f_n(T) = T^n$, $n \in \mathbb{N}_0$.
In particular: $f_0(T) = \mathrm{id}, f_1(T) = T, 0(T) = 0$.

(ii) $f \mapsto f(T)$ *is continuous, linear and multiplicative with norm* 1 *(as linear map). For all* $f, g \in C(\sigma(T))$*:*

$$fg(T) = f(T)g(T), \quad \overline{f}(T) = f(T^*).$$

(iii) *For every* $B \in L(H)$ *the following is equivalent:*

(a) $TB = BT$,

(b) $E_\lambda B = BE_\lambda$, $\lambda \in \mathbb{R}$,

(c) $f(T)B = Bf(T)$, $f \in C[m,M]$.

(iv) *Let* $f \in C(\sigma(T))$. *Then* $f(T)$ *is normal. In addition:*

(a) $|f(\lambda)| = 1$ *for all* $\lambda \in \sigma(T)$ $\implies$ $f(T)$ *is unitary;*

(b) $f(\lambda) \in \mathbb{R}$ *for all* $\lambda \in \sigma(T)$ $\implies$ $f(T)$ *is selfadjoint;*

(c) $f(\lambda) \geq 0$ *for all* $\lambda \in \sigma(T)$ $\implies$ $f(T) \geq 0$.

(v) *For* $x \in H$ *and* $f \in C(\sigma(T))$

$$\|f(T)x\|^2 = \int_{m-}^{M} |f(\lambda)|^2 \,\mathrm{d}\|E_\lambda x\|^2.$$

Proof. Observe that $\displaystyle\int_{m-}^{M} f(\lambda)\,\mathrm{d}E_\lambda \in L(H)$ and with norm $\leq \|f\|_\infty$ for $f \in C[m,M]$ by Theorem 2.19, so the map in (2.8) has norm ≤ 1. On the other hand, using (i), we see that its norm is ≥ 1 because $\|f_0(A)\| = \|\,\mathrm{id}\,\| = 1 = \|f_0\|_\infty$.
(i) follows immediately from the spectral theorem (Theorem 2.26) and (ii) is proved in Lemma 2.21 (iii) and (iv).
Now we prove (iii). The implication (iiic) $\implies$ (iiia) is clear.
(iiia) $\implies$ (iiib): Note that the functions $\lambda \mapsto \langle E_\lambda Bx\,, y\rangle$ and $\lambda \mapsto \langle BE_\lambda x\,, y\rangle = \langle E_\lambda x\,, B^*y\rangle$ are continuous in (m,M), take the value 0 in m and belong to $\mathrm{BV}[m,M]$. Note that for all $n \in \mathbb{N}_0$ and all $x, y \in H$

$$\begin{aligned}\int_{m-0}^{M} \lambda^n \,\mathrm{d}\langle E_\lambda Bx\,, y\rangle &= \langle A^n Bx\,, y\rangle = \langle BA^n x\,, y\rangle = \langle A^n x\,, B^* y\rangle \\ &= \int_{m-0}^{M} \lambda^n \,\mathrm{d}\langle E_\lambda x\,, B^* y\rangle.\end{aligned}$$

Hence, by the uniqueness claim in the Riesz representation theorem, $\langle E_\lambda Bx\,, y\rangle = \lambda \mapsto \langle BE_\lambda x\,, y\rangle = \langle E_\lambda x\,, B^*y\rangle$ for all $x, y \in H$ and all $\lambda \in [m,M]$, so (iiib) holds.
(iiib) $\implies$ (iiic): The claim is true for step functions f. Now let $f \in C[m,M]$ and $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$. Since $\int_{m-0}^{M} f_n \,\mathrm{d}E_\lambda$ converges to $\int_{m-0}^{M} f \,\mathrm{d}E_\lambda$ in the operator norm, we obtain

$$\begin{aligned}f(A)B &= \int_{m-0}^{M} f(\lambda)\ \mathrm{d}E_\lambda\ B = \lim_{n\to\infty} \int_{m-0}^{M} f_n(\lambda)\ \mathrm{d}E_\lambda\ B \\ &= \lim_{n\to\infty} B \int_{m-0}^{M} f_n(\lambda)\ \mathrm{d}E_\lambda = B \lim_{n\to\infty} \int_{m-0}^{M} f_n(\lambda)\ \mathrm{d}E_\lambda = B \int_{m-0}^{M} f_n(\lambda)\ \mathrm{d}E_\lambda.\end{aligned}$$

Now we prove (iv). By (ii)

$$f(A)f(A)^* = f\overline{f}(A) = \overline{f}f(A) = f(A)^*f(A).$$

If f is such that $|f(t)| = 1$ for all $t \in [m,M]$, then

$$f(A)f(A)^* = f(A)^*f(A) = f\overline{f}(A) = |f|^2(A) = \mathrm{id}\,.$$

If f is real valued, then $f(A)^* = \overline{f}(A) = f(A)$.
If $f(\lambda) \geq 0$ for all $\lambda \in [m,M]$, then

$$\langle f(A)x\,, x\rangle = \int_{m-0}^{M} f(t)\,\mathrm{d}\langle E_t x\,, x\rangle \geq 0, \qquad x \in H.$$

is real valued, then $f(A)^* = \overline{f}(A) = f(A)$.
Finally we show (v):

$$\begin{aligned}\|f(T)x\|^2 &= \langle f(T)x\,, f(T)x\rangle = \langle f(T)^* f(T)x\,, x\rangle = \langle (\overline{f}f)(T)x\,, x\rangle \\ &= \int_{m-}^{M} |f(\lambda)|^2 \,\mathrm{d}\langle E_\lambda x\,, x\rangle = \int_{m-}^{M} |f(\lambda)|^2 \,\mathrm{d}\|E_\lambda x\|^2.\end{aligned}$$ □

Remark 2.29. Let A be a bounded selfadjoint linear operator. Note that the map $f \mapsto f(A)$ gives representation of the Banach algebra of continuous functions in $[m,M]$ in the Banach algebra $L(H)$. We will show in Theorem 2.52 that E_λ is constant in $\rho(A)$. Hence, if f and g are continuous functions which are equal on $\sigma(A)$, then $f(A) = g(A)$. Therefore we obtain a representation of the Banach algebra $C(\sigma(A))$ by

$$C(\sigma(A)) \to L(H), \qquad f \mapsto f(A).$$

Remark 2.30 (Measurable functional calculus). Recall that in the proof of the spectral theorem, we assigned to every pair $(x,y) \in H \times H$ a non-decreasing function $\alpha_{xy} \in \mathrm{BV}[m,M]$. This function defines a finite regular Borel measure on $[m,M]$, denoted again by α_{xy}, as follows: $\alpha_{xy}((a,b]) := \alpha_{xy}(b) - \alpha_{xy}(a)$. Hence $\int_m^M f(t)\,\mathrm{d}\alpha_{xy}$ makes sense for every *bounded measurable* function

f. In particular, we can calculate $P(\Omega) = \int_m^M \chi_\Omega(t)\,\mathrm{d}\alpha_{xy}$ where $\Omega \in [m, M]$ is measurable and χ_Ω is its corresponding characteristic function. It can be shown that (Ω) is an orthogonal projection and that $P((-\infty, \lambda]) = E_\lambda$.
A formula for E_λ in terms of the resolvent of A will be proved in Theorem 2.49.

As an application of the spectral theorem we prove that every nonnegative bounded selfadjoint linear operator has a unique nonnegative square root (the case of unbounded nonnegative linear operators will be discussed in **??**). Moreover, we will show that the composition of positive commuting linear operators is again positive.

Corollary 2.31. *Let H be a Hilbert space and $A, B \in L(H)$ selfadjoint linear operators.*

(i) *There exists a unique $R \geq 0$ such that $R^2 = A$.*

(ii) *If $AB = BA$ and $A \geq 0$, $B \geq 0$, then $AB \geq 0$.*

Proof. (i) *Existence:* Let $g : \mathbb{R} \to \mathbb{R}$, $g(t) = \sqrt{|t|}$ and $R := g(A)$ defined as in Theorem 2.28. Then $R \geq 0$ and $R^2 = \int_{0-0}^{\infty} |g(t)|^2 \,\mathrm{d}E_t = \int_{0-0}^{\infty} t \,\mathrm{d}E_t = A$.
Uniqueness: Let $C \in L(H)$ such that $C \geq 0$ and $C^2 = A$. Let $M := \sup\{\langle Ax\,, x\rangle : x \in H, \|x\| = 1\} \geq 0$ and $\alpha \geq \max\{M, \|C\|^2\}$. Let $g : [0, \alpha] \to \mathbb{R}$, $g(t) := \sqrt{t}$. Choose polynomials $p_n : [0, \alpha] \to \mathbb{R}$ such that $\|p_n - g\|_\infty \to 0$ for $n \to \infty$. Now we define

$$q_n : [0, \sqrt{\alpha}] \to \mathbb{R}, \qquad q_n(t) := p_n(t^2),$$
$$\widetilde{g} : [0, \sqrt{\alpha}] \to \mathbb{R}, \qquad \widetilde{g} := g(t^2) = t.$$

Observe that $\|q_n - \widetilde{g}\|_\infty \to 0$ for $n \to \infty$, and that $q_n(C) = p_n(C^2) = p_n(A)$ (this is true as equality of polynomials; there is no functional calculus involved). It follows that

$$\begin{aligned}\|R - C\| &\leq \|R - p_n(A)\| + \|C - p_n(A)\| = \|R - p_n(A)\| + \|\widetilde{g}(C) - q_n(C)\| \\ &\leq \|g - p_n\|_\infty + \|\widetilde{g} - q_n\|_\infty \longrightarrow 0, \qquad n \to \infty.\end{aligned}$$

(ii) Let $\sqrt{A}$ be defined as in (i). Since B commutes with A, it commutes with $\sqrt{A}$ too (Theorem 2.28 (ii)), and we obtain

$$\langle ABx\,, x\rangle = \langle \sqrt{A}\sqrt{A}Bx\,, x\rangle = \langle \sqrt{A}Bx\,, \sqrt{A}x\rangle = \langle B\sqrt{A}x\,, \sqrt{A}x\rangle \geq 0, \qquad x \in H.$$

□

2.4 The spectral theorem for unitary operators

As before, we always assume that H is a complex Hilbert space.

Definition 2.32. $A \in L(H)$ is *unitary* if $AA^* = A^*A = \mathrm{id}_H$.

Proposition 2.33. *Let $A \in L(H)$. Then the following is equivalent:*

(i) *A is unitary.*

(ii) *$R(A) = H$ and $\langle Ax\,, Ay\rangle = \langle x\,, y\rangle$, $x, y \in H$.*

(iii) *$R(A) = H$ and $\|Ax\| = \|x\|$, $x \in H$.*

Proof. (i) $\implies$ (ii) $\implies$ (iii) is clear.
(iii) $\implies$ (ii) Using the polarisation formula:

$$\begin{aligned}\langle Ax\,, Ay\rangle &= \frac{1}{4}(\|Ax + Ay\|^2 - \|Ax - Ay\|^2 + \mathrm{i}\|Ax + \mathrm{i}Ay\|^2 - \mathrm{i}\|Ax - \mathrm{i}Ay\|^2) \\ &= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + \mathrm{i}\|x + \mathrm{i}y\|^2 - \mathrm{i}\|x - \mathrm{i}y\|^2) \\ &= \langle x\,, y\rangle\end{aligned}$$

(ii) $\implies$ (i) $\forall x, y \in H \langle x\,, \mathrm{id} - AA^*\rangle = \langle x\,, y\rangle - \langle Ax\,, Ay\rangle = 0$
$\implies$ $A^*A = \mathrm{id}_H$, in particular A is injective. Hence A bijective because it is surjective by assumption. It follows that $A^* = A^*A^{-1} = A^{-1}$, thus is unitary. □

Lemma 2.34. *Let $A, B \in L(H)$, A selfadjoint, and P be the orthogonal projection on $\ker(A)$. If $AB = BA$, then $BP = PB$.*

Proof. Let $x \in \ker A$. Then $Bx \in \ker A$ because $ABx = BAx = 0$. Since P is the projection on $\ker A$, we obtain $BPx = Bx = PBx$.

If $x \in \mathrm{rg}(A)$ then there exists a $y \in H$ such that $x = Ay$. It follows that $PBx = PBAy = PABy = 0 = BPx$.
By linearity and continuity $PBx = BPx = 0$ for all $x \in \overline{\mathrm{rg}(A)}$. The lemma is now proved because $H = \ker(A) \bigoplus \overline{\mathrm{rg}(A)} = \ker(A) \bigoplus \mathrm{rg}(A)$. □

Lemma 2.35. *Let $S, T \in L(H)$ selfadjoint operators and assume that $ST = TS$ and $S^2 = T^2$. Let P be the orthogonal projection on $\ker(S - T)$. Then:*

(i) $\ker S \subseteq \mathrm{rg}(P)$.

(ii) $S = (2P - \mathrm{id})T$.

Proof. (i) $\forall x \in H \|Sx\|^2 = \langle Sx\,, Sx\rangle = \langle S^2x\,, x\rangle = \langle T^2x\,, x\rangle = \|Tx\|^2$.
Then if $Sx = 0$, $Tx = 0$ and $(S - T)x = 0$, so $x \in R(P)$, and this means $Px = x$.

(ii) Observe $S(S - T) = (S - T)S$, hence $PS = SP$ by Lemma 2.34. Analogously $PT = TP$ is proved. Since $(S - T)(S + T) = S^2 - T^2 = 0$, it follows that

$\operatorname{rg}(S+T) \subseteq \ker(S-T) = \operatorname{rg}(P)$, consequently $P(S+T) = S+T$ and

$$\begin{aligned} & S+T = P(S+T) = [(S-T)+2T]P = (S-T)P + 2TP \\ \implies \quad & S = 2TP - \mathrm{id} = (2P - \mathrm{id})T. \end{aligned}$$ □

For every bounded selfadjoint linear operator A the linear operator e^{iA} as defined by the functional calculus is unitary. Now we will prove that every unitary linear operator is of this form.

Theorem 2.36. *Let U be a unitary operator on a Hilbert space H. Then there exists a selfadjoint operator $A \in L(H)$ with $\|A\| \leq \imath$ such that $U = e^{iA}$.*

Proof. Let $R := \operatorname{Re}(U) := \frac{1}{2}(U + U^*)$, $S := \operatorname{Im}(U) := \frac{1}{2i}(U - U^*)$. R, S have the following properties:

(i) $RS = SR, S^* = S, T^* = T, U = R + iS$.

(ii) $R^2 + S^2 = \frac{1}{4}(U^2 + (U^*)^2 + 2\,\mathrm{id} - U^2 - (U^*)^2 + 2\,\mathrm{id}) = \mathrm{id}$.

(iii) $\|R\|, \|S\| \leq \frac{1}{2}(\|U\| + \|U^*\|) = 1$.

Define $f : [-1,1] \to \mathbb{R}$, $\lambda \mapsto \sin(\arccos \lambda) = \sqrt{1 - \lambda^2}$, and

$$T := f(R) = \sin(\arccos R).$$

T satisfies:

(T1) $T^* = f(R)^* = \overline{f}(R) = f(R) = T$.

(T2) $RT = TR, ST = TS$ because S and T commute.

(T3) $T^2 = S^2$ because $T^2 + R^2 = \mathrm{id} - R^2 + R^2 = \mathrm{id} = S^2 + R^2$.

Let P be the orthogonal projection on $\ker(S-T)$, then $S = (2P - \mathrm{id})T$ and $\ker(S) \subseteq R(P)$ by Lemma 2.35.
Observe that $PR = RP$ by Lemma 2.34 because $R(S-T) = (S-T)R$. Then $P(\arccos R) = (\arccos R)P$.
Now define $A := (2P - \mathrm{id}) \arccos R$. A has the following properties:

(A1) $A = A^*$, $\|A\| \leq \|\arccos R\| \leq \pi$.

(A2) $A^2 = (2P - \mathrm{id})^2 (\arccos R)^2 = (4P^2 - 4P + \mathrm{id})(\arccos R)^2 = (\arccos R)^2$.

Now we will show: $\cos A = R$, $\sin A = S$.

The power series $\arccos(\lambda) = \sum_{n=0}^{\infty} g_n \lambda^n$ converges for $\lambda \in [-1,1]$. Define $h(\lambda^2) := \sum_{n=0}^{\infty} h_n \lambda^{2n}$ and let $g_N(R) := \sum_{n=0}^{N} g_n R^n$. We have $\|g_N(R) - \arccos(R)\| \to 0$ as

$N \to \infty$, so $\arccos(R) = \sum_{n=0}^{\infty} g_n R^n$.

Observe that $\lambda = \cos(\arccos \lambda) = h((\arccos \lambda)^2)$ for $\lambda \in [-1,1]$. If we replace λ by R we obtain $R = h((\arccos R)^2) = h(A^2) = \cos A$.

Similarly $\sin(\lambda) = \lambda \sum_{n=0}^{\infty} h_n \lambda^{2n}$, then

$$\begin{aligned} \sin A = A \sum_{n=0}^{\infty} h_n \lambda^{2n} &= (2P - \mathrm{id}) \arccos R \sum_{n=0}^{\infty} h_n (\arccos R)^{2n} \\ &= (2P - \mathrm{id}) \sin(\arccos R) = (2P - \mathrm{id})T = S. \end{aligned}$$

So we proved that $U = \cos A + i \sin A = e^{iA}$. □

Theorem 2.37 (Spectral theorem for unitary operators). *Let U be a unitary operator on a Hilbert space H. Then there exists a spectral resolution $(E_\lambda)_{\,lambda \in \mathbb{R}}$ such that Let $S, T \in L(H)$ selfadjoint operators and assume that*

(i) $E_\lambda = 0, \quad \lambda \leq -\pi,$
$E_\lambda = \mathrm{id}, \quad \lambda \geq \pi.$

(ii) *For polynomials P*

$$P(U) = \int_{-\pi-}^{\pi} p(e^{i\lambda}) \, dE_\lambda,$$

in particular

$$U = \int_{-\pi-}^{\pi} e^{i\lambda} \, dE_\lambda.$$

(iii) *For every $f \in C([-\pi, \pi])$ the operator*

$$f(U) := \int_{-\pi-}^{\pi} f(e^{i\lambda}) \, dE_\lambda$$

is well defined, belongs to $L(H)$ and the convergence of the integral is in operator norm.

Proof. Choose a selfadjoint operator $A \in L(H)$ as in Theorem 2.36 and let $(E_\lambda)_{\lambda \in \mathbb{R}}$ be its spectral resolution. Then $U = e^{iA}$ and $E_\lambda = 0$ for $\lambda < -\pi$ and $E_\lambda = \mathrm{id}$ for $\lambda \geq \pi$. The claim follows now from the spectral theorem for bounded selfadjoint operators. □

DRAFT

2.5 The Cayley transformation

The Cayley transform gives a bijection between selfadjoint linear operators and unitary linear operators. It will be used to prove the spectral theorem for (unbounded) selfadjoint linear operators in the next section, and later, in Section 3.1 to find selfadjoint extensions of symmetric linear operators.

In complex analysis the so-called Möbius transform

$$z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}, \qquad z \neq \mathrm{i},$$

maps the real line bijectively to the unit circle without 1. Its inverse is

$$w \mapsto \mathrm{i}\,\frac{1+w}{1-w}, \qquad w \neq 1.$$

The idea is to apply these formulas to selfadjoint linear operators instead of z and to unitary operators instead of w.

Remark 2.38. Let S be a symmetric operator on a complex Hilbert space H. Then for $a, b \in \mathbb{R}$ and $\lambda := a + ib$ the following holds:

$$\|(S-\lambda)x\|^2 = \|(S-a)x\|^2 + b^2\|x\|^2.$$

In particular, if $b \neq 0$, then:

(i) $S-\lambda$ is injective and $(S-\lambda)^{-1} : \operatorname{rg}(S-\lambda) \to H$ is bounded by $|b|^{-1}$;

(ii) S is closed, if and only if $(S-\lambda)^{-1}$ is closed, hence, by the closed graph theorem, if and only if $\operatorname{rg}(S-\lambda)$ is closed.

Definition 2.39. Let H be a complex Hilbert space and A a densely defined symmetric linear operator on H. Then the *Cayley transform* of A is defined by

$$U_A : \operatorname{rg}(A+\mathrm{i}) \to H, \qquad U_A := (A-\mathrm{i})(A+\mathrm{i})^{-1}.$$

Note that $A+\mathrm{i}$ is boundedly invertible in $\operatorname{rg}(A+\mathrm{i})$ by the remark above and that $\operatorname{rg}((A+\mathrm{i})^{-1}) = \mathcal{D}(A+\mathrm{i}) = \mathcal{D}(A-\mathrm{i})$. Therefore U_A is well-defined.

Proposition 2.40 (Properties of the Cayley transformation). *Let A be a symmetric operator on a complex Hilbert space H and U_A its Cayley transform.*

(i) *U_A is isometric and $\operatorname{rg}(U_A) = R(A-\mathrm{i})$. If A is closed, so is U_A.*

(ii) *$1 \neq \sigma_p(U_A)$ and $\operatorname{rg}(\mathrm{id} - U_A) = \mathcal{D}(A)$ is dense in H. The map*

$$(\mathrm{id} - U_A)^{-1} : \mathcal{D}(A) \to \mathcal{D}(U_A) \tag{2.9}$$

exists and is surjective.

(iii) $A = \mathrm{i}(\mathrm{id} + U)(\mathrm{id} - U)^{-1}$.

Proof. (i) Since $(A+\mathrm{i})^{-1}\big(\operatorname{rg}(A+\mathrm{i})\big) = \mathcal{D}(A)$ it follows that $\operatorname{rg}(U_A) = \operatorname{rg}(A-\mathrm{i})$. To show that U_A is an isometry we note that $\|(A-\mathrm{i})x\| = \|(A+\mathrm{i})x\|$ for all $x \in \mathcal{D}(A)$ by the symmetry of A. Therefore we obtain

$$\|U_A x\| = \|(A-\mathrm{i})(A+\mathrm{i})^{-1}x\| = \|(A+\mathrm{i})(A+\mathrm{i})^{-1}x\| = \|x\|, \qquad x \in \mathcal{D}(U_A).$$

Now we assume additionally that A is closed. Then, by Remark 2.38, $\mathcal{D}(U_A) = \operatorname{rg}(A-\mathrm{i})$ is closed. Since U_A is isometric, it is bounded, so by the closed graph theorem, U_A is closed.

(ii) Let $x \in \mathcal{D}(U_A)$ with $U_A x = x$. It follows that

$$(A-\mathrm{i})(A+\mathrm{i})^{-1}x = (A+\mathrm{i})(A-\mathrm{i})^{-1}x,$$

hence $2\mathrm{i}(A+\mathrm{i})^{-1}x = 0$. Since $A+\mathrm{i}$ is invertible, x must be 0, which proves that 1 is not an eigenvalue of U_A and the map (2.9) is well-defined.

To prove that $\operatorname{rg}(\mathrm{id} - U_A) \subseteq \mathcal{D}(A)$, fix $x \in \operatorname{rg}(A+\mathrm{i}) = \mathcal{D}(U_A)$. Then

$$(\mathrm{id} - U_A)x = (A+\mathrm{i}-(A-\mathrm{i}))(A+\mathrm{i})^{-1}x = 2\mathrm{i}(A+\mathrm{i})^{-1}x \in \mathcal{D}(A).$$

On the other hand, for $y \in \mathcal{D}(A)$ there exists let $x = \frac{1}{2\mathrm{i}}(A+\mathrm{i})y \in \operatorname{rg}(A+\mathrm{i})$. It follows that $y = 2\mathrm{i}(A+\mathrm{i})^{-1}x = (A+\mathrm{i}-(A-\mathrm{i}))(A+\mathrm{i})^{-1}x = (\mathrm{id} - U)x$.

Note that $\mathrm{id} - U_A$ is injective and $\operatorname{rg}(\mathrm{id} - U_A) = \mathcal{D}(A)$, hence $\operatorname{rg}(\mathrm{id} - U_A) = \mathcal{D}(A)$.

(iii) For every $x \in \mathcal{D}(A) = \mathcal{D}((\mathrm{id} - U_A)^{-1})$

$$\begin{aligned}\mathrm{i}(\mathrm{id} + U_A)\underbrace{(\mathrm{id} - U_A)^{-1}x}_{=\frac{1}{2\mathrm{i}}(A+\mathrm{i})x} &= \frac{1}{2}\big[\mathrm{id} + A(A-\mathrm{i})(A+\mathrm{i})^{-1}\big](A+\mathrm{i})x \\ &= \frac{1}{2}\big[A+\mathrm{i}+A-\mathrm{i}\big]x = Ax.\end{aligned}$$

□

Lemma 2.41. *Let $U : H \supseteq \mathcal{D}(U) \to H$ be a closed isometric linear operator such that $\operatorname{rg}(U - \mathrm{id})$ is dense in H. Then there exists exactly one closed symmetric operator $A : H \supseteq \mathcal{D}(A) \to H$ with $\mathcal{D}(A) = \operatorname{rg}(\mathrm{id} - U)$ such that U is the Cayley transform of A.*

Proof. Uniqueness. Assume that B is a symmetric linear operator on H whose Cayley transform is U. By Proposition 2.40 (iii) it follows that

$$B = \mathrm{i}(\mathrm{id} + U)(\mathrm{id} - U)^{-1} = A.$$

Existence. First we show that $1 \notin \sigma_{\mathrm{p}}(U)$. Let $x \in \mathcal{D}(U)$ with $Ux = x$. For all $y \in H$ is follows that $\langle x, (\mathrm{id} - U)y\rangle = \langle(\mathrm{id} - U)x, y\rangle = 0$. Since by assumption $\overline{\operatorname{rg}(U - \mathrm{id})} = H$, this implies $x = 0$, hence 1 is not an eigenvalue of U. Therefore we can define

$$A : H \supseteq \operatorname{rg}(\mathrm{id} - U) \to H, \qquad Ax = \mathrm{i}(\mathrm{id} + U)(\mathrm{id} - U)^{-1}x.$$

It is easy to see that $\operatorname{rg}(A) = \operatorname{rg}(U + \mathrm{id})$. Now we show that A is symmetric. Note that A is densely defined by assumption on U. For $x, y \in \mathcal{D}(A)$ there are $w, v \in \mathcal{D}(U)$ such that $x = (\mathrm{id} - U)w$ and $y = (\mathrm{id} - U)v$. Hence

$$\langle Ax\,, y\rangle = \mathrm{i}\langle(\mathrm{id} + U)(\mathrm{id} - U)^{-1}x\,, y\rangle = \mathrm{i}\langle(\mathrm{id} + U)w\,, (\mathrm{id} - U)v\rangle = \mathrm{i}(\langle Uw\,, v\rangle - \langle w\,, Uv\rangle)$$
$$\langle x\,, Ay\rangle = \overline{\langle Ay\,, x\rangle} = \overline{\mathrm{i}(\langle Uv\,, w\rangle - \langle v\,, Uw\rangle)} = -\mathrm{i}(\langle w\,, Uv\rangle - \langle Uw\,, v\rangle) = \langle Ax\,, y\rangle.$$

Next we show that A is closed. Fix a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$ for $n \to \infty$. With $z_n := (\mathrm{id} - U)^{-1}$ we find

$$\left.\begin{aligned} x_n &= (\mathrm{id} - U)z_n \\ -\mathrm{i}Ax_n &= (\mathrm{id} + U)z_n \end{aligned}\right\} \implies \begin{cases} z_n = \frac{1}{2}(x_n - \mathrm{i}Ax_n) \longrightarrow \frac{1}{2}(x - \mathrm{i}y) \\ Uz_n = \frac{1}{2}(-x_n - \mathrm{i}Ax_n) \longrightarrow \frac{1}{2}(-x - \mathrm{i}y). \end{cases}$$

Since U is closed, we obtain $z := \lim_{n\to\infty} z_n \in \mathcal{D}(U)$ and $Uz = \frac{1}{2}(-x - \mathrm{i}y)$. Again, by the closedness of U, we find $x = (\mathrm{id} - U)z \in \mathcal{D}(A)$ and

$$y = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \mathrm{i}(\mathrm{id} + U)z_n = \mathrm{i}(\mathrm{id} + U)z = \mathrm{i}(\mathrm{id} + U)(\mathrm{id} - U)^{-1}x = Ax.$$

Finally, we find that U is the Cayley transform of A. Let $x \in \mathcal{D}(A)$ and choose $y \in \mathcal{D}(U)$ such that $x = (\mathrm{id} - U)y$. Then $Ax = \mathrm{i}(\mathrm{id} + U)y$.
On the other hand, if $y \in \mathcal{D}(U)$, then $A((\mathrm{id} - U)y) = \mathrm{i}(\mathrm{id} + U)y$.
It follows that

$$(A + \mathrm{i})x = 2\mathrm{i}y, \qquad (A - \mathrm{i})x = 2\mathrm{i}Uy.$$

It follows that $\mathcal{D}(U) = \operatorname{rg}(A + \mathrm{i})$, $\operatorname{rg}(U) = \operatorname{rg}(A - \mathrm{i})$ and $U = (A - \mathrm{i})(A + \mathrm{i})^{-1}$. □

Corollary 2.42. *Let A be a symmetric and closed linear operator on a complex Hilbert space H with Cayley transform U. Then the following is equivalent:*

(i) *A is selfadjoint.*
(ii) *U is unitary.*

Proof. A is selfadjoint if and only if $\operatorname{rg}(A \pm \mathrm{i}) = H$. This is the case if and only if $\mathcal{D}(U) = \operatorname{rg}(U) = H$, that is, if and only if U is unitary. □

2.6 The spectral theorem for unbounded selfadjoint linear operators

Let H be a complex Hilbert space and $(E_\lambda)_{\lambda\in\mathbb{R}}$ a spectral family on H. We use the notation

$$\alpha_{x,y}(\lambda) = \langle E_\lambda x\,, y\rangle, \qquad x, y \in H,\ \lambda \in \mathbb{R}.$$

as in Section 2.3. In addition we set

$$\alpha_x(\lambda) = \alpha_{x,x}(\lambda), \qquad x \in H,\ \lambda \in \mathbb{R}.$$

Theorem 2.43. *Let H be a complex Hilbert space and $(E_\lambda)_{\lambda\in\mathbb{R}}$ a spectral family on H. For $x \in H$ and $f \in C(\mathbb{R}, \mathbb{C})$ the following is equivalent:*

(i) $\displaystyle\int_{-\infty}^{\infty} f(\lambda)\,\mathrm{d}E_\lambda\, x := \lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b f(\lambda)\,\mathrm{d}E_\lambda\, x$ *exists.*

(ii) $f \in L_2(\mathbb{R}, \mathrm{d}\alpha_x)$, *that is,* $\displaystyle\int_{-\infty}^{\infty} |f(\lambda)|^2\,\mathrm{d}\langle E_\lambda x\,, x\rangle$ *exists.*

(iii) *The map* $\varphi_x : H \to \mathbb{C}$, $\varphi_x(y) = \displaystyle\int_{-\infty}^{\infty} \overline{f(\lambda)}\,\mathrm{d}\langle E_\lambda y\,, x\rangle$ *is a bounded linear functional.*

Proof. (i) $\implies$ (iii) Let $y \in H$. Note that for $-\infty < a < b < \infty$

$$\int_a^b \overline{f(\lambda)}\,\mathrm{d}\langle E_\lambda y\,, x\rangle = \Big\langle \int_a^b \overline{f(\lambda)}\,\mathrm{d}E_\lambda\ y\,, x\Big\rangle = \Big\langle y\,, \int_a^b f(\lambda)\,\mathrm{d}E_\lambda\ x\Big\rangle.$$

In particular,

$$\begin{aligned}\varphi_x(y) &= \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_a^b \overline{f(\lambda)}\,\mathrm{d}\langle E_\lambda y\,, x\rangle = \Big\langle y\,, \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_a^b f(\lambda)\,\mathrm{d}E_\lambda\ x\Big\rangle \\ &= \Big\langle y\,, \int_{-\infty}^{\infty} f(\lambda)\,\mathrm{d}E_\lambda\ x\Big\rangle\end{aligned}$$

exists. It is cleary linear in y and bounded by $\Big\|\displaystyle\int_{-\infty}^{\infty} f(\lambda)\,\mathrm{d}E_\lambda\ x\Big\|$.

(iii) $\implies$ (ii) For $\alpha < \beta \in \mathbb{R}$ define $y_{\alpha,\beta} := \displaystyle\int_\alpha^\beta f(\lambda)\,\mathrm{d}E_\lambda x$. Note that $\|y_{\alpha,\beta}\|^2 = \displaystyle\int_\alpha^\beta |f(\lambda)|^2\,\mathrm{d}\langle E_\lambda x\,, x\rangle$ and

$$\begin{aligned}\|\varphi_x\|\,\|y_{\alpha,\beta}\| \geq |\varphi_x(y_{\alpha,\beta})| &= \int_{-\infty}^{\infty} \overline{f(\lambda)}\,\mathrm{d}\langle E_\lambda y_{\alpha,\beta}\,, x\rangle = \int_\alpha^\beta f(\lambda)\,\mathrm{d}\langle E_\lambda y_{\alpha,\beta}\,, x\rangle \\ &= \Big\langle y_{\alpha,\beta}\,, \int_\alpha^\beta f(\lambda)\,\mathrm{d}E_\lambda\ x\Big\rangle = \langle y_{\alpha,\beta}\,, y_{\alpha,\beta}\rangle = \|y_{\alpha,\beta}\|^2.\end{aligned}$$

Hence for all $\alpha < \beta$

$$\int_\alpha^\beta |f(\lambda)|^2\,\mathrm{d}\langle E_\lambda x\,, x\rangle = \|y_{\alpha,\beta}\|^2 \leq \|\varphi_x\|^2 < \infty.$$

Therefore also $\displaystyle\int_{-\infty}^{\infty} |f(\lambda)|^2\,\mathrm{d}\langle E_\lambda x\,, x\rangle < \infty$ and the integral in (i) exists.

(ii) $\Longrightarrow$ (i) For $-\infty < \alpha' < \alpha < \beta < \beta' < \infty$.

$$\begin{aligned}\left\| \int_{\alpha'}^{\beta'} f(\lambda)\ \mathrm{d}E_\lambda x - \int_\alpha^\beta f(\lambda)\ \mathrm{d}E_\lambda x \right\|^2 &= \left\| \int_{\alpha'}^{\alpha} f(\lambda)\ \mathrm{d}E_\lambda x + \int_\beta^{\beta'} f(\lambda)\ \mathrm{d}E_\lambda x \right\|^2 \\ &= \left\| \int_{\alpha'}^{\alpha} f(\lambda)\ \mathrm{d}E_\lambda x \right\|^2 + \left\| \int_\beta^{\beta'} f(\lambda)\ \mathrm{d}E_\lambda x \right\|^2 \\ &= \int_{\alpha'}^{\alpha} |f(\lambda)|^2\ \mathrm{d}E_\lambda x + \int_\beta^{\beta'} |f(\lambda)|^2\ \mathrm{d}E_\lambda x \longrightarrow 0, \qquad (\alpha, \alpha', \beta, \beta' \to 0).\end{aligned}$$

Hence the limit in (i) exists. □

For $f \in C(\mathbb{R}, \mathbb{C})$, we define (possibly unbounded) linear operators

$$f_E : \mathcal{D}(f_E) \subseteq H \to H, \quad f_E x := \int_{-\infty}^{\infty} f(\lambda)\ \mathrm{d}E_\lambda x,$$

for

$$\mathcal{D}(f_E) := \Big\{ x \in H : \int_{-\infty}^{\infty} |f(\lambda)|^2\ \mathrm{d}\langle E_\lambda x\,, x\rangle < \infty \Big\}.$$

Properties of such operators are collected in the following theorem.

Theorem 2.44. *Let H be a complex Hilbert space, $(E_\lambda)_{\lambda\in\mathbb{R}}$ a spectral family on H and $f, g \in C(\mathbb{R}, \mathbb{C})$ (it is sufficient to assume that f is measurable for every α_x).*

(i) *$\mathcal{D}(f_E)$ is dense in H and $\operatorname{rg}(E_\lambda - E_\mu) \subseteq \mathcal{D}(f_E)$ for $\mu < \lambda$.*

(ii) *$E_\lambda f_E \subseteq f_E E_\lambda$ for all $\lambda \in \mathbb{R}$.*

(iii) *If f is real valued, then f_E is selfadjoint.*

(iv) *If $f(t) = 0$ for all $t \in \mathbb{R}$, then $f_E = 0$. If $f(t) = 1$ for all $t \in \mathbb{R}$, then $f_E = \mathrm{id}$.*

(v) *If $x \in \mathcal{D}(f_E)$ and $y \in H$, then $\langle f_E x\,, y\rangle = \int_{-\infty}^{\infty} f(t)\ \mathrm{d}\langle E_t x\,, y\rangle$, in particular, the integral exists.*

(vi) *$f_E + g_E \subset (f+g)_E$ and $\mathcal{D}(f_E + g_E) = \mathcal{D}((|f| + |g|)_E)$.*

(vii) *$f_E g_E \subset (fg)_E$ and $\mathcal{D}(f_E g_E) = \mathcal{D}(g_E) \cap \mathcal{D}((fg)_E)$.*

Proof. (i), (ii) and (iii) are shown in Exercise 2.12. (iv) is clear.
(v). Let $x \in \mathcal{D}(f_E)$. Observe that $f \in \mathrm{L}_2(\mathbb{R}, \mathrm{d}\alpha_x)$ if and only if $\mathrm{Re}(f) \in \mathrm{L}_2(\mathbb{R}, \mathrm{d}\alpha_x)$ and $\mathrm{Im}(f) \in \mathrm{L}_2(\mathbb{R}, \mathrm{d}\alpha_x)$. Since

$$\int_\alpha^\beta f(t)\ \mathrm{d}E_t x = \int_\alpha^\beta \mathrm{Re}(f(t))\ \mathrm{d}E_t x + \mathrm{i} \int_\alpha^\beta \mathrm{Im}(f(t))\ \mathrm{d}E_t x$$

if $-\infty < \alpha < \beta < \infty$, so for all $x \in \mathcal{D}(f_E)$ it follows that

$$\int_{-\infty}^{\infty} f(t)\ \mathrm{d}E_t x = \int_{-\infty}^{\infty} \mathrm{Re}(f(t))\ \mathrm{d}E_t x + \mathrm{i} \int_{-\infty}^{\infty} \mathrm{Im}(f(t))\ \mathrm{d}E_t x,$$

and we can assume that f is real valued.
Observe that for a selfadjoint operator T and vectors $\xi, \eta \in \mathcal{D}(T)$ we have

$$\begin{aligned}\langle T\xi\,, \eta\rangle =& \frac{1}{2}\big[\langle T(\xi+\eta)\,, \xi+\eta\rangle - \langle T\xi\,, \xi\rangle - \langle T\eta\,, \eta\rangle\big] \\ &+ \frac{1}{2\mathrm{i}}\big[\langle T(\xi+\mathrm{i}\eta)\,, \xi+\mathrm{i}\eta\rangle - \langle T\xi\,, \xi\rangle - \langle T\eta\,, \eta\rangle\big].\end{aligned} \tag{2.10}$$

Applying this to $x = \xi$, $y = \eta$, $T = f_E$ and $T = E_t$, respectively, we obtain

$$\begin{aligned}\langle f_E x\,, y\rangle &= \frac{1}{2}\big[\langle f_E(x+y)\,, x+y\rangle - \langle f_E x\,, x\rangle - \langle f_E y\,, y\rangle\big] + \frac{\mathrm{i}}{2}\big[\dots\big] \\ &= \frac{1}{2}\int_{-\infty}^{\infty} f(t)\ \mathrm{d}\big[\langle E_t(x+y)\,, x+y\rangle - \langle E_t x\,, x\rangle - \langle E_t y\,, y\rangle + \mathrm{i}(\dots)\big] \\ &= \int_{-\infty}^{\infty} f(t)\ \mathrm{d}\langle E_t x\,, y\rangle.\end{aligned}$$

Now if $y \in H$, we define $y_{\alpha,\beta} = (E_\beta - E_\alpha)y$. By (i), $y_{\alpha,\beta} \in \mathcal{D}(f_E)$ and $y_{\alpha,\beta} \to y$ for $\alpha \to -\infty$ and $\beta \to \infty$. Moreover, $E_t y_{\alpha,\beta} = E_t y$ for $t \in (\alpha, \beta]$, $E_t y_{\alpha,\beta} = 0$ for $t \le \alpha$ and $E_t y_{\alpha,\beta} = y_{\alpha,\beta}$ for $t \ge \beta$. It follows that

$$\langle f_E x\,, y_{\alpha,\beta}\rangle = \int_{-\infty}^{\infty} f(t)\ \mathrm{d}\langle E_t x\,, y_{\alpha,\beta}\rangle = \int_\alpha^\beta f(t)\ \mathrm{d}\langle E_t x\,, y_{\alpha,\beta}\rangle = \int_\alpha^\beta f(t)\ \mathrm{d}\langle E_t x\,, y\rangle.$$

Since the left hand side converges for $\alpha \to -\infty$ and $\beta \to \infty$, so does the right hand side and the limits coincide:

$$\langle f_E x\,, y\rangle = \int_{-\infty}^{\infty} f(t)\ \mathrm{d}\langle E_t x\,, y\rangle, \qquad x \in \mathcal{D}(f_E),\ y \in H.$$

(vi) If $x \in \mathcal{D}(f_E + g_E) = \mathcal{D}(f_E) \cap \mathcal{D}(g_E)$, then $f \in L_2(\mathbb{R}, \mathrm{d}\alpha_x)$ and $g \in L_2(\mathbb{R}, \mathrm{d}\alpha_x)$ and

$$\int_\alpha^\beta f(t)\,\mathrm{d}E_t x + \int_\alpha^\beta g(t)\,\mathrm{d}E_t x = \int_\alpha^\beta (f+g)(t)\,\mathrm{d}E_t x$$

by linearity of the integral. Taking the limits $\alpha \to -\infty$ and $\beta \to \infty$ on both sides, we obtain $f_E x + g_E x = (f+g)_E x$. Clearly, $f, g \in L_2(\mathbb{R}, \mathrm{d}\alpha_x)$ if and only if $|f| + |g| \in L_2(\mathbb{R}, \mathrm{d}\alpha_x)$, hence $\mathcal{D}((f+g)_E) = \mathcal{D}((|f| + |g|)_E$.
(vii)... □

Corollary 2.45. *In the special case $f : \mathbb{R} \to \mathbb{C}$, $f(\lambda) := \lambda$,*

$$Ax = \int_{-\infty}^{\infty} \lambda\ \mathrm{d}E_\lambda x \quad \textit{for} \quad x \in \mathcal{D}(A) := \Big\{ x \in H : \int_{-\infty}^{\infty} \lambda^2\ \mathrm{d}\langle E_\lambda x\,, x\rangle < \infty \Big\} \tag{2.11}$$

is a selfadjoint linear operator.

Next we will prove the reverse of Corollary 2.45, that is, we will show that every selfadjoint linear operator has a representation as in (2.11). Note that in the case of a bounded selfadjoint operator, the integral representation converges in operator norm. If the operator is unbounded, then the integral converges only strongly since the spectral family will not have compact support.

Theorem 2.46 (Spectral theorem for unbounded selfadjoint operators).
Let H be a complex Hilbert space and $A(H \to H)$ a selfadjoint linear operator. Then there exists a spectral resolution $(E_\lambda)_{\lambda\in\mathbb{R}}$ such that

$$\int_{-\infty}^{\infty} \lambda \, \mathrm{d}\langle E_\lambda x\,, x\rangle = \langle Ax\,, x\rangle, \qquad x \in \mathcal{D}(A). \tag{2.12}$$

Uniqueness of the spectral resolution will follow from Stone's formula (see Theorem 2.49).

Proof. Since A is selfadjoint, its Cayley transform U is unitary by Corollary 2.42 and $A = \mathrm{i}(U + \mathrm{id})(U - \mathrm{id})^{-1}$. Let $(\widetilde{E}_\lambda)_{\lambda\in\Lambda}$ be the spectral resolution of $-U$ as in Theorem 2.37 such that

$$-U = \int_{-\pi-0}^{\pi} \mathrm{e}^{\mathrm{i}\lambda} \, \mathrm{d}\widetilde{E}_\lambda.$$

Since the spectral family is right continuous, it follows that $\widetilde{E}_\pi = \mathrm{id}$. Now we will show that $\widetilde{E}_{-\pi} = 0$.
To this end assume that $\widetilde{E}_{-\pi} \neq 0$. Then there exists an $x \neq 0$ in $\mathrm{rg}(\widetilde{E}_{-\pi})$. Hence

$$\begin{aligned} -Ux &= \int_{-\pi-0}^{\pi} \mathrm{e}^{\mathrm{i}\lambda} \, \mathrm{d}\widetilde{E}_\lambda x = \lim_{\varepsilon\searrow 0} \int_{-\pi-\varepsilon}^{\pi} \mathrm{e}^{\mathrm{i}\lambda} \, \mathrm{d}\widetilde{E}_\lambda x = \lim_{\varepsilon\searrow 0} \int_{-\pi-\varepsilon}^{-\pi} \mathrm{e}^{\mathrm{i}\lambda} \, \mathrm{d}\widetilde{E}_\lambda x \\ &= \lim_{\varepsilon\searrow 0} \int_{-\pi-\varepsilon}^{-\pi} \left(\mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\pi} \right) \mathrm{d}\widetilde{E}_\lambda x + \lim_{\varepsilon\searrow 0} \int_{-\pi-\varepsilon}^{-\pi} \mathrm{e}^{-\mathrm{i}\pi} \, \mathrm{d}\widetilde{E}_\lambda x. \end{aligned}$$

The first term vanishes because

$$\begin{aligned} \left\| \int_{-\pi-\varepsilon}^{-\pi} \left(\mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\pi} \right) \mathrm{d}\widetilde{E}_\lambda x \right\|^2 &= \int_{-\pi-\varepsilon}^{-\pi} \left| \mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\pi} \right|^2 \mathrm{d}\langle \widetilde{E}_\lambda x\,, x\rangle \\ &\le \sup\{ | \mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\pi} | \,:\, \lambda \in [-\pi-\varepsilon, -\pi] \} \|x\|^2 \longrightarrow 0, \qquad \varepsilon \searrow 0. \end{aligned}$$

The second term gives

$$\lim_{\varepsilon\searrow 0} \int_{-\pi-\varepsilon}^{-\pi} \mathrm{e}^{-\mathrm{i}\pi} \, \mathrm{d}\widetilde{E}_\lambda x = \lim_{\varepsilon\searrow 0} \mathrm{e}^{-\mathrm{i}\pi} (\widetilde{E}_{-\pi} - \widetilde{E}_{-\pi-\varepsilon}) x = \mathrm{e}^{-\mathrm{i}\pi} \, \widetilde{E}_{-\pi} x = \mathrm{e}^{-\mathrm{i}\pi} \, x = -x.$$

So in total we obtain $Ux = x$, which contradicts $1 \notin \sigma_{\mathrm{p}}(U)$ (see Proposition 2.40 (ii)). Similarly, one can show that $\mathrm{rg}\, E(\{\pi\}) = \{0\}$. It follows that

$$-U = \int_{(-\pi,\pi)} \mathrm{e}^{\mathrm{i}\lambda} \, \mathrm{d}\widetilde{E}_\lambda.$$

Now let $x \in \mathcal{D}(A) = \mathrm{rg}(\mathrm{id} - U)$ and choose $y \in H$ such that $x = (\mathrm{id} - U)y$. Then $Ax = \mathrm{i}(\mathrm{id} + U)y$ and consequently, using $U^{-1} = U^*$,

$$\begin{aligned} \langle Ax\,, x\rangle &= \mathrm{i} \left\langle (\mathrm{id} + U)y\,, (\mathrm{id} - U)y \right\rangle = \mathrm{i} \left\langle Uy\,, y\right\rangle - \mathrm{i}\left\langle y\,, Uy\right\rangle = \mathrm{i}\left\langle (U - U^{-1})y\,, y\right\rangle \\ &= -\mathrm{i} \int_{(-\pi,\pi)} (\mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}) \, \mathrm{d}\left\langle \widetilde{E}_\lambda y\,, y\right\rangle. \end{aligned}$$

On the other hand, using that $\widetilde{E}_\lambda$ and U commute, we have

$$\begin{aligned} \left\langle \widetilde{E}_\lambda x\,, x\right\rangle &= \left\langle \widetilde{E}_\lambda (\mathrm{id} - U)y\,, (\mathrm{id} - U)y \right\rangle = \left\langle \widetilde{E}_\lambda (\mathrm{id} - U^*)(\mathrm{id} - U)y\,, y \right\rangle \\ &= \int_{-\pi+0}^{\lambda} (1 + \mathrm{e}^{-\mathrm{i}t})(1 + \mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}\langle \widetilde{E}_t y\,, y\rangle. \end{aligned}$$

Applying the substitution rule, we obtain

$$\begin{aligned} \langle Ax\,, x\rangle &= -\mathrm{i} \int_{(-\pi,\pi)} (\mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}) \, \mathrm{d}\left\langle \widetilde{E}_\lambda y\,, y\right\rangle \\ &= -\mathrm{i} \int_{(-\pi,\pi)} \frac{(\mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda})}{(1 + \mathrm{e}^{-\mathrm{i}\lambda})(1 + \mathrm{e}^{\mathrm{i}\lambda})} \, (1 + \mathrm{e}^{-\mathrm{i}\lambda})(1 + \mathrm{e}^{\mathrm{i}\lambda}) \, \mathrm{d}\left\langle \widetilde{E}_\lambda y\,, y\right\rangle \\ &= -\mathrm{i} \int_{(-\pi,\pi)} \frac{(\mathrm{e}^{\mathrm{i}\lambda/2} + \mathrm{e}^{-\mathrm{i}\lambda/2})(\mathrm{e}^{\mathrm{i}\lambda/2} - \mathrm{e}^{-\mathrm{i}\lambda/2})}{(\mathrm{e}^{\mathrm{i}\lambda/2} + \mathrm{e}^{-\mathrm{i}\lambda/2})(\mathrm{e}^{-\mathrm{i}\lambda/2} + \mathrm{e}^{\mathrm{i}\lambda/2})} \, \mathrm{d}\left\langle \widetilde{E}_\lambda x\,, x\right\rangle \\ &= \int_{(-\pi,\pi)} \tan(\lambda/2) \, \mathrm{d}\left\langle \widetilde{E}_\lambda x\,, x\right\rangle \end{aligned}$$

Set $E_\lambda := \widetilde{E}_{2\arctan\lambda}$. Then Exercise 2.4 shows

$$\langle Ax\,, x\rangle = \int_{\mathbb{R}} \lambda \, \mathrm{d}\langle E_\lambda x\,, x\rangle \qquad \square$$

Theorem 2.47. *Let H be a complex Hilbert space and $A(H \to H)$ a selfadjoint linear operator and $(E_\lambda)_{\lambda\in\mathbb{R}}$ its spectral resolution from Theorem 2.46. Then $E_t x \in \mathcal{D}(A)$ for every $t \in \mathbb{R}$ and $x \in \mathcal{D}(A)$*

$$\mathcal{D}(A) = \left\{ x \in H : \int_{-\infty}^{\infty} \lambda^2 \, \mathrm{d}\langle E_\lambda x\,, x\rangle < \infty \right\} \quad \text{and} \quad Ax = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E_\lambda x$$

for $x \in \mathcal{D}(A)$.

Proof. Let $x \in \mathcal{D}(A) = \mathrm{rg}(\mathrm{id} - U_A)$ where U_A is the Cayley transform of A. Let $y \in H$ such that $x = (\mathrm{id} - U_A)y$. By construction, all E_λ commute with U, hence $E_\lambda x = E_\lambda(\mathrm{id} - U_A)y = (\mathrm{id} - U_A)E_\lambda y \in \mathrm{rg}(\mathrm{id} - U_A) = \mathcal{D}(A)$.
Now let $\mathcal{D} := \left\{ x \in H : \int_{-\infty}^{\infty} \lambda^2 \, \mathrm{d}\langle E_\lambda x\,, x\rangle < \infty \right\}$ and define $B : \mathcal{D} \subseteq H \to H$ by

$$Bx := \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E_\lambda x, \qquad x \in \mathcal{D}(B).$$

Observe that B is selfadjoint by Corollary 2.45. We have to show $A = B$. It suffices to show that $A \subseteq B$ because then $B = B^* \subseteq A^* = A$. Let $x, y \in \mathcal{D}(A)$. Then, by (2.12) and (2.10) we find

$$\langle Ax\,, y\rangle = \int_{-\infty}^{\infty} \lambda\, \mathrm{d}\langle E_\lambda x\,, y\rangle.$$

For arbitrary $y \in H$ we define $y_{\alpha,\beta} := (E_\beta - E_\alpha)y$. As in the proof of Theorem 2.44 (v) we obtain

$$\langle Ax\,, y_{\alpha,\beta}\rangle = \int_{-\infty}^{\infty} \lambda\ \mathrm{d}\langle E_\lambda x\,, y_{\alpha,\beta}\rangle = \int_{\alpha}^{\beta} \lambda\ \mathrm{d}\langle E_\lambda x\,, y_{\alpha,\beta}\rangle = \int_{\alpha}^{\beta} \lambda\ \mathrm{d}\langle E_\lambda x\,, y\rangle.$$

Since the left hand side converges for $\alpha \to -\infty$ and $\beta \to \infty$, so does the right hand side and we get

$$\langle Ax\,, y\rangle = \int_{-\infty}^{\infty} \lambda\ \mathrm{d}\langle E_\lambda x\,, y\rangle. \tag{2.13}$$

This shows that for every $x \in \mathcal{D}(A)$ the linear functional φ_x defined in Theorem 2.43 (iii) is bounded by $\|Ax\|$, in particular $x \in \mathcal{D}$. Hence (2.13) implies for all $y \in H$

$$\begin{aligned}\langle Ax\,, y\rangle &= \lim_{\alpha\to-\infty}\lim_{\beta\to\infty}\int_\alpha^\beta f(t)\ \mathrm{d}\langle E_t x\,, y\rangle = \lim_{\alpha\to-\infty}\lim_{\beta\to\infty}\Big\langle \int_\alpha^\beta f(t)\ \mathrm{d}E_t x\,, y\Big\rangle \\ &= \Big\langle \lim_{\alpha\to-\infty}\lim_{\beta\to\infty}\int_\alpha^\beta f(t)\ \mathrm{d}E_t x\,, y\Big\rangle = \langle Bx\,, y\rangle.\end{aligned}$$

Since this is true for all $x \in \mathcal{D}(A)$ and $y \in H$, we proved that $A \subseteq B$. □

In the rest of this section we want to prove a formula for the spectral family in terms of the selfadjoint operator. To this end, recall the Stieltjes inversion formula.

Theorem 2.48 (Stieltjes inversion formula). *Let $\omega : \mathbb{R} \to \mathbb{C}$ be a right continuous function of bounded variation with $\omega(t) \to 0$ for $|t| \to \infty$. The* Stieltjes transform *of ω is given by*

$$f : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}, \qquad z \mapsto \int_{\mathbb{R}} \frac{1}{t-z}\ \mathrm{d}\omega(t).$$

ω can be recovered from its Stieltjes transform by

$$\omega(t) = \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{t+\delta} f(s+\mathrm{i}\varepsilon) - f(s-\mathrm{i}\varepsilon)\,\mathrm{d}s.$$

Proof. See [Wei80, Theorem B1]. □

Theorem 2.49 (Stone's formula). *Let H be a complex Hilbert space and $A(H \to H)$ a selfadjoint linear operator on H. For all $x, y \in H$ and $t \in \mathbb{R}$ the following formula holds:*

$$\langle E(t)x\,, y\rangle = \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{t+\delta}\Big\langle \big((A-s-\mathrm{i}\varepsilon)^{-1} - (A-s+\mathrm{i}\varepsilon)^{-1}\big)x\,, y\Big\rangle\ \mathrm{d}s.$$

Proof. We apply the Stieltjes inversion formula to $\omega(t) = \langle E_t x\,, y\rangle$ and use that $\int_{\mathbb{R}} \frac{1}{t-z}\ \mathrm{d}\omega(t) = \int_{\mathbb{R}} \frac{1}{t-z}\ \mathrm{d}\langle E_t x\,, y\rangle = \langle (A-z)^{-1}x\,, y\rangle$ for $z \in \rho(A)$. Therefore we obtain

$$\begin{aligned}\big\langle E(t)x\,, y\big\rangle = \omega(t) &= \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{t+\delta}\big((\tau-s-\mathrm{i}\varepsilon)^{-1} - (\tau-s+\mathrm{i}\varepsilon)^{-1}\big)\ \mathrm{d}\omega(\tau)\ \mathrm{d}s \\ &= \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{t+\delta}\Big\langle\big[(A-s-\mathrm{i}\varepsilon)^{-1} - (A-s+\mathrm{i}\varepsilon)^{-1}\big]x\,, y\Big\rangle\ \mathrm{d}s.\end{aligned}$$ □

2.7 Spectrum and spectral resolution

Theorem 2.50. *Let H be a complex Hilbert space and $A(H \to H)$ a closed operator on H. The* discrete spectrum *of A is defined by*

$$\sigma_d(T) := \Big\{\lambda \in \sigma_p(T) : \lambda \textit{ is an isolated point in } \sigma(T) \textit{ and } \lambda \textit{ has finite multiplicity}\Big\}$$

and the essential spectrum[1] *of A is*

$$\sigma_{ess}(T) := \Big\{\lambda \in \mathbb{C} : \begin{array}{l}\lambda \textit{ is either an accumulation point of } \sigma(T) \\ \textit{or } \lambda \textit{ is an eigenvalue of infinite multiplicity}\end{array}\Big\}.$$

Observations 2.51. (i) $\sigma_{\mathrm{d}} \cup \sigma_{\mathrm{ess}}(T) \subseteq \sigma(T)$ and $\sigma_{\mathrm{d}}(T) \cap \sigma_{\mathrm{ess}}(T) = \emptyset$.

(ii) $\sigma_{\mathrm{ess}}(T)$ is closed.

(iii) $\sigma_{\mathrm{d}}(T)$ is not necessarily closed.

(iv) If $\dim H < \infty$, then $\sigma_{\mathrm{d}}(T) = \sigma(T)$ and $\sigma_{\mathrm{ess}}(T) = \emptyset$.

(v) If $\dim H = \infty$ and T is compact, then $\sigma_{\mathrm{d}}(T) = \sigma(T)\setminus\{0\}$, $\overline{\sigma_{\mathrm{d}}(T)} = \sigma(T)$ and $\sigma_{\mathrm{ess}}(T) = \{0\}$.

Next we will show the relation between the spectral resolution of a selfadjoint linear operator and its spectrum. As a corollary we will obtain that $\sigma_{\mathrm{d}} \cup \sigma_{\mathrm{ess}}(T) = \sigma(T)$ if T is selfadjoint.

As in Section 2.1 we use the notation

$$\alpha_{x,y}(t) = \langle E(t)x\,, y\rangle, \quad \alpha_x(t) = \alpha_{x,x}(t), \qquad x, y \in H,\ t \in \mathbb{R}.$$

[1] Note that there are many different definitions of the essential spectrum in the literature which do not necessarily coincide, see [?]

Recall that for a selfadjoint operator the spectrum and the approximative spectrum coincide, that is, $\lambda\sigma(A)$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \not\to 0$ and $(A-\lambda)x_n \to 0$.

Theorem 2.52. *Let H be a complex Hilbert space and $A(H \to H)$ be a selfadjoint linear operator on H with spectral resolution $(E(t))_{t\in\mathbb{R}}$. Then each of the following is equivalent:*

- *Spectrum:*
  - (i) $\lambda \in \sigma(T)$.
  - (ii) *There exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $x_n \not\to 0$ and $(T-\lambda)x_n \to 0$ for $n \to \infty$.*
  - (iii) $E(\lambda-\varepsilon) \neq E(\lambda+\varepsilon)$ *for all* $\varepsilon > 0$.
  - (iv) $(\lambda - z)^{-1} \in \sigma(T-z)^{-1}$ *for one/all* $z \in \rho(T)$.
- *Point spectrum:*
  - (i) $\lambda \in \sigma_p(T)$.
  - (ii) *There exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $x_n \not\to 0$ and $(T-\lambda)x_n \to 0$ for $n \to \infty$.*
  - (iii) $E(\lambda-) \neq E(\lambda)$ *(that is, E is not strongly continuous in λ).*
  - (iv) $(\lambda - z)^{-1} \in \sigma_p(T-z)^{-1}$ *for one/all* $z \in \rho(T)$ *and* $\ker(T-\lambda) = \ker\big((T-z)^{-1} - (\lambda-z)^{-1}\big)$.
- *Discrete spectrum:*
  - (i) $\lambda \in \sigma_d(T)$.
  - (ii) *There exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $x_n \not\to 0$ and $(T-\lambda)x_n \to 0$ for $n\to\infty$ and every bounded such sequence contains a convergent subsequence.*
  - (iii) *There exists an $\varepsilon > 0$ such that* $0 \neq \dim(\operatorname{rg} E((\lambda-\varepsilon, \lambda+\varepsilon))) \neq \infty$.
- *Essential spectrum:*
  - (i) $\lambda \in \sigma_{ess}(T)$.
  - (ii) *There exists a so-called* singular sequence *for A in λ, that is, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $x_n \not\to 0$, $x_n \xrightarrow{w} 0$ and $(T-\lambda)x_n \to 0$ for $n \to \infty$.*
  - (iii) $\dim(\operatorname{rg} E((\lambda-\varepsilon, \lambda+\varepsilon))) = \infty$ *for all* $\varepsilon > 0$.

  (ii) *is the so-called* Weyl criterion.

Another characterisation of the essential spectrum in terms of the sequences is given in Exercise 2.21.

Proof. • Spectrum:

(i) $\Longrightarrow$ (iii): Assume that there exists an $\varepsilon > 0$ such that $E(\lambda-\varepsilon) = E(\lambda+\varepsilon)$. Then $(t-\lambda)^2 \geq \varepsilon^2$ α_x-a.e. for all $x \in H$. In particular,

$$\|(A-\lambda)x\|^2 = \int_{\mathbb{R}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) \geq \varepsilon^2 \int_{\mathbb{R}} \mathrm{d}\alpha_x(t) = \varepsilon^2\|x\|^2.$$

Hence $\lambda \notin \sigma_{\mathrm{p}}(A)$ and, by the closed graph theorem, $\operatorname{rg}(A-\lambda)$ is closed. Observe that $\sigma_{\mathrm{c}}(A) = \emptyset$ since A is selfadjoint. Hence $\lambda \in \rho(T)$ in contradiction to the hypothesis.

(iii) $\Longrightarrow$ (ii): For every $n \in \mathbb{N}$ we choose an $x_n \in \operatorname{rg}(E((\lambda - \frac{1}{n}, \lambda + \frac{1}{n})))$ with $\|x_n\| = 1$. Then $(t-\lambda)^2 \leq \frac{1}{n^2}$ α_{x_n}-a.e. and therefore

$$\|(A-\lambda)x\|^2 = \int_{\mathbb{R}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) \leq \frac{1}{n^2} \longrightarrow 0, \qquad n \to \infty.$$

(ii) $\Longrightarrow$ (i): The hypothesis implies immediately that $(T-\lambda)$ is not boundedly invertible, hence $\lambda \in \sigma(T)$. hence $\lambda \notin \sigma_{\mathrm{p}}(A)$.

(i) $\Longleftrightarrow$ (iv): For fixed $z \in \rho(T)$

$$(T-z)^{-1} - (\lambda-z)^{-1} = (\lambda-z)^{-1}(\lambda-T)(T-z)^{-1} = (\lambda-z)^{-1}(T-z)^{-1}(\lambda-T).$$

$\lambda \in \rho(T)$ if and only if $\lambda - T$ is bijective, if and only if $(T-z)^{-1} - (\lambda-z)^{-1}$ is bijective.

• Point spectrum:

(i) $\Longrightarrow$ (ii): Let x be a non-zero eigenvector of A with eigenvalue λ and choose $x_n = s$ for all $n \in \mathbb{N}$.

(ii) $\Longrightarrow$ (i): Let $(x_n)_{n\in\mathbb{N}}$ as in the assumption and let $x \in H$ be its limit point. Then, by assumption, also $(Tx_n)_{n\in\mathbb{N}}$ converges with limit point λx. Since T is closed, it follows that $x \in \mathcal{D}(T)$ and $Tx = \lambda x$.

(i) $\Longrightarrow$ (iii): Assume that E is continuous in λ. Then $(t-\lambda)^2 \neq 0$ α_x-a.e. for all $x \in \mathcal{D}(A)$ and consequently

$$\|(A-\lambda)x\|^2 = \int_{\mathbb{R}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) > 0,$$

hence $\lambda \notin \sigma_{\mathrm{p}}(A)$.

(iii) $\Longrightarrow$ (i): Let $x \in E(\{\lambda\})$ with $\|x\| = 1$. Then

$$\|(A-\lambda)x\|^2 = \int_{\mathbb{R}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) = \int_{\mathbb{R}\setminus\{\lambda\}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) + \int_{\{\lambda\}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) = 0.$$

Note that the proof also shows that

$$E(\{\lambda\}) = \ker(A-\lambda).$$

- Discrete spectrum:
- Essential spectrum:

(i) $\Longrightarrow$ (iii): Assume that there exists an $\varepsilon > 0$ such that $E(\lambda-\varepsilon) = E(\lambda+\varepsilon)$. Then $(t-\lambda)^2 \geq \varepsilon^2$ α_x-a.e. for all $x \in H$. In particular,

$$\|(A-\lambda)x\|^2 = \int_{\mathbb{R}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) \geq \varepsilon^2 \int_{\mathbb{R}} \mathrm{d}\alpha_x(t) = \varepsilon^2 \|x\|^2.$$

Hence $\lambda \notin \sigma_{\mathrm{p}}(A)$ and, by the closed graph theorem, $\mathrm{rg}(A-\lambda)$ is closed. Observe that $\sigma_{\mathrm{c}}(A) = \emptyset$ since A is selfadjoint. Hence $\lambda \in \rho(T)$ in contradiction to the hypothesis.

(iii) $\Longrightarrow$ (ii): For every $n \in \mathbb{N}$ we choose an $x_n \in \mathrm{rg}(E((\lambda - \frac{1}{n}, \lambda + \frac{1}{n})))$ with $\|x_n\| = 1$. Then $(t-\lambda)^2 \leq \frac{1}{n^2}$ α_{x_n}-a.e. and therefore

$$\|(A-\lambda)x\|^2 = \int_{\mathbb{R}} (t-\lambda)^2 \, \mathrm{d}\alpha_x(t) \leq \frac{1}{n^2} \longrightarrow 0, \qquad n \to \infty.$$

(ii) $\Longrightarrow$ (i): The hypothesis implies immediately that $(T-\lambda)$ is not boundedly invertible, hence $\lambda \in \sigma(T)$. hence $\lambda \notin \sigma_{\mathrm{p}}(A)$. □

2.8 Appendix: Integration in Banach spaces

In the following we always assume that (Ω, Σ, μ) is a σ-finite measure space and X a Banach space. In most of our applications Ω will be the real line or a curve in $\mathbb{C}$. Good references for the Bochner integral are [HP74], [Yos95], [DU77], [AE09].

Definition 2.53. A function $f : \Omega \to X$ is called

(i) *simple function* $\iff$ $f = \displaystyle\sum_{k=}^{n} x_k \,\mathrm{id}_{B_k}$ where $x_j \in X$ and $B_j \in \Sigma$ with $\mu(B_j) < \infty$ for all $j = 1, \dots, n$.

(ii) *strongly measurable* $\iff$ there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ μ-a.e. (that is, $f_n(x) \to f(x)$ for μ-a.a. $x \in X$).

(iii) *separably valued* $\iff$ $\{f(s) : s \in \Omega\}$ is separable.

(iv) *countably valued* $\iff$ $\{f(s) : s \in \Omega\}$ is a countable.

(v) *weakly measurable* $\iff$ $s \mapsto \varphi(f(s))$ is measurable for every $\varphi \in X'$.

Theorem 2.54 (Pettis). *A function $f : \Omega \to X$ is strongly measurable if and only if f is weakly measurable and μ-a.e. separably valued.*

Proof. See, e.g., [HP74, Theorem 3.5.3]. □

Corollary 2.55. (i) *$f : \Omega \to X$ is strongly measurable $\iff$ f is the uniform limit μ-a.e. of a sequence of countably valued functions.*

(ii) *If X is separable, then strong measurability and weak measurability are equivalent.*

Now we will define integrals.

Definition 2.56. (i) For a simple function $f = \sum_{k=1}^{n} x_k \,\mathrm{id}_{B_k}$ we define the integral

$$\int_\Omega f \, \mathrm{d}\mu := \sum_{k=1}^{n} x_k \mu(B_k).$$

Obviously, the sum on the right hand side does not depend on the representation of f.

(ii) A countably valued function $f = \sum_{k=1}^{\infty} x_k \,\mathrm{id}_{B_k}$ is called *integrable* if and only if $\|f\|$ is Lebesgue-integrable. In this case we define

$$\int_\Omega f \, \mathrm{d}\mu := \sum_{k=1}^{\infty} x_k \mu(B_k).$$

(iii) A function $f = \sum_{k=1}^{\infty} x_k \,\mathrm{id}_{B_k}$ is called *Bochner integrable* if and only if there exists a sequence of countably valued integrable functions (f_n) such that $f_n \to f$ μ-a.e. and $\displaystyle\int_\Omega \|f_n - f\| \, \mathrm{d}\mu \to 0$ for $n \to \infty$. For a Bochner integrable function f we define

$$\int_\Omega f \, \mathrm{d}\mu = \lim_{n\to\infty} f_n \, \mathrm{d}\mu, \tag{2.14}$$

where $(f_n)_{n\in\mathbb{N}}$ is a sequence of countably valued integrable functions as above.

Observation 2.57. (i) If f is strongly measurable, then $\|f - f_n\|$ is measurable because it is limit of simple functions. Therefore $\displaystyle\int_\Omega \|f_n - f\| \, \mathrm{d}\mu$ is well-defined.

(ii) *Existence of the limit in* (2.14): It suffices to show that the sequence of the integrals is a Cauchy sequence. Let $n, k \in \mathbb{N}$. Since f_n and f_k are countably valued functions, we obtain

$$\begin{aligned} \left\| \int_\Omega f_n \, \mathrm{d}\mu - \int_\Omega f_k \, \mathrm{d}\mu \right\| &= \left\| \int_\Omega (f_n - f_k) \, \mathrm{d}\mu \right\| \leq \int_\Omega \|f_n - f_k\| \, \mathrm{d}\mu \\ &\leq \int_\Omega \|f_n - f\| \, \mathrm{d}\mu + \int_\Omega \|f_k - f\| \, \mathrm{d}\mu \longrightarrow 0, \qquad n, k \to \infty. \end{aligned}$$

(iii) *Uniqueness of the limit in* (2.14): The limit does not depend on the sequence $(f_n)_{n\in\mathbb{N}}$, because given two such sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$, we can form the sequence $(h_n)_{n\in\mathbb{N}} := (f_1, g_1, f_2, g_2, \dots)$, and it follows that

$$\lim_{n\to\infty} f_n \,\mathrm{d}\mu = \lim_{n\to\infty} h_n \,\mathrm{d}\mu = \lim_{n\to\infty} g_n \,\mathrm{d}\mu.$$

Theorem 2.58 (Bochner). *A function $f : \Omega \to X$ is Bochner integrable if and only if f is strongly measurable and $\int_\Omega \|f\| \,\mathrm{d}\mu < \infty$.*

Proof. See [HP74, Theorem 3.7.4] or [Yos95, Theorem 1 in V.5.]. □

Next, we list some important properties of the Bochner integral.

Theorem 2.59 (Bochner). *Let X, Y be Banach spaces, $T(X,Y)$ a closed linear operator and $f : \Omega \to X$ such that $f(s) \in \mathcal{D}(T)$ for all $s \in \Omega$. If f and $Tf : \Omega \to Y$ are Bochner integrable, then*

$$T \int_\Omega f \,\mathrm{d}\mu = \int_\Omega Tf \,\mathrm{d}\mu.$$

Proof. See [HP74, Theorem 3.7.12]. □

Theorem 2.60. *Let X, Y be Banach spaces and $T \in L(X,Y)$. If $f : \Omega \to X$ is Bochner integrable, then so is Tf and*

$$T \int_\Omega f \,\mathrm{d}\mu = \int_\Omega Tf \,\mathrm{d}\mu.$$

Proof. See [Yos95, Corollary 2 in V.5.]. □

Theorem 2.61 ([HP74, Theorem 3.7.6]). *If $f : \Omega \to X$ is Bochner integrable, then*

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\| \le \int_\Omega \|f\| \,\mathrm{d}\mu.$$

Theorem 2.62 ([HP74, Theorem 3.7.5, Theorem 3.7.7, Theorem 3.7.8]).

(i) *Let $f_n : \Omega \to X$ be a sequence of Bochner integrable functions such that $\int_\Omega \|f_n - f_m\| \,\mathrm{d}\mu \to 0$ for $n, m \to \infty$. Then there exists a non-unique Bochner integrable function f such that $\int_\Omega \|f_n - f\| \,\mathrm{d}\mu \to 0$ for $n \to \infty$. Two such functions are equal μ-a.e.*

(ii) *The Bochner integral is linear. The set of all Bochner integral functions $\Omega \to X$ is a normed space with norm $\|f\| := \int_\Omega \|f\| \,\mathrm{d}\mu$. If functions that coincide μ-a.e. are identified, then it becomes a Banach space.*

Theorem 2.63 (Dominated convergence theorem). *Let $f_n : \Omega \to X$ be a sequence of Bochner integrable functions that converges μ-a.e. to f. Assume that there exists a integrable function $g : \Omega \to \mathbb{R}$ such that $\|f_n(s)\| \le g(s)$ for all $n \in \mathbb{N}$ and μ-a.e. $s \in \Omega$. Then f is Bochner integrable and*

$$\lim_{n\in\mathbb{N}} \int_\Omega f_n \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu.$$

Proof. See [HP74, Theorem 3.7.9] □

Theorem 2.64 (Fubini). *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ-finite measure spaces and X a Banach space. Then $f : \Omega_1 \times \Omega_2 \to X$ is $\mu_1 \otimes \mu_2$-measurable, then the functions*

$$f_1 : \Omega_1 \to X, \quad f_1(s) = \int_{\Omega_2} f(s,t) \,\mathrm{d}\mu_2(t),$$
$$f_2 : \Omega_2 \to X, \quad f_2(s) = \int_{\Omega_1} f(s,t) \,\mathrm{d}\mu_1(t)$$

are defined almost everywhere in Ω_1 and Ω_2, respectively and

$$\int_{\Omega_1\times\Omega_2} f(s,t) \,\mathrm{d}\big(\mu_1(s) \otimes \mu_2(t)\big) = \int_{\Omega_1} f_1(s) \,\mathrm{d}\mu_1(s) = \int_{\Omega_2} f_2(t) \,\mathrm{d}\mu_2(t).$$

Proof. See [HP74, Theorem 3.7.13] □

Chapter 3

Selfadjoint extensions

3.1 Selfadjoint extensions of symmetric operators

Example 3.1. Let $H = L_2(0,1)$. We define T by

$$\mathcal{D}(T) := \{f \in L_2(0,1) : f \text{ abs. cont}, \ f' \in L_2(0,1), \ f(0) = f(1) = 0\},$$
$$Tf := \mathrm{i}f'.$$

T is a closed symmetric operator and $\mathcal{D}(T^*) = H^1(0,1)$ with $T^*f = \mathrm{i}f'$, hence $T \subsetneq T^*$.

Does T admit selfadjoint extensions? If so, how many and can we find formulas for them?

In general, if T is a symmetric operator and S is a selfadjoint extension, then we must have

$$T \subseteq S = S^* \subseteq T^*.$$

Let H be a complex Hilbert space, $T(H \to H)$ a symmetric operator and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By Remark 2.38 $T - \lambda$ is injective and $\mathrm{rg}(T-\lambda)$ is closed if and only if T is closed.

Recall that the Cayley transform gives a bijection

$$\text{symetric operators} \xleftarrow[\textit{transformation}]{\textit{Cayley}} \begin{cases} \text{isometric operators } U \\ \text{with } \overline{\mathrm{rg}(U-1)} = H \end{cases}$$

The relation between a symmetric operator T and its Cayley transform U is

$$U = (T - \mathrm{i})(T+\mathrm{i})^{-1}, \qquad \mathcal{D}(U_T) = \mathrm{rg}(T+\mathrm{i}), \quad \mathrm{rg}(U_T) = \mathrm{rg}(T-\mathrm{i}),$$
$$T = -\mathrm{i}(U+1)(U-1)^{-1} \qquad \mathcal{D}(T) = \mathrm{rg}(U-1), \quad \mathrm{rg}(U_T) = \mathrm{rg}(U+1).$$

In particular, T is closed if and only if both $\mathrm{rg}(U+1)$ and $\mathrm{rg}(U-1)$ are closed (in fact, $\mathrm{rg}(U+1)$ is closed $\iff$ T is closed $\iff$ $\mathrm{rg}(U-1)$ is closed), and

$$\begin{aligned} T \text{ closed} \quad &\iff \quad U \text{ closed}, \\ S \text{ symmetric extension of } T \quad &\iff \quad U_S \text{ isometric extension of } U. \\ T \text{ selfadjoint} \quad &\iff \quad U \text{ unitary}. \end{aligned}$$

So instead of looking for symmetric or selfadjoint extensions of a closed symmetric opertor T we try to find isometric or unitary extensions of its Cayley transform. The advantage is that the domain and the range of the Cayley transform are closed subspaces.

Theorem 3.2 (1st von Neumann formula). *Let H be a complex Hilbert space and S a closed symmetric linear operator on H with Cayley transform U. We define the subspaces*

$$N_+ := \mathrm{rg}(S+\mathrm{i})^\perp = \ker(S^* - \mathrm{i}) = \mathcal{D}(U)^\perp,$$
$$N_- := \mathrm{rg}(S-\mathrm{i})^\perp = \ker(S^* + \mathrm{i}) = \mathrm{rg}(U)^\perp.$$

Then $\mathcal{D}(S^*) = \mathcal{D}(S) \dot{+} N_+ \dot{+} N_-$, where $\dot{+}$ denotes the direct sum.

Proof. Let $x \in \mathcal{D}(S^*)$. We have to show that there exist unique elements $x_0 \in \mathcal{D}(S)$ and $x_\pm \in N_\pm$ such that $x = x_0 + x_+ x_-$.

Existence: Choose $x_0 \in \mathcal{D}(S)$ and $y \in \mathrm{rg}(S+\mathrm{i})^\perp = \ker(S^*+\mathrm{i}) \subseteq \mathcal{D}(S^*)$ such that

$$\begin{aligned}(S^* + \mathrm{i})x = (S+\mathrm{i})x_0 + y &= (S+\mathrm{i})x_0 + \frac{1}{2\mathrm{i}}(S^*+\mathrm{i})y + \frac{1}{2\mathrm{i}}(S^* - \mathrm{i})y \\ &= (S+\mathrm{i})x_0 + \frac{1}{2\mathrm{i}}(S^*+\mathrm{i})y.\end{aligned}$$

Define $x_+ = \frac{1}{2\mathrm{i}}y \in \mathrm{rg}(S+\mathrm{i})^\perp$ and $x_- = x - x_0 - \frac{1}{2\mathrm{i}}y \in \ker(S^*+\mathrm{i}) = \mathrm{rg}(S-\mathrm{i})^\perp$.

Uniqueness: It suffices to show that $x_0 - x_+ - x_- = 0$ with $x_0 \in \mathcal{D}(S)$ and $x_\pm \in N_\pm$ only if $x_0 = x_\pm = 0$. By assumption

$$0 = S^*(x_0 - x_+ - x_-) = Sx_0 - S^*x_+ - S^*x_- = Sx_0 - \mathrm{i}x_+ + \mathrm{i}x_-.$$

Hence

$$\left.\begin{aligned} x_0 &= x_- + x_+ \\ Sx_0 &= \mathrm{i}x_+ - \mathrm{i}x_- \end{aligned}\right\} \quad \Longrightarrow \quad (S+\mathrm{i})x_0 = \mathrm{i}x_+ - \mathrm{i}x_- + \mathrm{i}x_- + \mathrm{i}x_+ = 2\mathrm{i}x_+.$$

Hence $x_+ \in \mathrm{rg}(S+\mathrm{i}) \cap \mathrm{rg}(S+\mathrm{i})^\perp = \{0\}$. Similarly it follows that $x_- = 0$. Then also $x_0 = x_+ + x_- = 0$. ☐

Definition 3.3. Let S be a linear operator on a Banach space X. For $z \in \mathbb{C}$ we define the *deficiency number* $n(S,z) := \dim\big(\mathrm{rg}(S-z)^\perp\big)$. For symmetric operators S we set

$$n_+(S) := n(S, -\mathrm{i}) = \dim \mathrm{rg}(S+\mathrm{i})^\perp, \quad n_-(S) := n(S, \mathrm{i}) = \dim \mathrm{rg}(S-\mathrm{i})^\perp.$$

Corollary 3.4. *Let S be symmetric. Then*

$$\begin{aligned} S \text{ is essentially selfadjoint} \quad &\iff \quad n_+(S) = n_=(S) = 0, \\ S \text{ is selfadjoint} \quad &\iff \quad n_+(S) = n_=(S) = 0 \text{ and } S \text{ is closed.} \end{aligned}$$

Definition 3.5. Let S be a symmetric linear operator on a Banach space X and T a symmetric extension of S. T is called an

$$\begin{aligned} m\text{-dimensional extension of } S \quad &\iff \quad \dim\big(\mathcal{D}(T)/\mathcal{D}(S)\big) = m, \\ m\text{-dimensional restriction of } S^* \quad &\iff \quad \dim\big(\mathcal{D}(S^*)/\mathcal{D}(T)\big) = m, \end{aligned}$$

where $\dim(U/V) = m$ for subspaces $U, V \subseteq H$ if and only if there exists an m-dimensional subspace D such that $U = V \dot{+} D$.

Theorem 3.6 (2nd von Neumann formula). *Let H be a complex Hilbert space and S a closed symmetric linear operator on H with Cayley transform U.*

(i) *S has symmetric extensions if $n_+(S) > 0$ and $n_-(S) > 0$.*

(ii) *Every m-dimensional symmetric extension T of S is of the form*

$$\begin{aligned} \mathcal{D}(T) &= \mathcal{D}(S) \dot{+} \{y + \widetilde{V}y : y \in \widetilde{N}_+\}, \\ T(x + y + \widetilde{V}y) &= Sx + \mathrm{i}y - \mathrm{i}\widetilde{V}y \quad \text{for } x \in \mathcal{D}(S),\ y \in \widetilde{N}_+, \end{aligned} \tag{3.1}$$

where $\widetilde{N}_+$ is an m-dimensional subspace of N_+ and $\widetilde{V} : \widetilde{N}_+ \to N_-$ is an isometry. If $m < \infty$, then T has deficiency indices $n_\pm(T) = n_\pm(S) - m$.

Every operator of the form (3.1) is a selfadjoint extension of S.

(iii) *If $n_\pm(S) < \infty$, then S has selfadjoint extensions if and only if $n_+(S) = n_-(S)$.*

Proof. Let U_S be the Cayley transform of S and T a symmetric operator with Cayley transform U_T. Observe that

$$\begin{aligned} T \text{ symmetric extension of } S \quad &\iff \quad U_T \text{ isometric extension of } U_S, \\ T \text{ selfadjoint extension of } S \quad &\iff \quad U_T \text{ unitary extension of } U_S. \end{aligned}$$

Existence of a symmetric extension as in (3.1): Suppose that $n_+(S) > 0$ and $n_-(S) > 0$ and chose $p \in \mathbb{N}$ with $m \leq \min\{n_+(S), n_-(S)\}$. By assumption, we can choose p-dimensional subspaces $\widetilde{N}_+$ of $N_+ = \mathrm{rg}(S + \mathrm{i})^\perp = \mathcal{D}(U_S)^\perp$ and $\widetilde{N}_-$ of $N_- = \mathrm{rg}(S - \mathrm{i})^\perp = \mathrm{rg}(U_S)^\perp$ and a unitary operator $\widetilde{V} : \widetilde{N}_+ \to N_-$. Now we define an extension U_T of U_S by

$$\begin{aligned} &U_T : \mathcal{D}(U_S) \oplus \widetilde{N}_+ \to \mathrm{rg}(U_S) \oplus \widetilde{N}_-, \\ &U_T(x + y) = U_S(x) - \widetilde{V}y \qquad \text{for} \quad x \in \mathcal{D}(U_S) \quad \text{and} \quad y \in \widetilde{N}_+. \end{aligned}$$

Then U_T is closed isometry and $\overline{\mathrm{rg}(U_T - \mathrm{id})} \subseteq \overline{\mathrm{rg}(U_S - \mathrm{id})} = H$, so, by Lemma 2.41 and Proposition 2.40, it is the Cayley transform of the closed symmetric operator $T = \mathrm{i}(\mathrm{id} + U_T)(1 - U_T)^{-1}$. Its domain is given by

$$\begin{aligned} \mathcal{D}(T) &= \mathrm{rg}(\mathrm{id} - U_T) = (\mathrm{id} - U_T)\Big(\mathrm{rg}(S + \mathrm{i}) \oplus \widetilde{N}_+\Big) \\ &= (\mathrm{id} - U_T)\big(\mathrm{rg}(S + \mathrm{i})\big) \dot{+} (\mathrm{id} - U_T)\widetilde{N}_+ = \mathcal{D}(S) \dot{+} (\mathrm{id} + \widetilde{V})\widetilde{N}_+ \\ &= \mathcal{D}(S) \dot{+} \{y + \widetilde{V}y : y \in \widetilde{N}_+\}. \end{aligned}$$

Observe that, as a consequence of Theorem 3.2, this is a direct sum because the second space is a subspace of $N_+ \dot{+} N_-$. Moreover, $\dim\{y + \widetilde{V}y : y \in \widetilde{N}_+\} = \dim \widetilde{N}_+ = p$, hence $\mathcal{D}(T)/\mathcal{D}(S) = p$ and, if $p < \infty$,

$$\begin{aligned} n_+(S) &= \dim\big(\mathrm{rg}(S + \mathrm{i})^\perp\big) = \dim\big(\mathcal{D}(U_S)^\perp\big) = \dim\big(\mathcal{D}(U_T)^\perp \oplus \widetilde{N}_+\big) \\ &= n_+(T) + p, \\ n_-(S) &= \dim\big(\mathrm{rg}(S - \mathrm{i})^\perp\big) = \dim\big(\mathrm{rg}(U_S)^\perp\big) = \dim\big(\mathrm{rg}(U_T)^\perp \oplus \widetilde{N}_-\big) \\ &= n_-(T) - p, \end{aligned}$$

and for $x \in \mathcal{D}(S)$ and $y \in \widetilde{N}_+$

$$\begin{aligned} T(x_0 + y + \widetilde{V}y) &= S(x_0) + T(y + \widetilde{V}y) \\ &= S(x_0) + \mathrm{i}(U_T + \mathrm{id})(\mathrm{id} - U_T)^{-1} \overbrace{(\mathrm{id} + \widetilde{V})y}^{=(\mathrm{id} - U_T)y} \\ &= S(x_0) + \mathrm{i}(U_T + \mathrm{id})y = S(x_0) + \mathrm{i}y - \mathrm{i}\widetilde{V}y. \end{aligned}$$

Since the isometry $\widetilde{V}$ was arbitrary, we proved that every operator of the form (3.1) is a selfadjoint extension of S and that (i) is true.

Now assume that T is a symmetric extension of S. Then U_T is a isometric extension of U_S. Note that $\mathcal{D}(U_S)$, $\mathcal{D}(U_T)$, $\mathrm{rg}(U_S)$ and $\mathcal{D}(U_T)$ are closed, hence there exist closed subspaces $\widetilde{N}_\pm$ such that $\mathcal{D}(U_T) = \mathcal{D}(U_S) \oplus \widetilde{N}_+$, $\mathrm{rg}(U_T) = \mathrm{rg}(U_S) \oplus \widetilde{N}_-$. Therefore, setting $\widetilde{V} = -U_T|_{\widetilde{N}_+}$, we obain

$$\begin{aligned} &U_T : \mathcal{D}(U_S) \oplus \widetilde{N}_+ \to \mathrm{rg}(U_S) \oplus \widetilde{N}_- \\ &U_T(x + y) = U_S x - \widetilde{V}y \qquad \text{for } x \in \mathcal{D}(S) \text{ and } y \in \widetilde{N}_+. \end{aligned}$$

As before, it follows that T is of the form (3.1).

If T is a selfadjoint extension of S, then U_T is a unitary extension of U_S. Hence U_T maps $\mathcal{D}(U_T) \ominus \mathcal{D}(U_S) = \mathcal{D}(U_S)^\perp$ unitarily to $\mathrm{rg}(U_T) \ominus \mathrm{rg}(U_S) = \mathrm{rg}(U_S)^\perp$. Consequently, $n_+(S) = \dim \mathcal{D}(U_S)^\perp = \dim \mathrm{rg}(U_S)^\perp = n_-(S)$. □

Recall that $n_\pm(T) = 0$ if T is selfadjoint.

Theorem 3.7. *Let H be a complex Hilbert space and S a closed symmetric linear operator on H with $n_+(S) = n_-(S) = m < \infty$. Let T a linear operator on H. Then*

DRAFT

(i) *T is a selfadjoint extension of S* $\iff$ *T is an m-dimensional symmetric extension of S.*

(ii) *T is a selfadjoint restriction of S^** $\iff$ *T is an m-dimensional symmetric restriction of S^*.*

Proof. (i) If T is a selfadjoint extension of S. Then $0 = n_\pm(T) = n_\pm(S) - \dim(\mathcal{D}(T)/\mathcal{D}(S))$, hence $m = n_\pm(S) = \dim(\mathcal{D}(T)/\mathcal{D}(S))$.
On the other hand, if T is an m-dimensional extension of S, then its deficiency indices are zero, that is, $\dim(\operatorname{rg}(T \pm \mathrm{i})^\perp) = 0$. Hence T is essentially selfadjoint. Since a finite dimensional extension of a closed operator is closed, it follows that T is selfadjoint.
(ii) Observe that $\dim S/S^* = 2m$. By assumption $S \subseteq T \subseteq S^*$, so T is an m-dimensional restriction of S^* if and only if it is an m-dimensional extension of S. □

Definition 3.8. A symmetric operator T is a *maximal symmetric operator* if it has no proper symmetric extensions. Clearly, a closed symmetric operator is maximal symmetric if and only if at least one of its deficiency numbers is zero.

It can be shown that for every closed symmetric operator $S(H \to H)$ with deficieny indices $n_+(S) = m$ and $n_-(S) = n$ there exists a Hilbert space $\widetilde{H} \supseteq H$ and a closed symmetric linear operator $\widetilde{S}(\widetilde{H} \to \widetilde{H})$ with deficiency indices $n_+(\widetilde{S}) = n_-(\widetilde{S}) = n_+(S) + n_+(S)$ such that $S = P\widetilde{S}P$ where $P \in L(\widetilde{H})$ is the orthogonal projection on H.
Since $\widetilde{S}$ has equal deficiency indices, it has a selfadjoint extension T. Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral resolution of T. Then

$$Sx = \int_{-\infty}^{\infty} \lambda \,\mathrm{d}E_\lambda x, \qquad x \in \mathcal{D}(S).$$

$(E_\lambda)_{\lambda\in\mathbb{R}}$ is called a *generalized spectral resolution of* S. If S is maximal symmetric, then $x \in \mathcal{D}(S)$ if and only if $\int_{-\infty}^{\infty} \lambda^2 \,\mathrm{d}\|E_\lambda x\|^2 < \infty$. For details, see [Yos95, XI.15] and [AG93].

Let us review the example at the beginning of the section.

Example. Let $H = L_2(0,1)$. We define T by

$$\mathcal{D}(T) := \{f \in L_2(0,1) : f \text{ abs. cont}, f' \in L_2(0,1),\ f(0) = f(1) = 0\},$$
$$Tf := \mathrm{i}f'.$$

T is a closed symmetric operator and $\mathcal{D}(T^*) = H^1(0,1)$ with $T^*f = \mathrm{i}f'$.
In order to determine if T has selfadjoint extensions, it suffices to calculate $n_\pm(T) = \dim\ker(T^* \pm \mathrm{i})$. It is easy to see that

$$\ker(T^* + \mathrm{i}) = \operatorname{span}\{\varphi_+\}, \quad \ker(T^* - \mathrm{i}) = \operatorname{span}\{\varphi_-\},$$

where $\varphi_\pm(t) = \mathrm{e}^{\pm t}$. Hence $n_+(T) = n_-(T) = 1$, so T admits selfadjoint extensions.
To find all selfadjoint extensions of T, we have to find all unitary maps $\ker(T^* + \mathrm{i}) \to \ker(T^* - \mathrm{i})$. Obviously they are given by

$$U_\vartheta : \ker(T^* + \mathrm{i}) \to \ker(T^* - \mathrm{i}), \quad U_\vartheta\varphi_+ = \Big(\frac{\mathrm{e}^2 - 1}{1 - \mathrm{e}^{-2}}\Big)\, \mathrm{e}^{\mathrm{i}\vartheta}\, \varphi_-$$

for arbitrary $\vartheta \in \mathbb{R}$. Therefore every selfadjoint extension of T is of the form

$$\mathcal{D}(\widetilde{T}_\vartheta) = \mathcal{D}(T) + \operatorname{span}\Big\{\varphi_+ + \Big(\frac{\mathrm{e}^2 - 1}{1 - \mathrm{e}^{-2}}\Big)^{\frac12} \mathrm{e}^{\mathrm{i}\vartheta}\, \varphi_-\Big\},$$

$$\widetilde{T}_\vartheta\Big(f_0 + \alpha\Big(\varphi_+ + \Big(\frac{\mathrm{e}^2 - 1}{1 - \mathrm{e}^{-2}}\Big)^{\frac12} \mathrm{e}^{\mathrm{i}\vartheta}\, \varphi_-\Big)\Big) = f_0' + \mathrm{i}\alpha\varphi_+ - \mathrm{i}\alpha\Big(\frac{\mathrm{e}^2 - 1}{1 - \mathrm{e}^{-2}}\Big)^{\frac12} \mathrm{e}^{\mathrm{i}\vartheta}\, \varphi_-.$$

An example of a symmetric operator T with $n_+(T) \neq n_-(T)$ is given in Exercise 3.1.

3.2 Deficiency indices and points of regular type

Recall that for a closed linear operator S and $z \in \mathbb{C}$ we defined the deficiency indices $n(S, z) := \dim\operatorname{rg}(S - z)^\perp$.

Definition 3.9. Let H be a Hilbert space and S a linear operator on H. A point $z \in \mathbb{C}$ is called a *point of regular type of* S if

$$\exists\, c_z > 0 \ \text{ such that } \ \|(z - S)x\| \geq c_z\|x\| \ \text{ for all } \ x \in \mathcal{D}(S).$$

The set

$$\Gamma(S) := \{z \text{ is of regular type of } S\}$$

is the *regularity domain on* S.

In the case when S is closed, the following is easy to see:

- $\rho(S) \subseteq \Gamma(S)$,
- $z \in \Gamma(S) \iff S - z$ is injective and $\operatorname{rg}(z - S)$ is closed.

Proposition 3.10. *Let S be a linear operator on a Hilbert space H. Then*

(i) *$\Gamma(S)$ is open.*

(ii) *S is symmetric* $\implies$ $\mathbb{C} \setminus \mathbb{R} \subseteq \Gamma(S)$.

(iii) *S is isometric* $\implies$ $\mathbb{C} \setminus \{|z| = 1\} \subseteq \Gamma(S)$.

Proof. (i) Fix $z_0 \in \Gamma(S)$. Then also the open ball with radius c_{z_0} centred in z_0 lies in $\Gamma(S)$ because for $z \in \mathbb{C}$ with $|z - z_0| < c_{z_0}$

$$\|(S-z)x\| \geq \|(S-z_0)x\| - |z-z_0|\,\|x\| \geq (\underbrace{c_{z_0} - |z-z_0|}_{>0})\|x\|.$$

(ii) For every $z \in \mathbb{C} \setminus \mathbb{R}$ the map $(z-S)^{-1} : \operatorname{rg}(z-S) \to \mathcal{D}(S)$ exists and is bounded by $|\operatorname{Im} z|^{-1}$. By the closed graph theorem $\operatorname{rg}(z-S)$ is closed, so $z \in \Gamma(S)$.

(iii) Let $z \in \mathbb{C}$ with $|z| \neq 1$. Then, for all $x \in \mathcal{D}(S)$,

$$\|(S-z)x\| \geq |\,\|Sx\| - |z|\,\|x\|\,| = \underbrace{|1-|z||}_{>0}\,\|x\|. \qquad \square$$

Theorem 3.11. *Let S be a closable linear operator on a complex Hilbert space H. The following holds.*

(i) *The deficiency numbers $n(S,z)$ are locally constant in $\Gamma(S)$. In particular they are constant in connected components of $\Gamma(S)$.*

(ii) *If S is symmetric, then $n(S,z)$ is constant in the upper and in the lower half plane (but in general $n(S,\mathrm{i}) \neq n(S,-\mathrm{i})$).*

(iii) *If S is isometric, then $n(S,z)$ is constant inside and outside of the unit circle (but in general $n(S,0) \neq n(S,2)$).*

Proof. (ii) and (iii) follow immediately form (i) and Proposition 3.10. So we only have to show (i).

Case 1. S is closed. Let $z_0 \in \Gamma(S)$. Since S is closed, $\operatorname{rg}(z_0 - S)$ is closed. We will show that $n(S,z_0) = n(S,z)$ for all z with $|z - z_0| < \frac{c_{z_0}}{2}$.
Recall that for closed subspaces U, V of H with $U \cap V^\perp = \{0\}$, $\dim U \leq \dim V$. In particular, if $V \cap U^\perp + U \cap V^\perp = \{0\}$, then $\dim U = \dim V$. We will apply this to $U = \operatorname{rg}(z_0 - S)^\perp$ and $V = \operatorname{rg}(z-S)^\perp$. So we only have to show

$$\operatorname{rg}(z_0-S)^\perp \cap \operatorname{rg}(z-S) = \operatorname{rg}(z-S)^\perp \cap \operatorname{rg}(z_0-S) = \{0\}$$

for all z with $|z-z_0| < \frac{c_{z_0}}{2}$. Recall that by the proof of Proposition 3.10, $z \in \Gamma(S)$ and $c_z \geq |\,c_{z_0} - |z-z_0|\,| = \frac{c_{z_0}}{2}$.
Assume $\operatorname{rg}(z-S)^\perp \cap \operatorname{rg}(z_0-S) \neq \{0\}$. Then there exists an $x \in \mathcal{D}(S) \setminus \{0\}$ such that $(z_0 - S)x \perp \operatorname{rg}(z-S)$. Using that $(z-S)x \perp (z_0-S)x$ we obtain the contradiction

$$\begin{aligned}\|(z-S)x\| &\leq \Big(\|(S-z)x\|^2 + \|S-z_0)x\|^2\Big)^{\frac{1}{2}} = \|(S-z)x\| + \|(S-z_0)x\| \\ &= \|(z_0-z)x\| \leq \frac{|z_0-z|}{c_z}\|(S-z)x\| < \frac{c_{z_0}}{2}\frac{1}{c_{z_0}}\|(S-z)x\| = \|(S-z)x\|.\end{aligned}$$

Now assume $\operatorname{rg}(z_0-S)^\perp \cap \operatorname{rg}(z-S) \neq \{0\}$. Then there exists an $x \in \mathcal{D}(S) \setminus \{0\}$ such that $(z-S)x \perp \operatorname{rg}(z_0-S)$. Using that $(z-S)x \perp (z_0-S)x$ we obtain the contradiction

$$\begin{aligned}\|(z_0-S)x\| &\leq \Big(\|(S-z)x\|^2 + \|S-z_0)x\|^2\Big)^{\frac{1}{2}} = \|(S-z)x - (S-z_0)x\| \\ &= \|(z_0-z)x\| \leq \frac{|z_0-z|}{c_{z_0}}\|(S-z)x\| \\ &< \frac{c_{z_0}}{2}\frac{1}{c_z}\|(S-z)x\| = \frac{1}{2}\|(S-z)x\|.\end{aligned}$$

Case 2. S is closable. Let $\overline{S}$ be the closure of S. By case 1, it suffices to show that $\Gamma(S) = \Gamma(\overline{S})$ and that $n(S,z) = n(\overline{S},z)$ for all $z \in \Gamma(S)$. The inclusion $\Gamma(\overline{S}) \subseteq \Gamma(S)$ is obvious. If $z \in \Gamma(S)$ and $x \in \mathcal{D}(S)$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $Sx_n \to \overline{S}x$. Hence

$$\|(z-\overline{S})x\| = \lim_{n\to\infty}\|(z-S)x_n\| \geq c_z \lim_{n\to\infty}\|x_n\| = c_z\|x\|,$$

showing that $\Gamma(S) = \Gamma(\overline{S})$. Moreover, $\operatorname{rg}(z-S)^\perp = \operatorname{rg}(z-\overline{S})^\perp$ because $\operatorname{rg}(z-S)$ is dense in $\operatorname{rg}(z-S)$. Hence $n(S,z) = n(\overline{S},z)$ for all $z \in \Gamma(S) = \Gamma(\overline{S})$. $\square$

Corollary 3.12. *Let S be a symmetric operator on a complex Hilbert space H. The following holds.*

(i) *S is essentially selfadjoint* $\iff$ $n_+(S) = n_-(S) = 0$.

(ii) *S is selfadjoint* $\iff$ *S is closed and $n_+(S) = n_-(S) = 0$.*

Proof. This follows immediately from the fact that a symmetric operator S is essentially selfadjoint if and only if $\operatorname{rg}(S \pm \mathrm{i})$ is dense in H. $\square$

Corollary 3.13. *For a S closed symmetric operator on a complex Hilbert space H the following holds.*

(i) *S has real points of regular type* $\implies$ *S has a selfadjoint extension.*

(ii) *S is semibounded* $\implies$ *S has a selfadjoint extension.*

Proof. (i) By assumption and Proposition 3.10 (i), $\Gamma(S)$ is connected, hence $n(S,\mathrm{i}) = n(S,-\mathrm{i})$ by Theorem 3.11. Therefore S has selfadjoint extensions by Theorem 3.6.

(ii) Without restriction we assume that S is semibounded from below. Then there exists a $\gamma \in \mathbb{R}$ such that $\langle Sx\,, x\rangle \geq \gamma$ for all $x \in \mathcal{D}(S)$. For all $\lambda < \gamma$ we obtain

$$\|(\lambda - S)x\|\,\|x\| \geq \langle(\lambda-S)x\,, x\rangle \geq (\lambda-\gamma)\|x\|^2.$$

Hence $(\infty,\gamma) \subseteq \Gamma(S)$ and the assertion follows from (i). $\square$

Theorem 3.14. *Let S be a symmetric operator on a complex Hilbert space H with defect indices $n_+(S) = n_-(S) = m < \infty$. Let T_1 and T_2 be selfadjoint extensions of S with spectral resolutions E_1 and E_2. Let $I \subseteq \mathbb{R}$ be an open or closed interval and $k_j := \dim \operatorname{rg}(E_j(I))$ for $j = 1, 2$.*

If $k_1 < \infty$, then $k_2 < \infty$ and $|k_1 - k_2| < \infty$.

Proof. Assume that $I = (\alpha, \beta)$ with $-\infty < \beta < \alpha < \infty$. Note that $\operatorname{rg}(E_j(I)) \subseteq \mathcal{D}(T_j)$ for $j = 1, 2$ and that $\dim\big(\mathcal{D}(T)/\mathcal{D}(\overline{S})\big) = m$. Let us assume that $k_2 > k_1 + m$. Then

$$\dim\big(\operatorname{rg}(E_2(I)) \cap \mathcal{D}(\overline{S})\big) \geq k_2 - m > k_1 \tag{3.2}$$

and for every $x \in \big(\operatorname{rg}(E_2(I)) \cap \mathcal{D}(\overline{S})\big) \setminus \{0\}$

$$\begin{aligned}\Big\|\Big(T_1 - \frac{\alpha+\beta}{2}\Big)x\Big\|^2 &= \Big\|\Big(T_2 - \frac{\alpha+\beta}{2}\Big)x\Big\|^2 \\ &= \int_{(\alpha,\beta)} \Big(t - \frac{\alpha+\beta}{2}\Big)^2 \,\mathrm{d}\langle E(t)x\,, x\rangle < \Big(\frac{\beta-\alpha}{2}\Big)^2 \|x\|^2.\end{aligned}$$

So, for all $x \in \big(\operatorname{rg}(E_2(I)) \cap \mathcal{D}(\overline{S})\big) \setminus \{0\}$

$$\Big\|\Big(T_1 - \frac{\alpha+\beta}{2}\Big)x\Big\| < \frac{\beta-\alpha}{2}\|x\|. \tag{3.3}$$

By (3.2) there exists an $x \in \operatorname{rg}(E_2(I)) \cap \mathcal{D}(\overline{S})$ with $\|x\| - 1$ and $x \perp \operatorname{rg}(E_1(I)$. For this x

$$\Big\|\Big(T_1 - \frac{\alpha+\beta}{2}\Big)x\Big\|^2 = \int_{\mathbb{R}\setminus(\alpha,\beta)} \Big(t - \frac{\alpha+\beta}{2}\Big)^2 \,\mathrm{d}\langle E(t)x\,, x\rangle \geq \frac{\beta-\alpha}{2}\|x\| \tag{3.4}$$

in contradiction to (3.3). This proves $k_2 \leq k_1 + m < \infty$. Applying the same to reasoning to k_2, we find $k_1 \leq k_2 + m$, so $|k_1 - k_2| \leq m$.

If I is a closed interval of the form $[\alpha, \beta]$ with $-\infty < \beta \leq \alpha < \infty$, then we have to change "<" to "≤" in (3.3) and "≥" to ">" in (3.4). □

Corollary 3.15. *With the assumptions and notation as in Theorem 3.18, it follows that if $\sigma(T_1) \cap (a, b)$ consists only of discrete eigenvalues with total multiplicity k_1, then $\sigma(T_2) \cap (a, b)$ consists only of discrete eigenvalues with total multiplicity $k_2 \leq k_1 + m$.*

Theorem 3.16. *Let S be a symmetric operator on a complex Hilbert space H with defect indices $n_+(S) = n_-(S) = m < \infty$. Let $\lambda \in \mathbb{C}$ and assume that there exists a $c > 0$ such that*

$$\|(S - \lambda)x\| \geq c\|x\|, \qquad x \in \mathcal{D}(S).$$

Then for every selfadjoint extension T of S the set $\sigma(T) \cap (\lambda - c,\ \lambda + c)$ is empty or consists of isolated eigenvalues with total multiplicity $\leq m$.

Proof. Let E be the spectral resolution of T. We have to show that

$$\dim\big(E(\lambda - c,\ \lambda + c)\big) = \dim\big(E(\lambda - c - 0,\ \lambda + c)\big) \leq m.$$

If this was not true, then there exists an $x_0 \in \operatorname{rg}\big(E(\lambda - c - 0,\ \lambda + c)\big) \cap \mathcal{D}(\overline{S})$ with $x_0 \neq 0$ because $\dim\big(\mathcal{D}(T)/\mathcal{D}(\overline{S})\big) = m$ and $\operatorname{rg}\big(E(\lambda - c - 0,\ \lambda + c)\big) \subseteq \mathcal{D}(T)$, leading to the contradiction

$$\begin{aligned}c\|x_0\| &\leq \|(\overline{S} - \lambda_0)x_0\| = \|(T - \lambda_0)x_0\| \\ &= \Big(\int_{|t-\lambda|<c} |t - \lambda_0|^2 \,\mathrm{d}\langle E_t x_0\,, x_0\rangle\Big)^{\frac{1}{2}} < c\|x_0\|.\end{aligned}$$

□

Remark. If $\lambda \in \Gamma(S) \cap \mathbb{R}$, then Exercise 3.3 shows that S has a selfadjoint extension T with $\lambda \in \sigma_{\mathrm{p}}(T)$ and $\dim\ker(T - \lambda) \leq m$.

Corollary 3.17. *Let S be a semibounded symmetric operator on a complex Hilbert space H with defect indices $n_+(S) = n_-(S) = m < \infty$. Without restriction we assume that $S \geq \gamma$ for some $\gamma \in \mathbb{R}$. Let T be a selfadjoint extension T of S. Then*

(i) *$\sigma(T) \cap (\infty, \gamma)$ consists of isolated eigenvalues of total multiplicity $\leq m$.*

(ii) *T is semibounded from below.*

Proof. (i): Let $\lambda < \gamma$ and $c := \gamma - \lambda$. For all $x \in \mathcal{D}(S)$ we obtain

$$\begin{aligned}\|(S - \lambda)x\| &\geq \langle (S - \lambda)x\,, x\rangle = \langle (S - \gamma)x\,, x\rangle + (\gamma - \lambda)\|x\|^2 \\ &> (\gamma - \lambda)\|x\|^2 = c\|x\|^2.\end{aligned}$$

Hence, by Theorem 3.16, the set $\sigma(T) \cap (\lambda - c,\ \lambda + c) = \sigma(T) \cap (2\lambda - \gamma,\ \gamma)$ consists only of isolated eigenvalues of total multiplicity $\leq m$. Since this is true for all $\lambda < \gamma$, the claim is proved.

(ii) is an immediate consequence of (i). □

Theorem 3.18. *Let S be a symmetric operator on a complex Hilbert space H with defect indices $n_+(S) = n_-(S) = m < \infty$. Let T_1 and T_2 be selfadjoint extensions of S with spectral resolutions E_1 and E_2.*

(i) *For every $z \in \rho(T_1) \cap \rho(T_2)$ the range of the operator $(T_1 - z)^{-1} - (T_2 - z)^{-1}$ is at most m-dimensional.*

(ii) *$\sigma_{ess}(T_1) = \sigma_{ess}(T_2)$.*

Let $I \subseteq \mathbb{R}$ be an open or closed interval and $k_j := \dim \operatorname{rg}(E_j(I))$ for $j = 1, 2$.

If $k_1 < \infty$, then $k_2 < \infty$ and $|k_1 - k_2| < \infty$.

Proof. (i) Observe that every $z \in \rho(T_j)$ belongs to $\Gamma(S)$, hence $\dim(\operatorname{rg}(S-z))^\perp = m < \infty$. Let P be the orthogonal projection on $(\operatorname{rg}(S-z))^\perp$. Then for all $x \in H$ and all $z \in \rho(T_1) \cap \rho(T_2)$

$$\begin{aligned}
&(T_1-z)^{-1}x - (T_2-z)^{-1}x \\
&\qquad = \Big((T_1-z)^{-1} - (T_2-z)^{-1}\Big)(1-P)x + \Big((T_1-z)^{-1} - (T_2-z)^{-1}Px \\
&\qquad = \Big((\overline{S}-z)^{-1} - (\overline{S}-z)^{-1}\Big)(1-P)x + \Big((T_1-z)^{-1} - (T_2-z)^{-1}\Big)Px \\
&\qquad = \Big((T_1-z)^{-1} - (T_2-z)^{-1}\Big)Px.
\end{aligned}$$

So we showed that $(T_1-z)^{-1} - (T_2-z)^{-1} = \big((T_1-z)^{-1} - (T_2-z)^{-1}\big)P$ which implies that the dimension of its range is less or equal than $\dim \operatorname{rg} P$.
(ii) Let $\lambda \in \mathbb{C} \setminus \sigma_{\text{ess}}(T_1)$. Then there exists $\varepsilon > 0$ such that $\dim \operatorname{rg}(E_1(\lambda-\varepsilon, \lambda+\varepsilon)) < \infty$. Using theorem 3.14 it follows that $\dim \operatorname{rg}(E_2(\lambda-\varepsilon, \lambda+\varepsilon)) < \infty$, implying that $\lambda \notin \sigma_{\text{ess}}(T_2)$.

Alternative proof: Since T_1 and T_2 are selfadjoint, $\mathrm{i} \in \rho(T_1) \cap \rho(T_2)$. The operator $(T_1-\mathrm{i})^{-1} - (T_2-\mathrm{i})^{-1}$ is bounded and is compact because its range is finite-dimensional by (i). Hence, by Exercise **??**, $\sigma_{\text{ess}}(T_1) = \sigma_{\text{ess}}(T_2)$. □

DRAFT

Chapter 4

Perturbation Theory

4.1 Closed operators

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Definition 4.1. Let X, Y, Z be normed spaces and $T(X \to Y)$, $S(X \to Z)$ be linear operators. The operator S is called *T-bounded* (or *relatively bounded with respect to T*) if and only if $\mathcal{D}(S) \supseteq \mathcal{D}(T)$ and there exist $a, b \geq 0$ such that

$$\|Sx\| \leq a\|x\| + b\|Tx\| \qquad \text{for all } x \in \mathcal{D}(T). \tag{4.1}$$

The infimum of all $b \geq 0$ such that (4.1) holds for some $a \geq 0$, is called the *T-bound* of S.

For example, if S is bounded, it is T-bound with relative bound 0.

Remark 4.2. Note that (4.1) is equivalent to the existence of $\alpha, \beta \geq 0$ such that

$$\|Sx\|^2 \leq \alpha^2\|x\|^2 + \beta^2\|Tx^2\| \qquad \text{for all } x \in \mathcal{D}(T). \tag{4.2}$$

The infimum of all $\beta \geq 0$ such that (4.2) holds, is equal to the T-bound of T.

Next we will give a criterion for relative boundedness.

Theorem 4.3. *Let X, Y, Z be Banach spaces and $T(X \to Y)$, $S(X \to Z)$ be linear operators with $\mathcal{D}(S) \supseteq \mathcal{D}(T)$. Assume that T is closed and S is closable. Then S is T-bounded.*

Proof. □

Theorem 4.4. *Let X, Y be Banach spaces and $T(X \to Y)$, $S(X \to Y)$ be linear operators. If S is T-bounded with relative bound < 1, then the following holds:*

(i) *$T + S$ is closable if and only if T is closable. In this case $\mathcal{D}(\overline{T + S}) = \mathcal{D}(\overline{T})$.*

(ii) *$T + S$ is closed if and only if T is closed.*

Proof. □

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We call a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T)$ *T-bounded* if both $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ are bounded. The notion *T-convergent* is defined analogously.

Definition 4.5. Let X, Y, Z be normed spaces, and $T(X \to Y)$, $S(X \to Z)$ be linear operators. The operator S is called *T-compact* (or *relatively compact with respect to T*) if and only if $\mathcal{D}(S) \supseteq \mathcal{D}(T)$ and every T-bounded sequence $(x_n)_{n\in\mathbb{N}}$ contains a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Sx_{n_k})_{k\in\mathbb{N}}$ converges.

Proposition 4.6. *If S is T-compact, then S is T-bounded.*

Proof. □

At the end of this section we will show that, under additional conditions, the T-bound of S is 0.

Theorem 4.7. *Let X, Y be Banach spaces and $T(X \to Y)$, $S(X \to Y)$ be linear operators. Assume that T is closable and that S is T-compact. Then the following holds:*

(i) *S is $T + S$-compact.*

(ii) *$T + S$ is closable.*

(iii) *$\mathcal{D}(\overline{T + S}) = \mathcal{D}(\overline{T})$.*

(iv) *If T is closed, then $T + S$ is closed.*

Proof. □

Now we will prove a stronger version of Proposition 4.6.

Theorem 4.8. *Let X, Y, Z be Banach spaces and $T(X \to Y)$, $S(X \to Y)$ be linear operators. Assume that S is T-compact and assume that in addition at least one of the following conditions hold:*

(i) *S is closable.*

(ii) *X and Y are Hilbert spaces and T is closable.*

Then S is T-bounded with relative bound 0.

Proof. □

4.2 Selfadjoint operators

Theorem 4.9. *Let H be a complex Hilbert space, $T(H \to H)$ a selfadjoint linear operator and $S(H \to H)$ with $\mathcal{D}(T) \subseteq \mathcal{D}(S)$. Then the following is equivalent:*

(i) *S is T-bounded.*

(ii) $c := \limsup_{\eta\to\infty} \|S(T - \mathrm{i}\eta)^{-1}\| < \infty$.

In this case, the lim inf *is a limit and the limit is equal to the T-bound of S.*

Proof. □

Theorem 4.10 (Kato-Rellich). *Let H be a complex Hilbert space, $T(H \to H)$ a linear operator and $S(H \to H)$ a symmetric linear operator with $\mathcal{D}(T) \subseteq \mathcal{D}(S)$. Assume that S has T-bound < 1. Then the following holds:*

(i) *If T is selfadjoint, then so is $T + S$.*

(ii) *If T is essentially selfadjoint, then so is $T + S$ and $\mathcal{D}(\overline{T+S}) = \mathcal{D}(\overline{T})$.*

Proof. □

4.3 Stability of the essential spectrum

Theorem 4.11 (Weyl). *Let S, T be selfadjoint operators on a complex Hilbert space H and assume that*

$$(S - z)^{-1} - (T - z)^{-1} \tag{4.3}$$

is compact for some $z \in \rho(S) \cap \rho(T)$. Then $\sigma_{ess}(S) = \sigma_{ess}(T)$.

Note that (4.3) holds for one $z \in \rho(S) \cap \rho(T)$ if and only if it holds for all $z \in \rho(S) \cap \rho(T)$.

Proof. □

As an immediate corollary we obtain

Corollary 4.12. *Let T be selfadjoint and K compact and selfadjoint. Then $\sigma_{ess}(T) = \sigma_{ess}(T + K)$.*

Proof. Note that by Theorem 4.7, $T + K$ is selfadjoint. Let $\lambda \in \rho(T) \cap \rho(T + K)$. For example, we can choose $\lambda = \mathrm{i}$. Then

$$(T - \lambda)^{-1} - (T + K - \lambda)^{-1} = (T - \lambda)^{-1} K (T + K - \lambda)^{-1}$$

is compact. Hence the assertion follows from Weyl's theorem (Theorem 4.11). □

We have defined the essential spectrum only for selfadjoint operators. For a non-selfadjoint operator T it can be defined as

$$\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}.$$

A linear operator S is called *semi-Fredholm* if $\mathrm{rg}(\lambda - T)$ is closed and $\dim(\ker(\lambda - T)) < \infty$ or $\mathrm{codim}(\mathrm{rg}(\lambda - T)) < \infty$.
With this definition, Theorem 4.11 and Corollary 4.12 are valid also for non-selfadjoint linear operators.

For the next theorem, however, we need selfadjointness and symmetry of the operators involved.

Theorem 4.13. *Let T be selfadjoint and S a symmetric, T-compact linear operator. Then the following holds.*

(i) *$T + S$ is selfadjoint and $\sigma_{ess}(T) = \sigma_{ess}(T + S)$.*

(ii) *T and $T + S$ have the same singular sequences.*

Proof. That $T + S$ is selfadjoint follows from Theorem 4.7.

Let $\lambda \in \sigma_{\mathrm{ess}}(T)$ and $(x_n)_{n\in\mathbb{N}}$ a singular sequence for T and λ (see Theorem **??**). It follows that $(Tx_n)_{n\in\mathbb{N}}$ converges weakly to 0. In particular, $(x_n)_{n\in\mathbb{N}}$ converges weakly to 0 in $(\mathcal{D}(T), \|\cdot\|_T)$. Since S is T-compact, $(Sx_n)_{n\in\mathbb{N}}$ converges to zero. Consequently $(T + S - \lambda)x_n = (T - \lambda)x_n + Sx_n \to 0$ for $n \to \infty$.

Now let $\lambda \in \sigma_{\mathrm{ess}}(T + S)$ and $(x_n)_{n\in\mathbb{N}}$ a singular sequence for $T + S$ and λ. Since S (and hence $-S$) is $T + S$-compact by Theorem 4.7, by what we already showed we find that $(x_n)_{n\in\mathbb{N}}$ is also a singular sequence for $T + S - S = T$ and λ. □

4.4 Application: Schrödinger operators

The following is taken mostly from [Kat95, V S5].
Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. We want to find realisations of Δ on the space $H := L(\mathbb{R}^3)$.
The *minimal operator* T_0 is Laplace operator with the compactly supported infinitely differentiable functions, that is

$$T_0 f = \Delta f \quad \text{for all} \quad f \in \mathcal{D}(T_0) := C_c^\infty(\mathbb{R}^3). \tag{4.4}$$

Recall that the Fourier transformation is a unitary operator on $L(R^3)$ and its restriction to the space $\mathcal{S}$ of the test functions (Schwartz functions). Recall that $f \in \mathcal{S}$ if and only if it is infinitely differentiable and for every $\alpha \in \mathbb{N}_0^3$ and $p \in \mathbb{N}_0$ there exists a constant $C_{\alpha,p,f}$ such that

$$(1 + |x|^{2p})^{\frac{1}{2}} |D^\alpha f(x)| \le C_{\alpha,p,f}, \qquad x \in \mathbb{R}^3.$$

The restriction of the Fourier transformation maps $\mathcal{S}$ bijectively on itself.

Theorem 4.14 (The free Schrödinger operator). *T_0 is essentially selfadjoint. Its closure H_0 is*

$$H_0 = \mathcal{F}^{-1} M_{k^2} \mathcal{F}, \qquad \mathcal{D}(H_0) = \mathcal{F}^{-1}\mathcal{D}(M_{k^2}),$$

where $\mathcal{F}$ is the Fourier transformation and M_{k^2} is the maximal operator of multiplication by $|k|^2 = k_1^2 + k_2^2 + k_3^2$ in $L(\mathbb{R}^3, \mathrm{d}k^3)$.

Proof. Since the Fourier transformation is unitary and M_{k^2} is selfadjoint, so is H_0.

Note that T_0 is symmetric, hence it is closable. We have to show that $\overline{T_0} = H_0$. We define two auxiliary operators:

$$T_1 = -\Delta|_{\mathcal{S}}, \qquad M^0_{k^2} = M_{k^2}|_{C_c^\infty(\mathbb{R}^3)}.$$

<u>*Step 1.* $\overline{T_1} = \overline{T_0}$.</u>

It suffices to show $T_1 \subseteq \overline{T_1}$. Let $w \in C_c(\mathbb{R}^3)$ such that $0 \le w \le 1$ and $w(x) = 1$ for all $|x| \le 1$. For $n \in \mathbb{N}$ we define $w_n(s) := w(\frac{x}{n})$. Fix $f \in \mathcal{D}(T_1) = \mathcal{S}$. We define $f_n := w_n f \in C_c^\infty(\mathbb{R}^3) = \mathcal{D}(T_0)$. Note that $f(x) = f_n(x)$ for $|x| \le n$. Hence $f_n \to f$ because

$$\begin{aligned}\|f - f_n\|^2 &= \int_{\mathbb{R}^3} |f(x) - f_n(x)|^2 \,\mathrm{d}x \le \int_{|x|\ge n} |f(x) - f_n(x)|^2 \,\mathrm{d}x \\ &\le 2\int_{|x|\ge n} |f(x)|^2 \,\mathrm{d}x \longrightarrow \infty, n \to \infty.\end{aligned}$$

To show that also $T_0 f_n \to T_1 f$ follows because

$$\Delta(w_n f)(x) = w_n(x)\Delta f(x) + \frac{2}{n}\nabla w(x/n)\cdot\nabla f(x) + \frac{1}{n^2} f(x)\Delta w(x/n).$$

Note that $|\nabla w(x/n)|$ and $|\Delta w(x/n)|$ are bounded with bound independent of n and that $|\nabla f|,\ \Delta f \in L_2(\mathbb{R}^3)$ because $f \in \mathcal{S}$. Hence we obtain

$$\begin{aligned}\|Tf - Tf_n\| \le \Bigg(&\int_{|x|\ge n} |1 - w(x/n)||\Delta f(x)| \,\mathrm{d}x \\ &+ \frac{2}{n}\int_{\mathbb{R}^3} |\nabla f(x)||\nabla w(x/n)| \,\mathrm{d}x + \frac{1}{n^2}\int_{\mathbb{R}^3} |f(x)||\Delta w(x/n)| \,\mathrm{d}x\Bigg)\end{aligned}$$

and all terms tend to 0 for $n \to \infty$. We have shown that $f \in \mathcal{D}(\overline{T_0})$.

<u>*Step 2.* $M_{k^2} = \overline{M_{k^2}|_{\mathcal{S}}}$.</u>

It suffices to show that $M_{k^2} \subseteq \overline{M^0_{k^2}}$. Let $f \in \mathcal{D}(M_{k^2})$. Then the function $g := (1 + (M_{k^2})^2)^{\frac{1}{2}} f$ belongs to $L_2(\mathbb{R}^3)$. Therefore there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^3)$ such that $\varphi_n \to g$ and let $f_n = (1 + (M_{k^2})^2)^{-\frac{1}{2}}\varphi_n$. Then $f_n \in C_c^\infty(\mathbb{R}^3)$ for all $n \in \mathbb{N}$. Moreover $f_n \to f$ and $M_{k^2} f_n \to M_{k^2} f$. Hence we have shown that $f \in \mathcal{D}(\overline{M^0_{k^2}})$.

In summary it follows that

$$H_0 = \mathcal{F}^{-1} M_{k^2}\mathcal{F} = \mathcal{F}^{-1}\overline{M_{k^2}|_{\mathcal{S}}}\mathcal{F} = \overline{\mathcal{F}^{-1} M_{k^2}|_{\mathcal{S}}\mathcal{F}} = \overline{T_1} = \overline{T_0}. \qquad \square$$

Since the Fourier transformation is unitary, the spectra of M_{k^2} and H_0 are equal. So we have the following corollary.

Corollary 4.15. $\sigma(H_0) = \sigma_{ess}(H_0) = [0, \infty)$.

In the next proposition we collect some properties of functions belonging to $\mathcal{D}(H_0)$.

Proposition 4.16 (Properties of $u \in \mathcal{D}(H_0)$). *Let $u \in \mathcal{D}(H_0)$.*

(i) $\|\mathcal{F}u\| \le \dfrac{\pi}{\sqrt{\alpha}}\|(H_0 + \alpha^2)^{-1}\|^2 < \infty$ *for all* $\alpha > 0$.

(ii) *There exists a constant $c > 0$, such that for all $\alpha > 0$ and all $u \in H$*

$$|u(x)| \le c(\alpha^{-\frac{1}{2}}\|H_0 x\| + \alpha^{\frac{3}{2}}\|u\|).$$

(iii) *For $\alpha > 0$ and $\gamma \in (0, \frac{1}{2})$ there exists a constant $c > 0$, such that for all $u \in H$ and all $x, y \in \mathbb{R}^3$*

$$|u(x) - u(y)| \le c|x - y|^\gamma(\alpha^{-(\frac{1}{2}-\gamma)}\|H_0 x\| + \alpha^{\frac{3}{2}+\gamma}\|u\|),$$

that is, u is Hölder continuous.

Proof. (i) Let $u \in \mathcal{D}(H_0)$. Then the function $k \mapsto (1 + k^2)(\mathcal{F}f)(k)$ belongs to $L_2(\mathbb{R}^3)$. Therefore, using Hölder's inequality, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^3} |\mathcal{F}u(k)|^2 \,\mathrm{d}k\right)^2 &= \left(\int_{\mathbb{R}^3} \frac{1}{k^2+1}(1 + k^2)|\mathcal{F}u(k) \,\mathrm{d}k\right)^2 \\ &\le \left(\int_{\mathbb{R}^3} \frac{1}{(1+k^2)^2} \,\mathrm{d}k\right)\int_{\mathbb{R}^3} \left((1 + k^2)^2|\mathcal{F}u(k)|^2 \,\mathrm{d}k\right)^2 \\ &= \frac{\pi^2}{\alpha}\|(M_{k^2} + \alpha)\mathcal{F}(u)\| = \frac{\pi^2}{\alpha}\|(H_0 + \alpha)(u)\|.\end{aligned}$$

(ii) Using the estimate from (i) we find for $u \in \mathcal{D}(H_0)$

$$\begin{aligned}|u(x)| = |\mathcal{F}^{-1}\mathcal{F}u(x)| &= (2\pi)^{-\frac{3}{2}}\left|\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}kx}\,\mathcal{F}u(k) \,\mathrm{d}k\right| \\ &\le (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}kx}\,|\mathcal{F}u(k)|^2 \,\mathrm{d}k \le \alpha^{-\frac{1}{2}}\pi^2(2\pi)^{-\frac{3}{2}}\|(H_0 + \alpha^2 u\| \\ &\le c(\alpha^{-\frac{1}{2}}\|H_0 u\| + \alpha^{\frac{3}{2}}\|u\|).\end{aligned}$$

(iii) We note that

$$|\,\mathrm{e}^{\mathrm{i}kx} - \mathrm{e}^{\mathrm{i}ky}\,| = |1 - \mathrm{e}^{\mathrm{i}k(x-y)}\,| \le \min\{2, |k|\,|x - y|\} \le 2^{1-\gamma}(|k|\,|x - y|)^\gamma.$$

For $\gamma \in (0, \frac{1}{2})$ we have that $\int_{\mathbb{R}^3} |k|^\gamma |\mathcal{F}u(k)| \,\mathrm{d}k = \int_{\mathbb{R}^3} \frac{|k|^\gamma}{1+k^2}(1+|k|^2)|\mathcal{F}u(k)| \,\mathrm{d}k < \infty$.

$$\begin{aligned}
|u(x) - u(y)| &= (2\pi)^{-\frac{3}{2}} \left| \int_{\mathbb{R}^3} \left(\mathrm{e}^{\mathrm{i}kx} - \mathrm{e}^{\mathrm{i}ky}\right) \mathcal{F}u(k) \,\mathrm{d}k \right| \\
&\le (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} |\,\mathrm{e}^{\mathrm{i}kx} - \mathrm{e}^{\mathrm{i}ky}\,|\, |\mathcal{F}u(k)| \,\mathrm{d}k \\
&\le (2\pi)^{-\frac{3}{2}} 2^{1-\gamma} |x-y| \int_{\mathbb{R}^3} |k|^\gamma \,|\mathcal{F}u(k)| \,\mathrm{d}k \\
&\le (2\pi)^{-\frac{3}{2}} 2^{1-\gamma} |x-y| \left(\int_{\mathbb{R}^3} \frac{|k|^\gamma}{1+|k|^2} \,\mathrm{d}k \right) \left(\int_{\mathbb{R}^3} (1+|k|^2)|\mathcal{F}u(k)| \,\mathrm{d}k \right) \\
&\le (2\pi)^{-\frac{3}{2}} 2^{1-\gamma} |x-y| C(\gamma) \frac{\pi^2}{\alpha} \|(H_0 + \alpha^2)u\|^2. \qquad \square
\end{aligned}$$

21 Oct 2010

Schrödinger operators with potential

In the following we will always assume

$$q = q_0 + q_1$$

where $q_0 \in L_\infty(\mathbb{R}^3)$ and $q_1 \in L_2(\mathbb{R}^3)$. The maximal multiplication operators on $L_2(\mathbb{R}^3)$ associated to these functions will be denoted by Q, Q_0, Q_1 respectively. Let T_0 be defined as before in (4.4). Then the operator

$$S_0 := T_0 + Q$$

is well-defined because $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^3) \subseteq \mathcal{D}(A)$.

Theorem 4.17. *S_0 is essentially selfadjoint and $H := \overline{S}_0 = H_0 + Q$.*

Proof. We will show that Q is T-bounded with relative bound 0. By the Kato-Rellich theorem (Theorem 4.10) the assertion is then proved. Let $u \in \mathcal{D}(H_0)$. By Proposition 4.16, u is bounded, hence $u \in \mathcal{D}(Q_1)$ and

$$\|Q_1 u\| = \|q_1 u\| \le \|u\|_\infty \|q_1\|_2 \le c\|q_1\| \left(\alpha^{\frac{3}{2}}\|u\| + \alpha^{-\frac{1}{2}}\|H_0 u\|\right).$$

Moreover, $\|Q_0 u\| = \|\|q_0 u\| \le \|u\|_2 \|q_0\|_\infty$. It follows that $\mathcal{D}(H_0) \subseteq \mathcal{D}(Q)$ and

$$\|Qu\| \le \|Q_1 u\| + \|Q_0 u\| \le \left(c\|q_1\|\alpha^{\frac{3}{2}} + \|q_0\|_\infty\right)\|u\| + c\|q_1\|\alpha^{-\frac{1}{2}}\|H_0 u\|.$$

Since α can be taken arbitrarily large, the theorem is proved. $\square$

Theorem 4.18. *Assume that the conditions of Theorem 4.17 hold. Additionally, assume that $q_0(x) \to 0$ for $|x| \to \infty$. Then Q is H_0-compact and $\sigma_{ess}(H) = [0, \infty)$.*

Proof. By Theorem 4.13 and Corollary 4.15 it suffices to show that Q is T_0-compact. First assume that $Q_0 = 0$. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T_0) \subseteq \mathcal{D}(H_0)$ such that $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ are bounded. We have to show that $(Qx_{n_k})_{k\in\mathbb{N}}$ converges for some subsequence. By Proposition 4.16 it follows that $(x_n)_{n\in\mathbb{N}}$ is Hölder continuous and therefore equicontinuous. By assumption it is also uniformely bounded. Hence, by the Arzelá-Ascoli theorem, for every compact ball $B_N(0)$ there exists a subsequence that converges uniformly in $B_N(0)$. Using a diagonal sequence argument, we obtain a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges on $\mathbb{R}^3$ uniformly to some bounded continuous function v. Note that v belongs to $L_2(\mathbb{R}^3)$. Therefore, because Q_1 is a multiplication operator with an L_2-function, $Q_1 u_{n_k} \to Q_1 v$.

If $Q_0 \neq 0$ then we can choose a sequence $\tilde{q}_n$ of compactly supported bounded functions which converge uniformly to q_0. Let $\widetilde{Q}_n$ be the corresponding multiplication operators. Note that $\|\widetilde{Q}_n - Q_0\| \to 0$ for $n \to \infty$. By what is already shown it follows that $\widetilde{Q}_n + Q_1$ is T_0-compact. Hence $(\widetilde{Q}_n + Q_1)(H_0 - 1)^{-1}$ is compact. Then also $(\widetilde{Q}_0 + Q_1)(H_0 - 1)^{-1}$ is compact since it is the limit of compact operators as can be seen from

$$\begin{aligned}
&\|(\widetilde{Q}_n + Q_1)(H_0 - 1)^{-1} - (Q_0 + Q_1)(H_0 - 1)^{-1}\| \\
&\qquad = \|(\widetilde{Q}_n + Q_0)(H_0 - 1)^{-1}\| \le \|\widetilde{Q}_n + Q_0\| \, \|(H_0 - 1)^{-1}\|. \qquad \square
\end{aligned}$$

Note that for example the Coulomb potential $q(x) = \frac{\mathrm{e}}{|x|}$ satisfies the conditions of Theorem 4.18.

Chapter 5

Operator semigroups

5.1 Motivation

This chapter follows very closely [EN00].

Definition 5.1. A *semigroup* is a set M with an associative operation on M. A semigroup with a neutral element is called *monoid* (or *semigroup with a neutral element*).

Example.
- $(\mathbb{R}_+, +)$ with the usual addition on $\mathbb{R}_+ := [0, \infty)$
- $(\mathbb{R}_+, *)$ with $s * t := \mathrm{e}^{s+t}$, $s, t \geq 0$; associativity of $*$ follows from associativity of $(\mathbb{R}_+, +)$.

In this chapter we will deal with semigroups of linear operators with some additional properties.

There are two ways to access semigroups: Using the functional equation (FE) or the initial value problem ACP.

Semigroups for autonomous systems

A physical system is described by a point in a phase space X. Which space is appropriate as phase space, depends on the given system. Points in phase space are called states of the given system.

Let z_0 be a point in the phase space X describing the given system at time t_0, then the system after time $t > 0$ will be in some state $(z_0)_t$. We assume that the new state does not depend on the initial time t_0 or the history of the state, but only on the initial state z_0 and the elapsed time t. In this case the system is called *autonomous*.

Consequently, in an autonomous system we find for every initial value $z_0 \in X$ at time t_0 and for all $s, t > 0$:

$$\begin{aligned} z_0 &:= \text{state with initial value } z_0 \text{ at time } t_0 \\ (z_0)_t &:= \text{state with initial value } z_0 \text{ after time } t \\ ((z_0)_t)_s &:= \text{state with initial value } (z_0)_t \text{ after time } s \\ &= \text{with initial value } z_0 \text{ after time } t+s \\ &= (z_0)_{t+s} \end{aligned}$$

Let us write $U(t)z_0$ instead of $(z_0)_t$, $t > 0$. We obtain

$$\begin{aligned} U(s+t)z_0 &= U(s)U(t)z_0, \qquad s,\, t > 0 \\ U(0)z_0 &= z_0. \end{aligned} \tag{5.1}$$

If this is true for every possible $z_0 \in X$, this yields the functional equation

$$\begin{aligned} U(s+t) &= U(s)U(t), \qquad s,\, t > 0, \\ U(0) &= \mathrm{id}\,. \end{aligned} \tag{FE}$$

Hence the set of all $\{U(t) : t > 0\}$ with the operation given in (FE) is a semigroup with neutral element (associativity follows from the associativity of the addition in $\mathbb{R}_+$).

Examples 5.2.

1. *Mass on a spring.*

We consider a particle with mass $m > 0$ hanging on an ideal spring with Hook's constant $k > 0$ (that is, we neglect friction and assume that Hook's law holds for arbitrarily large amplitudes and momenta). The system is described completely by the position x and the momentum p of the particle at a given time t_0. For the phase space we can therefore choose $X = \mathbb{R} \times \mathbb{R} = \text{position} \times \text{momentum}$. Without restriction we assume $t_0 = 0$. The equation of motion is

$$m\ddot{x} = -kx, \quad p = m\dot{x}, \quad t \geq 0, \qquad x(0) = x_0, \quad p(0) = p_0,$$

or, written as first order system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}, \qquad \begin{pmatrix} x \\ p \end{pmatrix}(0) = \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}. \tag{5.2}$$

By the theorem of Picard-Lindelöf the system has a unique solution. It is given by the Picard-Lindelöf iteration as

$$\begin{pmatrix} x \\ p \end{pmatrix}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}^n \begin{pmatrix} x_0 \\ p_0 \end{pmatrix},$$

In this case the time evolution is given by

$$U(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}^n =: \exp\left(t\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}\right)$$

In this simple one-dimensional example we observe:

- All initial values $(x_0,\, p_0)^t$ are allowed.
- The solutions exist and are unique for all $t \geq 0$ and they are continuous for $t \searrow 0$.
- The solutions depend continuously on the initial value $(x_0,\, p_0)^t$.
- Also $t < 0$ is allowed.
- The asymptotic behaviour of the solutions depend on the eigenvalues of the matrix $\left(\begin{smallmatrix} 0 & m^{-1} \\ -k & 0 \end{smallmatrix}\right)$.
- It is easy to check the the functional equation (FE) holds.

2. *Heat conducting rod.*

Let $f(x,t)$ be the temperature in an ideal heat conducting rod of length L at position $x \in [0, L]$ and time $t \geq 0$. As phase space we choose $X = C([0, L])$ or $X = L_p(0, L)$. If we disregard boundary conditions, we are led to the following initial value problem

$$\begin{aligned} \frac{\partial f}{\partial t} &= \kappa\, \frac{\partial^2 f}{\partial x^2}, \qquad t \geq 0,\ x \in [0, L], \\ f(\,\cdot\,, 0) &= \varphi_0 \in X. \end{aligned}$$

Instead of treating this initial value problem as a partial differential equation in $(0, L) \times \mathbb{R}_+$, we can consider it as a first order problem in the space X:

$$\begin{aligned} \frac{d}{dt}\varphi &= A\varphi, \qquad t \geq 0, \\ \varphi(0) &= \varphi_0, \end{aligned} \tag{ACP}$$

where A is the unbounded operator $A = \kappa\, \frac{\partial^2}{\partial x^2}$ in the space X (in order to define A, we have to specify its domain $\mathcal{D}(A)$; here the boundary conditions enter) and φ is a map $\mathbb{R}_+ \to X$ (here $\varphi(t) = f(\,\cdot\,, t)$, $t \geq 0$).

If X is a Banach space and A a linear operator in X, then a problem of the form (ACP) is called an *abstract Cauchy problem*.

Formally, the solution of (ACP) is again "$\varphi(t) = \mathrm{e}^{tA}\, \varphi_0$". In contrast to the first example, however, where the linear operator $\left(\begin{smallmatrix} 0 & m^{-1} \\ -k & 0 \end{smallmatrix}\right)$ is bounded, here we have the following problems and questions:

- If A is unbounded, then only initial values $\varphi_0 \in \mathcal{D}(A)$ are allowed in (ACP).
- If $\varphi_0 \in \mathcal{D}(A)$, does (ACP) then have a solution $\varphi(\cdot)$?
- How does the time asymptotic of solutions depend on the spectrum of A?
- What is a solution of (ACP)?

3. *More examples.*

Many partial differential equations can be treated as above, for instance the Schrödinger equation

$$\frac{\partial}{\partial t}\, \Psi = \mathrm{i}\, \Delta\Psi + \mathrm{i}\, V\Psi$$

or the Navier-Stokes equation

$$\begin{aligned} \frac{\partial}{\partial t}\, \Psi - \Delta\Psi + (\Psi \cdot \nabla)\Psi + \nabla p &= 0, \\ \div \Psi &= 0 \\ \Psi|_{t=0} &= \Psi_0. \end{aligned}$$

We are going to deal with existence and uniqueness of solutions of problems of the form (ACP). This will depend on properties of the operator A. The main theorems are the generation theorems by Hille and Yosida (Theorem 5.31), by Lumer and Phillips (Theorem 5.44) and by Stone (Theorem 5.47).

5.2 Basic definitions and properties

Definition 5.3. Let X be a Banach space.

(i) A family $\mathcal{T} = (T(t))_{t \geq 0} \subseteq L(X)$ is called a *semigroup* (more precisely a *1-parameter operator semigroup*) if

$$\begin{aligned} T(t+s) &= T(t)T(s), \qquad t,\, s \geq 0 \\ T(0) &= \mathrm{id}\,. \end{aligned} \tag{5.3}$$

(ii) A family $\mathcal{S} = (S(t))_{t \in \mathbb{R}} \subseteq L(X)$ is called a *group* (more precisely a *1-parameter group*) if

$$\begin{aligned} S(t+s) &= S(t)S(s), \qquad t,\, s \in \mathbb{R} \\ S(0) &= \mathrm{id}\,. \end{aligned} \tag{5.4}$$

Definition 5.4. Let X be a Banach space and $\mathcal{T} = (T(t))_{t \geq 0}$ a semigroup on X. Let us consider the map

$$T : \mathbb{R}_+ \to L(X), \qquad t \mapsto T(t).$$

(i) $\mathcal{T}$ is called a *uniformly continuous semigroup* if T is continuous with respect to the operator norm;
that is, for every $t_0 \geq 0$ and every $\varepsilon > 0$ exists a $\delta > 0$ such that $\|T(t_0) - T(t)\| < \varepsilon$ for all $t \geq 0$ with $|t - t_0| < \delta$.

(ii) $\mathcal{T}$ is called *strongly continuous* or a *C_0-semigroup*[1], if T is strongly continuous;
that is, for every $x \in X$ the map $\mathbb{R}_+ \to X$, $t \mapsto T(t)x$ is continuous,
that is, for every $x \in X$, $t_0 \geq 0$ and $\varepsilon > 0$ exists a $\delta > 0$ such that $\|T(t)x - T(t_0)x\| < \varepsilon$ for all $t \geq 0$ with $|t - t_0| < \delta$.

Examples. (i) Let $X = \mathbb{C}$ and $a \in \mathbb{C}$. Then $T(t) = \mathrm{e}^{at}$ defines a strongly continuous semigroup.

(ii) Let $X = \mathbb{C}^n$ and $A \in M_n(\mathbb{C})$. Then $T(t) = \mathrm{e}^{At}$ defines a strongly continuous semigroup on X.

Example 5.5 (Multiplication semigroup). Let $X = C(K)$ where K is compact subset of $\mathbb{C}$ and fix $q \in C(K)$. Then $(T(t)f)(\xi) = \mathrm{e}^{tq(\xi)} f(\xi)$ defines a uniformly continuous semigroup on $C(K)$.

Example 5.6 (Translation semigroup). Consider the function spaces

(i) $X = L_\infty(\mathbb{R})$

(ii) $X = BUC(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} : f \text{ bounded and uniformly continuous}\}$

(iii) $X = L_p(\mathbb{R})$.

In each case, the translation operators are defined by

$$T(t)f(\xi) = f(\xi + t), \qquad t \geq 0,\ \xi \in \mathbb{R}.$$

In all three cases $T(t) \in L(X)$, $t \geq 0$, and $\mathcal{T} = (T(t))_{t \geq 0}$ satisfies (5.3), hence it is a semigroup on X.
In case (i), is $\mathcal{T}$ not strongly continuous, hence it cannot be continuous in norm. For instance, let

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(\xi) = \begin{cases} 1, & \xi \geq 0, \\ -1, & \xi < 0, \end{cases}$$

then $f \in L_\infty(\mathbb{R})$ and $\|T(t)f - T(0)f\|_\infty = 2$, $t > 0$, consequently $T(\cdot)f$ is not continuous in 0 ($\mathcal{T}$ is not strongly continuous in 0).
In the cases (ii) and (iii) $\mathcal{T}$ is a strongly continuous by not norm continuous semigroup on X. It can be shown that $\|T(t) - \mathrm{id}\,\| = 2$ for $t > 0$.

Proposition 5.7. *Let X be a Banach space and $\mathcal{T} = (T(t))_{t \geq 0}$ a semigroup on X. Then the following is equivalent:*

(i) *T is strongly continuous.*

(ii) *T is strongly continuous in* 0.

[1] C_0 stands for Cesàro-summable.

(iii) *There exist $\delta > 0$, $M \geq 1$ and a dense subset $D \subseteq X$ such that*

(a) $\|T(t)\| \leq M, \quad t \in [0, \delta]$,

(b) $\lim_{t \searrow 0} T(t)x = x, \quad x \in D$.

If (iii) (a) *holds, then, with* $\omega = \frac{\log M}{\delta}$,

$$\|T(t)\| \leq M\ \mathrm{e}^{t\omega}, \qquad t \geq 0. \tag{5.5}$$

Proof. First we show the estimate (5.5): For every $t \in \mathbb{R}_+$ there exists an $n \in \mathbb{N}_0$ and $\tau \in [0, \delta)$ such that $t = \tau + n\delta$. Using the semigroup property of $\mathcal{T}$ and the estimate (iii)(a) and $0 < n \log M \leq \frac{t}{\delta} \log M = t\omega$, we find

$$\begin{aligned} \|T(t)\| &= \|T(\tau + n\delta)\| = \|T(\tau)\underbrace{T(\delta)\cdots T(\delta)}_{n\text{-times}}\| \leq \|T(\tau)\|\,\|T(\delta)\|^n \leq MM^n \\ &= M\,\mathrm{e}^{n \log M} \leq M\,\mathrm{e}^{t\omega} \end{aligned}$$

(ii) $\Rightarrow$ (iii) We only have to show (iii)(a). Assume there exist no $\delta > 0$ and $M \geq 1$ such that (iii)(a) holds. Then there is a sequence $(t_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_+$ with $t_n \searrow 0$ and $\|T(t_n)\| \to \infty$ for $n \to \infty$. By the uniform boundedness principle, there exists an $x \in X$ such that $\|T(t_n)x\| \to \infty$, $n \to \infty$. Consequently $T(t_n)x \not\to x = T(0)x$, in contradiction to the strong continuity of T in 0.
(iii) $\Rightarrow$ (ii) Let $(t_n)_{n \in \mathbb{N}}$ with $t_n \searrow 0$, $n \to \infty$; without restriction we can assume $t_n \leq \delta$, $n \in \mathbb{N}$. By assumption $\|T(t_n)\| \leq M$, $n \in \mathbb{N}$, and $T(\cdot)x|_K$ is continuous for every $x \in D$. For arbitrary $x \in X$ and $\varepsilon > 0$ choose $y \in D$ such that $\|x - y\| < \min\{\varepsilon/3,\ \varepsilon/(3M)\}$ and choose $N \in \mathbb{N}$ large enough such that $\|T(t_n)y - y\| < \varepsilon/3$ for $n \geq N$. This implies

$$\begin{aligned} \|T(t_n)x - x\| &\leq \|T(t_n)(x - y)\| + \|T(t_n)y - y\| + \|y - x\| \\ &\leq \|T(t_n)\|\,\|x - y\| + \|T(t_n)y - y\| + \|y - x\| < \varepsilon. \end{aligned}$$

Since $(t_n)_{n \in \mathbb{N}}$ and $\varepsilon > 0$ can be chosen arbitrary, the claim $\lim_{t \searrow 0} \|T(t)x - x\| = 0$ is proved.
(ii) $\Rightarrow$ (i) Let $t_0,\ h > 0$ and $x \in X$ be given.
Right continuity of $\mathcal{T}$ in t_0: Since $\mathcal{T}$ is strongly continuous in 0, it follows that

$$\|T(t_0 + h)x - T(t_0)x\| \leq \|T(t_0)\|\,\|T(h)x - x\| \longrightarrow 0. \qquad h \searrow 0,$$

Right continuity of $\mathcal{T}$ in t_0: We already showed "(ii) $\Rightarrow$ (iii)", hence $\|T(t)\| \leq M\,\mathrm{e}^{t\omega}$, $t \geq 0$, for appropriate $M \geq 1$ and $\omega \in \mathbb{R}$. This implies

$$\|T(t_0)x - T(t_0 - h)x\| \leq \underbrace{\|T(t_0 - h)\|}_{\text{bounded}}\,\underbrace{\|T(h)x - x\|}_{\to 0,\ h \to 0} \longrightarrow 0, \qquad h \searrow 0.$$

(i) $\Rightarrow$ (ii) is clear. □

Definition 5.8. A strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on a Banach space X is called

(i) *bounded*, if we can choose $\omega = 0$ in (5.5);

(ii) *contractive* or a *contraction semigroup* if we can choose $\omega = 0$ and $M = 1$ in (5.5);

(iii) *isometric* if $\|T(t)x\| = \|x\|,\ t \geq 0,\ x \in X$.

Definition 5.9. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. Then

$$\omega_0 = \omega_0(\mathcal{T}) := \inf\{\omega \in \mathbb{R} : \exists\ M \geq 1 \text{ such that } \ \|T(t)\| \leq M\, \mathrm{e}^{t\omega},\ t \geq 0\} \quad (5.6)$$

is the *growth bound* or the *type* of $\mathcal{T}$.

Remarks 5.10. • It is possible that $\omega_0 = -\infty$: Every nilpotent semigroup has growth bound $-\infty$. (A semigroup is nilpotent, if there exists a $t_0 \geq 0$ such that $T(t) = 0$ for all $t \geq t_0$.)

For instance, let $X = L_p(0, a)$ for some $a \in (0, \infty)$ and

$$(T(t)f)(\xi) := \begin{cases} f(t-\xi), & t \leq \xi \leq a, \\ 0, & \text{else}, \end{cases} \qquad f \in X.$$

Obviously, $\mathcal{T} = (T(t))_{t\geq 0}$ is a semigroup on X and $\omega_0(\mathcal{T}) = -\infty$.

- In general, the infimum in (5.6) is not a minimum.
- In general M has to be chosen > 1, independently how large ω is chosen.

5.3 Uniformly continuous semigroups

Definition 5.11. Let X be a Banach space and $A \in L(X)$. Let

$$\exp(tA) := \mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad t \in \mathbb{R}. \quad (5.7)$$

Then the family $(\exp(tA))_{t\geq 0}$ is the *semigroup generated by* A, and $(\exp(tA))_{t\in\mathbb{R}}$ the *group generated by* A.

The following proposition shows that Definition 5.11 makes sense.

Proposition 5.12. *Let X be a Banach space and $A \in L(X)$.*

(i) $\exp(tA)$ *converges absolutely and* $\exp(tA) \in L(X)$ *for all* $t \in \mathbb{R}$.

(ii) $\exp(0 \cdot A) = \mathrm{id}$.

(iii) $\exp((t+s)A) = \exp(tA)\, \exp(sA),\ \ s, t \geq 0$.

(iv) $\mathbb{R} \to L(X),\ t \mapsto \exp(tA)$ *is continuous.*

(v) *If $S \in L(X)$, such that S^{-1} exists and $S^{-1} \in L(X)$, then*

$$\exp(S^{-1}AS) = S^{-1}\exp(A)\, S. \quad (5.8)$$

(vi) *If $B \in L(X)$ with $AB = BA$, then*

$$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A). \quad (5.9)$$

(i)–(iv) show that $(\exp(tA))_{t\geq 0}$ is a uniformly continuous semigroup.

Proof. (i) For $k < m \in \mathbb{N}$ we have

$$\left\| \sum_{n=0}^{m} \frac{t^n}{n!} A^n - \sum_{n=0}^{k} \frac{t^n}{n!} A^n \right\| = \left\| \sum_{n=k+1}^{m} \frac{t^n}{n!} A^n \right\| \leq \sum_{n=k+1}^{m} \frac{t^n}{n!} \|A^n\| \longrightarrow 0, \qquad k, m \to \infty,$$

because $\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A\|^n = \mathrm{e}^{t\|A\|}$. Consequently, the sequence $\left(\sum_{n=0}^{k} \frac{t^n}{n!} A^n\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L(X)$, hence it converges in $L(X)$ (because $L(X)$ is a Banach space).
(ii) is clear.
(iii) follows from (vi).
(iv) For $t, h \in \mathbb{R}$ we have that

$$\begin{aligned} \|\exp((t+h)A) - \exp(tA)\| &\leq \|\exp(tA)\|\ \|\exp((hA) - \mathrm{id}\,\| \\ &= \|\exp(tA)\| \left\| \sum_{n=1}^{\infty} \frac{h^n}{n!} A^n \right\| \leq \|\exp(tA)\|\, |h|\ \|A\| \sum_{n=0}^{\infty} \frac{h^n}{(n+1)!}\ \|A^n\| \\ &\leq |h|\ \|A\|\ \|\exp(tA)\|\ \|\exp(h\,\|A\|)\,\|. \end{aligned}$$

Therefore $\exp((t+h)A) \longrightarrow \exp(tA)$ for $h \to 0$.
(v) Since the series are absolutely convergent, we obtain

$$\exp(S^{-1}AS) = \sum_{n=0}^{\infty} \frac{1}{n!} (S^{-1}AS)^n = S^{-1} \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n \right) S = S^{-1}\exp(A)S.$$

(vi) Since the series are absolutely convergent, we obtain with Cauchy's product formula

$$\begin{aligned} \exp(A)\exp(B) &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n\right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} B^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} A^k \frac{1}{(n-k)!} B^{n-k} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n \\ &= \exp(A+B). \end{aligned}$$

□

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Example 5.13 (Matrix semigroups). Let $X = \mathbb{C}^n$ and $A \in L(X) = M_n(\mathbb{C})$. Then Proposition 5.12 yields a technique how to calculate exponentials of matrices. There exists a $S \in Gl(n, \mathbb{C})$ such that SAS^{-1} has Jordan normal form , that is, $A = S^{-1}(D+N)S$ with a diagonal matrix D and a nilpotent matrix N such that $ND = DN$. Then

$$\begin{aligned} \exp(tA) &= \exp(S^{-1}(tD+tN)S) = S^{-1}\exp(tD+tN)S = S^{-1}\exp(tD)\exp(tN)S \\ &= S^{-1}\exp(tD)\underbrace{\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}N^n\right)}_{\substack{\text{only finitely}\\ \text{many terms!}}} S. \end{aligned}$$

For calculations we use: If A is of block diagonal form

$$A = \begin{pmatrix} A_1 & \cdots & \cdots & 0 \\ \vdots & A_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A_j \end{pmatrix} =: \operatorname{diag}(A_1, \dots, A_n)$$

where $A_k \in M(n_k, \mathbb{C})$, $n_k \in \mathbb{N}$, with $\sum_{k=1}^{j} n_k = n$, then

$$\exp(tA) = \operatorname{diag}\big(\exp(tA_1), \dots, \exp(tA_n)\big), \qquad t \in \mathbb{R}.$$

In particular, for a Jordan block of length m

$$J = \begin{pmatrix} \lambda & 1 & & & \\ 0 & \lambda & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda \end{pmatrix}$$

we obtain

$$\exp(tJ) = e^{t\lambda}\begin{pmatrix} 1 & t & \cdots & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & t \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

The asymptotic behaviour of $\exp(tA)x$ depends on the Jordan structure of A.

Example. Let $m > 0$, $k \in \mathbb{R}$ and $A := \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}$ (see Example 5.2.1). Choose $\kappa \in \mathbb{C}$ such that $\kappa^2 = -k$ and let $S := \frac{1}{\sqrt{2}}\begin{pmatrix} \kappa & \sqrt{m}^{-1} \\ -\sqrt{m} & \kappa^{-1} \end{pmatrix}$. Then

$$SAS^{-1} = \frac{1}{2}\begin{pmatrix} \kappa & \sqrt{m}^{-1} \\ -\sqrt{m} & \kappa^{-1} \end{pmatrix}\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}\begin{pmatrix} \kappa^{-1} & -\sqrt{m}^{-1} \\ \sqrt{m} & \kappa \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{m} & 0 \\ 0 & -\frac{\kappa}{m} \end{pmatrix}.$$

Recall that $k \geq 0$, whence $\kappa \in \mathrm{i}\mathbb{R}$. The solutions $\exp(tA)\begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$ are periodic with period $\omega = \frac{2\pi m}{|\kappa|}$ because

$$\begin{aligned} \exp\left((t+\omega)A\right) &= S^{-1}\exp\left((t+\omega)\begin{pmatrix} \frac{\kappa}{m} & 0 \\ 0 & -\frac{\kappa}{m} \end{pmatrix}\right) S \\ &= S^{-1}\begin{pmatrix} \exp\left((t+\omega)\frac{\kappa}{m}\right) & 0 \\ 0 & \exp\left(-(t+\omega)\frac{\kappa}{m}\right) \end{pmatrix} S \\ &= S^{-1}\begin{pmatrix} \exp(t\frac{\kappa}{m}) & 0 \\ 0 & \exp(-t\frac{\kappa}{m}) \end{pmatrix} S = \exp(tA). \end{aligned}$$

So far, we only considered the functional equation (FE). From Proposition 5.12 we know that for $A \in L(X)$ the group $(\exp(tA))_{t\in\mathbb{R}}$ is continuous. The following proposition shows that it is even differentiable.

Proposition 5.14. *Let X be a Banach space, $A \in L(X)$ and $\mathcal{T} = (T(t))_{t\geq 0}$ the semigroup generated by A (i. e., $T(t) = \exp(tA)$, $t \geq 0$). Then the following holds:*

(i) *The map $\mathbb{R} \to L(X)$, $t \mapsto T(t)$, is differentiable and with derivative*

$$\frac{d}{dt}T(t) = AT(t) = T(t)A, \qquad t \in \mathbb{R}.$$

(ii) *If $S : \mathbb{R} \to L(X)$ is a solution of*

$$U(0) = \mathrm{id}, \qquad \frac{d}{dt}U(t) = AU(t), \quad t \in \mathbb{R}, \tag{5.10}$$

then $S = T$.

Proof. (i) Because of

$$\frac{T(t+h) - T(t)}{h} = T(t)\,\frac{T(h) - \mathrm{id}}{h} = \frac{T(h) - \mathrm{id}}{h}\,T(t), \quad t \in \mathbb{R},\ h \in \mathbb{R}\setminus\{0\},$$

is suffices to show the differentiability in $t = 0$ with $\frac{d}{dt}T(0) = A$. This follows from

$$\begin{aligned} \left\|\frac{1}{h}\left(T(h) - T(0)\right) - A\right\| &= \left\|\frac{1}{h}\sum_{n=1}^{\infty}\frac{h^n}{n!}A^n - A\right\| = \left\|\frac{1}{h}\sum_{n=2}^{\infty}\frac{h^n}{n!}A^n\right\| \\ &\leq |h|\,\|A\|^2\sum_{n=0}^{\infty}\frac{h^n\|A\|^n}{(n+2)!} \leq |h|\,\|A\|^2\,\exp(h\,\|A\|) \longrightarrow 0, \end{aligned}$$

for $|h| \to 0$.

(ii) Observe that $T(0) = S(0)$ by assumption. For arbitrary $t_0 \in \mathbb{R}$ it follows that

$$\frac{d}{dt}\left(T(t)S(t_0-t)\right) = \left(\frac{d}{dt}T(t)\right)S(t_0-t) + T(t)\left(\frac{d}{dt}S(t_0-t)\right) = AT(t)S(t_0-t) - \underbrace{T(t)A}_{=AT(t)}S(t_0-t) = 0.$$

Suppose that $T(t)S(t_0-t)$ are not constant with respect to t. Then there exists $\tau \in \mathbb{R}$, $x \in X$ and $\varphi \in X'$ such that $\varphi\big((T(\tau)S(t_0-\tau)x\big) \neq \varphi\big((T(0)S(t_0)x\big)$. But for arbitrary $x \in X$ and $\varphi \in X'$ the calculation above gives $\frac{d}{dt}\varphi\big((T(t)S(t_0-t))x\big) = 0,\ t \in \mathbb{R}$, hence $\varphi\big((T(t)S(t_0-t))x\big)$ is constant in t. Consequently, $T(t)S(t_0-t)$ is constant with respect to t and therefore

$$T(t_0) = T(t_0)S(t_0-t_0) = T(0)S(t_0-0) = S(t_0).$$

Since $t_0 \in \mathbb{R}$ was arbitrary, the assertion is proved. □

Corollary 5.15. *If X is a Banach space, $x_0 \in X$, $A \in L(X)$ and $(T(t))_{t\geq 0}$ the group generated by A, then $T(\cdot)x_0$ is the unique solution of the initial value problem*

$$x(0) = x_0, \qquad \frac{d}{dt}x = Ax, \qquad t \in \mathbb{R}.$$

Theorem 5.16 (Characterisation of uniformly continuous semigroups).

Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a semigroup on X. Then $\mathcal{T}$ is a uniformly continuous semigroup on X if and only if there exists an $A \in L(X)$ such that $T(t) = \exp(tA)$, $t \geq 0$. The operator A is uniquely determined by $\mathcal{T}$; T is differentiable and

$$\frac{d}{dt}T(t) = AT(t) = T(t)A, \qquad t \geq 0. \tag{5.11}$$

Proof. If $A \in L(X)$, then $(\exp(tA))_{t\geq 0}$ is a uniformly continuous semigroup and satisfies (5.11) by Proposition 5.12.

Now assume that $\mathcal{T} = (T(t))_{t\geq 0}$ is a uniformly continuous semigroup on X. Define

$$V(t) := \int_0^t T(s)\,\mathrm{d}s, \qquad t \geq 0. \tag{5.12}$$

Since T is continuous, we obtain

$$\frac{1}{t}V(t) = \frac{1}{t}\int_0^t T(s)\,\mathrm{d}s \longrightarrow T(0) = \mathrm{id}, \qquad t \searrow 0.$$

Hence there exists an t_0 such that $V(t)$ is boundedly invertible for all $t \in (0, t_0]$ (use that $V(t)$ has bounded inverse if and only if $t^{-1}V(t)$ has bounded inverse and that $t^{-1}V(t) = \mathrm{id} - (\mathrm{id} - t^{-1}V(T)))$.
Moreover, (5.12) shows that V is continuously differentiable because for $h > 0$ we have that, for $h \searrow 0$,

$$\frac{1}{h}\left(V(t+h) - V(t)\right) = \frac{1}{h}\int_t^{t+h} T(s)\,\mathrm{d}s = T(t)\,\frac{1}{h}\int_0^h T(s)\,\mathrm{d}s \longrightarrow T(t),$$

$$\frac{1}{h}\left(V(t-h) - V(t)\right) = \frac{1}{h}\int_{t-h}^{t} T(s)\,\mathrm{d}s = T(t-h)\,\frac{1}{h}\int_0^h T(s)\,\mathrm{d}s \longrightarrow T(t).$$

Differentiability of T follows from

$$T(t) = V(t_0)^{-1}V(t_0)T(t) = V(t_0)^{-1}\int_0^{t_0} T(s+t)\,\mathrm{d}s = V(t_0)^{-1}\int_t^{t+t_0} T(s)\,\mathrm{d}s = V(t_0)^{-1}\left(V(t+t_0) - V(t)\right), \qquad t \geq 0,$$

because V is differentiable. In particular, it follows that

$$\frac{d}{dt}T(t) = V(t_0)^{-1}\frac{d}{dt}\left(V(t+t_0) - V(t)\right) = V(t_0)^{-1}\left(T(t+t_0) - T(t)\right) = V(t_0)^{-1}\left(T(t_0) - \mathrm{id}\right)T(t)$$

Obviously, the operator $A := V(t_0)^{-1}(T(t_0) - \mathrm{id})$ is linear and bounded. By Proposition 5.14 we obtain that $T(t) = \exp(tA)$, $t \geq 0$. □

Definition 5.17. Let A and $\mathcal{T}$ be as in Theorem 5.16. Then A is called the *(infinitesimal) generator* of $\mathcal{T}$.

For a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ and its generator A we have

$$Ax = \lim_{t\searrow 0}\frac{1}{t}\left(T(t) - \mathrm{id}\right)x, \qquad x \in X, \tag{5.13}$$

$$T(t)x = \sum_{n=0}^{\infty}\frac{t^n}{n!}A^n x, \qquad x \in X,\ t \geq 0. \tag{5.14}$$

Example (Multiplication semigroups on $\mathbf{C_0(\Omega)}$).

Definition 5.18. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $q \in C(\Omega)$. Then the operator M_q, defined by

$$M_q f := qf, \qquad f \in \mathcal{D}(M_q) := \{f \in C_0(\Omega) : qf \in C_0(\Omega)\},$$

is the *multiplication operator induced by q* on

$$C_0(\Omega) := \{f \in C(\Omega) : \forall\, \varepsilon > 0\ \ \exists\, K_\varepsilon \subseteq \Omega \text{ compact such that } |f(\xi)| < \varepsilon,\ \xi \in \Omega \setminus K_\varepsilon\},$$

with the norm $\|f\| = \sup\{|f(\xi)| : \xi \in \Omega\}$.

Proposition 5.19. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $q \in C(\Omega)$. Then the following holds:*

(i) *$M_q : \mathcal{D}(M_q) \subseteq C_0(\Omega) \to C_0(\Omega)$ is densely defined and closed.*

(ii) *M_q is bounded $\iff$ q is bounded.*

(iii) *M_q is boundedly invertible $\iff$ q is boundedly invertible, in this case $(M_q)^{-1} = M_{q^{-1}}$.*

(iv) $\sigma(M_q) = \overline{q(\Omega)}$.

Proof. See, e.g., [EN00, Proposition I.4.2]. □

Definition 5.20. Let $q \in C(\Omega)$ with $\omega := \sup \mathrm{Re}(q(\xi)) < \infty$ and define $\widetilde{q}_t(\xi) := \mathrm{e}^{tq(\xi)}$, $t \geq 0$, $\xi \in \Omega$. We denote the corresponding multiplication operator by

$$T_q(t) := M_{\widetilde{q}_t}, \qquad t \geq 0.$$

Obviously $\widetilde{q}_t \in C(\Omega)$, $t \geq 0$, and therefore for every $t \geq 0$ the operator $T_q(t)$ is a multiplication operator on $C_0(\Omega)$. It is clear that $\mathcal{T}_q = (T_q(t))_{t\geq 0}$ is a semigroup on $C_0(\Omega)$, because for all $f \in C_0(\Omega)$, $s, t \geq 0$ and $\xi \in \Omega$

$$\begin{aligned}\big(T_q(s)T_q(t)f\big)(\xi) &= \mathrm{e}^{tq(\xi)}\big(T_q(s)f\big)(\xi) = \mathrm{e}^{sq(\xi)}\,\mathrm{e}^{tq(\xi)}\,f(\xi) = \mathrm{e}^{(s+t)q(\xi)}\,f(\xi)\\ &= \big(T_q(s+t)f\big)(\xi),\end{aligned}$$

and

$$\|T_q(t)\| \leq \mathrm{e}^{t\omega}, \qquad t \geq 0.$$

What are necessary and sufficient conditions on q such that $\mathcal{T}_q$ is a uniformly continuous or a strongly continuous semigroup? If $\mathcal{T}_q$ is uniformly continuous, what is its generator?

Proposition 5.21. *With the definitions from Definition 5.20 the following is true:*

(i) *$\mathcal{T}_q$ is uniformly continuous if and only if q is bounded. In this case M_q is the generator of $\mathcal{T}_q$.*

(ii) *If q is unbounded (but still $\sup\{\mathrm{Re}(q(\xi)) : \xi \in \Omega\} < \infty$), then $\mathcal{T}_q$ is a strongly continuous semigroup on $C_0(\Omega)$.*

Proof. (i) Assume that q is bounded. Then M_q is bounded and for all $t \geq 0$, $f \in C_0(\Omega)$ and $\xi \in \Omega$

$$\begin{aligned}\big(T_q(t)f\big)(\xi) \;&=\; \mathrm{e}^{tq(\xi)}\,f(\xi) \;=\; \sum_{n=0}^{\infty}\frac{t^n q(\xi)^n}{n!}f(\xi) \;=\; \sum_{n=0}^{\infty}\frac{t^n}{n!}(M_{q^n}f)(\xi)\\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\big((M_q)^n f\big)(\xi) = \Big(\Big(\sum_{n=0}^{\infty}\frac{t^n}{n!}(M_q)^n\Big)f\Big)(\xi) \;=\; \big(\exp(tM_q)f\big)(\xi).\end{aligned}$$

Hence $T_q(t) = \exp(tM_q)$, $t \geq 0$, and therefore $\mathcal{T}_q$ is a uniformly continuous semigroup by Theorem 5.16.

Assume that q is unbounded. Then there exists a sequence $(\xi_n) \subseteq \Omega$ such that $|q(\xi_n)| \to \infty$. Let $t_n := \frac{1}{|q(\xi_n)|}$, $n \in \mathbb{N}$. If $\mathcal{T}_q$ was uniformly continuous, then, for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|T(t_n)f - f\| < \varepsilon$, $n \geq N$, $f \in C_0(\Omega)$. For every $n \in \mathbb{N}$ we choose a function $f_n \in C_0(\Omega)$ such that $f_n(\xi_n) = 1$ and $\|f_n\| = 1$. Define $\delta := \min\{|\,\mathrm{e}^z - 1|\, z \in \mathbb{C},\ |z| = 1\} > 0$. Then, for all $n \in \mathbb{N}$

$$\begin{aligned}\|T(t_n)f_n - f_n\| &= \sup\{|\,\mathrm{e}^{t_n q(\xi)}\,f_n(\xi) - f_n(\xi)| \,:\, \xi \in \Omega\} \geq |\,\mathrm{e}^{t_n q(\xi_n)}\,f_n(\xi_n) - f_n(\xi_n)|\\ &= |\,\mathrm{e}^{t_n q(\xi_n)} - 1|\,|f_n(\xi_n)| = |\,\mathrm{e}^{t_n q(\xi_n)} - 1|\, \geq \delta,\end{aligned}$$

hence $\mathcal{T}_q$ is not uniformly continuous.

(ii) Let $f \in C_0(\Omega)$. We have to show that $\mathbb{R}_+ \to X$, $t \mapsto T_q(t)f$ is continuous. By Proposition 5.7 is suffices to show the continuity in 0. Fix $\varepsilon > 0$. By assumption there is a compact set $K_\varepsilon \subset \Omega$ such that

$$|f(\xi)| < \frac{\varepsilon\,\|f\|}{\mathrm{e}^{|\omega|} + 1}, \qquad \xi \in \Omega \setminus K_\varepsilon,$$

where $\omega = \sup\{\mathrm{Re}(q(\xi)) \,:\, \xi \in \Omega\}$. Since K_ε is compact and q is continuous, there exists a $t_0 \in (0,1)$ such that

$$\big|\,1 - \exp(tq(\xi))\,\big| < \varepsilon, \qquad t \in [0, t_0],\ \xi \in K_\varepsilon.$$

Hence, for all $t \in [0, t_0]$,

$$\begin{aligned}\|T(t)f - f\| &= \sup\{|\,\mathrm{e}^{tq(\xi)}\,f(\xi) - f(\xi)| \,:\, \xi \in \Omega\}\\ &= \sup\{|(\mathrm{e}^{tq(\xi)} - 1)f(\xi)| \,:\, \xi \in K_\varepsilon\} + \sup\{|(\mathrm{e}^{tq(\xi)} - 1)f(\xi)| \,:\, \xi \in \Omega \setminus K_\varepsilon\}\\ &\leq \|f\|\,\sup\{|\,\mathrm{e}^{tq(\xi)} - 1| \,:\, \xi \in K_\varepsilon\} + \frac{\varepsilon\,\|f\|}{\mathrm{e}^{|\omega|} + 1}\sup\{|\,\mathrm{e}^{tq(\xi)} - 1| \,:\, \xi \in \Omega \setminus K_\varepsilon\}\\ &< \varepsilon\,\|f\| + \varepsilon\,\|f\| = 2\varepsilon\,\|f\|.\end{aligned}$$

□

5.4 Strongly continuous semigroups

In chapter 5.2 we already saw: If $\mathcal{T} = (T(t))_{t\geq 0}$ is a semigroup on a Banach space X, then

$$\mathcal{T} \text{ uniformly continuous} \iff T(t) = \exp(tA) \text{ for some } A \in L(X) \text{ and all } t \geq 0.$$

Moreover, T is differentiable and $\frac{d}{dt}T(0) = A$.

Now we use the latter property to assign a uniquely defined generator to strongly continuous semigroups.

DRAFT

Definition 5.22. Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup on X. The operator A, defined by

$$\mathcal{D}(A) := \Big\{ x \in X \,:\, \lim_{h\searrow 0} \tfrac{1}{h}(T(h)x - x) \text{ exists } \Big\},$$
$$Ax := \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x), \qquad x \in \mathcal{D}(A),$$

is called the (*infinitesimal*) *generator or* $\mathcal{T}$.

Remark. If $\mathcal{T}$ is a uniformly continuous semigroup, then this definition coincides with the definition of the generator in Definition 5.20.

Lemma 5.23. *Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ strongly continuous semigroup on X. For $x \in X$ we define the map $\tau_x : \mathbb{R}_+ \to X,\ t \mapsto \tau_x(t) = T(t)x$. Then the following is equivalent:*

(i) *τ_x is differentiable.*

(ii) *τ_x is differentiable in 0.*

In this case $\dot\tau_x(t) = T(t_0)\dot\tau_x(0)$.

Proof. (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (i) Let $t_0 > 0$, $h \in (0, t_0)$ and $x \in X$ such τ_x is differentiable in 0. The differentiability from the right of τ_x in t_0 follows from

$$\frac{1}{h}\big(\tau_x(t_0+h) - \tau_x(t_0)\big) = T(t_0)\,\frac{1}{h}\big(\tau_x(h) - \tau_x(0)\big) \longrightarrow T(t_0)\frac{d}{dt}\tau_x(0), \qquad h \to 0.$$

Differentiability from the left of τ_x in t_0 follows from

$$\begin{aligned}\frac{1}{h}\big(\tau_x(t_0) - &\tau_x(t_0-h)\big)\\ &= T(t_0-h)\left(\frac{1}{h}\big(\tau_x(h) - \tau_x(0)\big) - \frac{d}{dt}\tau_x(0)\right) + T(t_0-h)\frac{d}{dt}\tau_x(0)\\ &\longrightarrow T(t_0)\frac{d}{dt}\tau_x(0), \qquad h \to 0,\end{aligned}$$

because the first term converges to 0 (since $T(t_0 - h)$ is bounded uniformly bounded for $h \in (0, t_0)$ and the term in brackets tends to 0 by hypothesis). ☐

Corollary 5.24. *If $\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X with generator A, then*

$$\mathcal{D}(A) = \{x \in X \,:\, t \mapsto T(t)x \ \text{ is differentiable}\}.$$

Proposition 5.25. *Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ strongly continuous semigroup on X with generator A. Then the following holds:*

(i) *A is a linear operator.*

(ii) *If $x \in \mathcal{D}(A)$, then $T(t)x \in \mathcal{D}(A)$ for all $t \geq 0$ and the map $\tau_x : \mathbb{R}_+ \to X,\ t \mapsto T(t)x$ is differentiable with derivative*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \qquad t \geq 0.$$

(iii) *If $t \geq 0$ and $x \in X$, then $\int_0^t T(s)x \,\mathrm{d}s \in \mathcal{D}(A)$.*

(iv) *If $t \geq 0$, then*

$$T(t)x - x = A\int_0^t T(s)x \,\mathrm{d}s, \qquad x \in X, \tag{5.15}$$
$$T(t)x - x = \int_0^t T(s)Ax \,\mathrm{d}s, \qquad x \in \mathcal{D}(A). \tag{5.16}$$

Proof. (i) is clear.
(ii) If $x \in \mathcal{D}(A)$, then τ_x is differentiable with $\frac{d}{dt}\tau_x(0) = Ax$ and $\frac{d}{dt}T(t)x = \frac{d}{dt}\tau_x(t) = T(t)\frac{d}{dt}\tau(0) = T(t)Ax$. Hence also

$$\lim_{h\searrow 0}\frac{1}{h}\Big(T(h)T(t)x - T(t)x\Big) = \lim_{h\searrow 0}\frac{1}{h}\Big(T(t+h)x - T(t)x\Big) = T(t)Ax$$

exists and consequently $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T(t)Ax$.
(iii) and (5.15): Let $t \geq 0$, $h > 0$ and $x \in X$. The assertions follow from

$$\begin{aligned}\frac{1}{h}\Big(T(h)\int_0^t T(s)x \,\mathrm{d}s - \int_0^t T(s)x \,\mathrm{d}s\Big) &= \frac{1}{h}\Big(\int_h^{t+h} T(s)x \,\mathrm{d}s - \int_0^t T(s)x \,\mathrm{d}s\Big)\\ &= \frac{1}{h}\Big(\int_t^{t+h} T(s)x \,\mathrm{d}s - \int_0^h T(s)x \,\mathrm{d}s\Big) \longrightarrow T(t)x - T(0)x, \quad h \to 0.\end{aligned}$$

(iv) and (5.16): Let $x \in \mathcal{D}(A)$, $t \geq 0$ and $h > 0$. Define

$$\varphi_h : [0,t] \to X, \quad \varphi_h(s) = T(s)\frac{T(h)x - x}{h}$$

Then φ_h converges uniformly to $T(\cdot)Ax$ on $[0,t]$ for $h \to 0$. Hence we obtain

$$\begin{aligned}A\int_0^t T(s)x \,\mathrm{d}s &= \lim_{h\searrow 0}\,\frac{1}{h}\big(T(h) - \mathrm{id}\big)\int_0^t T(s)x \,\mathrm{d}s = \lim_{h\searrow 0}\,\int_0^t \frac{1}{h}\big(T(h) - \mathrm{id}\big)T(s)x \,\mathrm{d}s\\ &= \lim_{h\searrow 0}\,\int_0^t \varphi_h(s) \,\mathrm{d}s = \int_0^t \lim_{h\searrow 0}\,\varphi_h(s) \,\mathrm{d}s = \int_0^t T(s)Ax \,\mathrm{d}s.\end{aligned}$$ ☐

Recall that the semigroup $\mathcal{T}$ determines uniquely its generator A by Definition 5.22. Now we will show that the generator A determines uniquely the corresponding semigroup $\mathcal{T}$.

Proposition 5.26. *Let X be a Banach space, $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup and A its generator. Then $\mathcal{D}(A) \subseteq X$ is dense, A is closed and A determines the semigroup $\mathcal{T}$ uniquely.*

Proof. Since for every $x \in X$ the map $\mathbb{R}_+ \to X$, $t \mapsto T(t)x$ is continuous, 5.25 shows

$$\mathcal{D}(A) \ni \frac{1}{t}\int_0^t T(s)x \,\mathrm{d}s \longrightarrow x, \quad t \searrow 0.$$

Hence we proved that $\overline{\mathcal{D}(A)} = X$.

Given a sequence $(x_n)_n \subseteq \mathcal{D}(A)$ and $x, y \in X$ such that $x_n \to x$ and $Ax_n \to y$ for $n \to \infty$, we have to show that $x \in \mathcal{D}(A)$ and $Ax = y$. Note that

$$\begin{aligned}\frac{1}{t}\left(T(t)x - x\right) &= \lim_{n\to\infty}\frac{1}{t}\left(T(t)x_n - x_n\right) \overset{(5.16)}{=} \lim_{n\to\infty}\int_0^t T(s)Ax_n \,\mathrm{d}s \\ &\overset{(*)}{=} \int_0^t \lim_{n\to\infty} T(s)Ax_n \,\mathrm{d}s \overset{(+)}{=} \int_0^t T(s)y \,\mathrm{d}s,\end{aligned}$$

where $(*)$ holds because the map $[0,t] \to X$, $s \mapsto T(s)Ax_n$ converges uniformly to $s \mapsto T(s)y$, and $(+)$ follows because $T(s)$ is closed. Hence, by definition of A, $x \in \mathcal{D}(A)$ and

$$Ax = \lim_{t\searrow 0}\frac{1}{t}\left(T(t)x - x\right) = \lim_{t\searrow 0}\frac{1}{t}\int_0^t T(s)y \,\mathrm{d}s = y.$$

Let $\mathcal{S} = (S(t))_{t\geq 0}$ be a strongly continuous semigroup with generator A. We have to show that $S(t) = T(t)$, $t \geq 0$. For $x \in \mathcal{D}(A)$ and $t > 0$ we define $\eta : [0,t] \to X$, $\eta(s) := T(t-s)S(s)x$ (cf. the proof of Proposition 5.14). The function η is differentiable because for $s \in (0,t)$ and small enough $|h|$

$$\begin{aligned}&\frac{1}{h}\left(\eta(s+h) - \eta(s)\right) = \frac{1}{h}\Big(T(t-s-h)S(s+h)x - T(t-s)S(s)x\Big) \\ &= \underbrace{T(t-s-h)}_{\text{unif. bdd. w.r.t. } h}\frac{1}{h}\Big(S(s+h)x - S(s)x\Big) + \frac{1}{h}\Big(T(t-s-h) - T(t-s)\Big)\underbrace{S(s)x}_{\in\mathcal{D}(A)} \\ &\longrightarrow T(t-s)AS(s)x - T(t-s)AS(s)x = 0.\end{aligned}$$

Therefore η is constant on $[0,t]$ and it follows that

$$T(t)x = \eta(0) = \eta(t) = S(t)x.$$

Since $T(t)$ and $S(t)$ are bounded and $\mathcal{D}(A)$ is dense in X, we obtain $T(t) = S(t)$. □

Remark. Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup on X with generator A. A *classical solution* of

$$\frac{d}{dt}x = Ax(t), \quad t \geq 0, \qquad x(0) = x_0, \tag{5.17}$$

is a map $u : \mathbb{R}_+ \to X$ which is continuously differentiable, $u(t) \in \mathcal{D}(A)$ for all $t \geq 0$, and u solves the initial value problem (5.17). For an initial value $x_0 \in \mathcal{D}(A)$, the unique classical solution of (5.17) is $T(\cdot)x_0$. For $k \in \mathbb{N}$ and $x_0 \in \mathcal{D}(A^k)$ we have

$$T(\cdot)x_0 \in C^k([0,\infty), X) \cap C^{k-1}([0,\infty), \mathcal{D}(A)).$$

Lemma 5.27 (Scaling). *Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup on X with generator A. For every $\lambda \in \mathbb{C}$ and $\alpha > 0$, the family $\mathcal{S} = (S(t))_{t\geq 0}$ defined by $S(t) = \mathrm{e}^{t\lambda}T(\alpha t)$ is a strongly continuous semigroup on X with generator $B = \alpha A + \lambda\,\mathrm{id}$.*

Proof. Straightforward computation. □

Theorem 5.28. *Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup on X with generator A. Choose $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq M\,\mathrm{e}^{\omega t}$, $t \geq 0$ (cf. Proposition 5.7). Then the following holds:*

(i) *Fix $\lambda \in \mathbb{C}$. If for all $x \in X$ the improper integral*

$$R(\lambda)x := \int_0^\infty \mathrm{e}^{-s\lambda}\,T(s)x \,\mathrm{d}s \tag{5.18}$$

exists, then $\lambda \in \rho(A)$ and $R(\lambda) = R(\lambda, A)$.

(ii) *If $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > \omega$, then $\lambda \in \rho(A)$ and $R(\lambda) = R(\lambda, A)$ and we have the estimates*

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{\operatorname{Re}(\lambda) - \omega}, \tag{5.19}$$

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\operatorname{Re}(\lambda) - \omega)^n}, \quad n \in \mathbb{N}. \tag{5.20}$$

Proof. (i) Without restriction we can assume that $\lambda = 0$ (otherwise we rescale according to Lemma 5.27).
First we show that $\operatorname{rg}(R(0)) \subseteq \mathcal{D}(A)$ and $AR(0)x = -x$ for all $x \in X$. This follows from the definition of A and

$$\begin{aligned}\frac{1}{h}(T(h) - \mathrm{id})R(0)x &= \frac{1}{h}(T(h) - \mathrm{id})\int_0^\infty T(s)x \,\mathrm{d}s \\ &= \frac{1}{h}\int_0^\infty T(s+h)x \,\mathrm{d}s - \frac{1}{h}\int_0^\infty T(s)x \,\mathrm{d}s \\ &= -\frac{1}{h}\int_0^h T(s)x \,\mathrm{d}s \longrightarrow -x, \qquad \text{for } h \searrow 0.\end{aligned}$$

Now we show that $R(0)Ax = -x$ for all $x \in \mathcal{D}(A)$. We compute for $x \in \mathcal{D}(A)$

$$\begin{aligned} R(0)Ax &= \lim_{t\to\infty} \int_0^t T(s)Ax \,\mathrm{d}s \overset{(1)}{=} \lim_{t\to\infty} A \int_0^t T(s)x \,\mathrm{d}s \overset{(2)}{=} A \lim_{t\to\infty} \int_0^t T(s)x \,\mathrm{d}s \\ &= AR(0)x = -x. \end{aligned}$$

Note that (1) follows from Proposition 5.25 (iv) and (2) holds because A is closed.

(ii) Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega$. By (i) it suffices to show that $R(\lambda)x$ exists for all $x \in X$. This and the estimate (5.19) hold because for all $t \geq 0$

$$\begin{aligned} \left\| \int_0^t \mathrm{e}^{-s\lambda} T(s)x \,\mathrm{d}s \right\| &\leq \int_0^t \left\| \mathrm{e}^{-s\lambda} T(s)x \right\| \,\mathrm{d}s \leq M \|x\| \int_0^t \left|\mathrm{e}^{-s\lambda}\right| \mathrm{e}^{s\omega} \,\mathrm{d}s \\ &\leq M \|x\| \int_0^t \mathrm{e}^{s(\omega - \mathrm{Re}(\lambda))} \,\mathrm{d}s = M \|x\| \frac{1 - \mathrm{e}^{t(\omega - \mathrm{Re}(\lambda))}}{\mathrm{Re}(\lambda) - \omega} \\ &\leq \frac{M\|x\|}{\mathrm{Re}(\lambda) - \omega}. \end{aligned}$$

Now let $n \geq 2$. Using the von Neumann series, we have

$$(R(\lambda, A))^n = (\lambda - A)^{-n} = \frac{(-)^n}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} (\lambda - A)^{-1},$$

hence, with (5.18),

$$\begin{aligned} \|(R(\lambda, A))^n\| &= \frac{1}{(n-1)!} \left\| \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} \int_0^\infty \mathrm{e}^{-s\lambda} T(s)x \,\mathrm{d}s \right\| \\ &= \frac{1}{(n-1)!} \left\| \int_0^\infty s^{n-1} \mathrm{e}^{-s\lambda} T(s)x \,\mathrm{d}s \right\| \\ &\leq \frac{M\|x\|}{(n-1)!} \int_0^\infty s^{n-1} \mathrm{e}^{s(\omega - \mathrm{Re}(\lambda))} \,\mathrm{d}s = \frac{M\|x\|}{(\mathrm{Re}(\lambda) - \omega)^n}. \end{aligned}$$ □

Theorem 5.28 shows that the spectrum of a generator always lies in a left semi-plane of the complex plane.

Definition 5.29.

- If the integral in (5.18) exists, then it is called the *Laplace transform* of $T(\cdot)x$.
- If A is the generator of a strongly continuous semigroup $\mathcal{T}$, then

$$s(A) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}.$$

  is called the *spectral bound* of A.

If A is the generator of a strongly continuous semigroup $\mathcal{T}$, then

$$-\infty \leq s(A) \leq \omega_0(\mathcal{T}) < \infty.$$

Indeed, if $\mathrm{Re}(\lambda) > \omega$ then $\lambda \in \rho(A)$ by Theorem 5.28, so the spectral bound must be less or equal to ω.

Example 5.30 (Multiplication semigroup). Let $\Omega \in \mathbb{C}$ be a domain and $q \in C(\Omega, \mathbb{C})$ such that $\omega := \sup\{\mathrm{Re}(q(\xi)) : \xi \in \Omega\} < \infty$. Then

$$T_q(t) := M_{t\,\mathrm{e}^q}, \quad t \geq 0,$$

defines a strongly continuous semigroup $\mathcal{T}_q = (T_q(t))_{t\geq 0}$ on $X = C_0(\Omega)$, see Proposition 5.21.

Now we show that the generator of $\mathcal{T}_q$ is the multiplication operator M_q.

Proof. Let A be the generator of $\mathcal{T}_q$. Then, for all $f \in \mathcal{D}(A)$ and $\xi \in \Omega$,

$$(Af)(\xi) = \lim_{h\searrow 0} \frac{\mathrm{e}^{hq(\xi)} f(\xi) - f(\xi)}{h} = f(\xi) \lim_{h\searrow 0} \frac{\mathrm{e}^{hq(\xi)} - 1}{h} = f(\xi)q(\xi) = (M_q f)(\xi).$$

This proves that $A \subseteq M_q$. Observe that $\lambda \in \rho(A) \cap \rho(M_q)$ for large enough λ by assumption. Therefore we also have $M_q \subseteq A$. □

5.5 Generation theorems

Proposition 5.26 and Theorem 5.28 give necessary conditions for a linear operator to be generator of a strongly continuous semigroup. It must be densely defined, its spectrum must lie in a left half-plane of $\mathbb{C}$ and the powers of the resolvent must satisfy certain estimates. Now we show that this is sufficient.

Theorem 5.31 (Hille-Yosida-Phillips). *For a Banach space X, $A(X \to X)$ a densely defined linear operator and constants $M \geq 1$, $\omega \in \mathbb{R}$ the following is equivalent:*

(i) *A generates a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on X with*

$$\|T(t)\| \leq M\,\mathrm{e}^{t\omega}, \qquad t \geq 0.$$

(ii) *A is densely defined and closed, $\{\lambda \in \mathbb{R} : \lambda > \omega\} \subseteq \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \qquad n \in \mathbb{N},\ \lambda > \omega. \tag{5.21}$$

(iii) *A is densely defined and closed, $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > \omega\} \subseteq \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\mathrm{Re}\,\lambda - \omega)^n}, \qquad n \in \mathbb{N},\ \mathrm{Re}\,\lambda > \omega. \tag{5.22}$$

The idea of the proof is to approximate the operator A by bounded operators. For $n \in \mathbb{N}$, $n > \omega$ define the so-called *Yosida approximants*

$$A_n := n\,A\,R(n, A) = n^2\,R(n, A) - n. \tag{5.23}$$

Lemma 5.32. *Let X be a Banach space and $A(X \to X)$ a densely defined linear operator. Assume that there are $\omega \in \mathbb{R}$ and $M \geq 1$ such that (ii) from Theorem 5.31 is satisfied. For $\lambda > \omega$ let $A_\lambda := \lambda AR(\lambda, A)$ as in (5.23). Then $A_\lambda \in L(X)$ for all $\lambda > \omega$ and*

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x, \qquad x \in X, \tag{5.24}$$

$$\lim_{\lambda\to\infty} A_\lambda x = Ax, \qquad x \in \mathcal{D}(A). \tag{5.25}$$

Proof. A_λ is bounded because

$$A_\lambda = \lambda(A - \lambda + \lambda)(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda$$

Observe that $\|\lambda R(\lambda, A)\| \leq \frac{|\lambda| M}{\lambda - \omega}$, so $\lambda R(\lambda, A)$ is uniformly bounded in the interval $(\omega + 1, \infty)$ (i. e., there is a $c \in \mathbb{R}$ such that $\|\lambda R(\lambda, A)\| \leq c$ for all $\lambda > \omega + 1$). Since $\mathcal{D}(A)$ is dense in X, is suffices to prove (5.24) $x \in \mathcal{D}$. For such x we find

$$\|\lambda R(\lambda, A)x - x\| = \|R(\lambda, A)Ax\| \leq \frac{M}{\lambda - \omega}\|Ax\| \longrightarrow 0, \quad \lambda \to \infty.$$

From (5.24) we obtain for $x \in \mathcal{D}(A)$

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda AR(\lambda, A)x = \lim_{\lambda\to\infty} \lambda R(\lambda, A)Ax = Ax. \qquad \square$$

Proof of 5.31. (i) $\Rightarrow$ (iii) follows from Proposition 5.26 and Theorem 5.28.
(iii) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (i) Let $\mathbb{N}_{>\omega} := \{n \in \mathbb{N} : n > \omega\}$. For $n \in \mathbb{N}_{>\omega}$ let $\mathcal{T}_n = (T_n(t))_{t\geq 0}$ be the uniformly continuous semigroup generated by A_n. We will show that $T_n(t)$ converges strongly to some $T(t) \in L(X)$ for $n \to \infty$ and that $\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup with generator A.

Step 1: *Estimate for* $\|T_n(t)\|$.
For $t \geq 0$, $n \in \mathbb{N}_{>\omega}$ and $\omega_1 := \sup\left\{\frac{n|\omega|}{(n-\omega)} : n \in \mathbb{N}_{>\omega}\right\} < \infty$ we obtain

$$\|T_n(t)\| = e^{-tn}\,\|e^{tn^2 R(n,A)}\| \leq e^{-tn}\sum_{j=0}^{\infty}\frac{t^j n^{2j}}{j!}\|R(n, A)^j\|$$

$$\leq M e^{-tn}\sum_{j=0}^{\infty}\frac{t^j n^{2j}}{(n-\omega)^j j!} = M e^{-tn} e^{tn^2/(n-\omega)} = M e^{nt\omega/(n-\omega)} \leq M e^{t\omega_1}.$$

Step 2: Using the series representations we easily see that $T_n(t)A_m = A_m T_n(t)$ for all m, $n \in \mathbb{N}_{>\omega}$ and $t \geq 0$. Proposition 5.25 (ii) yields

$$T_n(t)x - T_m(t)x = \int_0^t \frac{d}{ds}\left(T_n(s)T_m(t-s)x\right)\,ds = \int_0^t T_n(s)T_m(t-s)\left(A_n x - A_m x\right)\,ds.$$

For $x \in \mathcal{D}(A)$ we use the estimate from Step 1 and formula 5.25 to obtain for $t \geq 0$

$$\|T_n(t)x - T_m(t)x\| \leq M^2\,\|A_n x - A_m x\| \int_0^t e^{2s\omega_1}\,ds \longrightarrow 0, \qquad n, m \to \infty. \tag{5.26}$$

Step 3: *For all $y \in X$ there exists $T(t)y := \lim_{n\to\infty} T_n(t)y$ where the convergence is uniform on intervals $[0, t_0]$ with $t_0 > 0$. In addition, $T(\cdot)y \in$* $C([0, t_0], X)$. (To keep notation simple, we write $T(\cdot)$ instead $T(\cdot)|_{[0,t_0]}$, etc.)
Fix $y \in X$ and $\varepsilon > 0$. Since $\mathcal{D}(A)$ is dense in $\subseteq X$, there exists an $x \in \mathcal{D}(A)$ such that $\|x - y\| < \varepsilon$. On finite intervals $[0, t_0]$, convergence in (5.26) is uniform with respect to t, hence there exists an $N \in \mathbb{N}_{>\omega}$ with $\|T_n(t)x - T_m(t)x\| < \varepsilon$ for all n, $m \geq N$ and $t \in [0, t_0]$. Consequently, for n, $m \geq N$ and $t \in [0, t_0]$,

$$\begin{aligned}\|T_n(t)y - T_m(t)y\| &\leq \|T_n(t)x - T_m(t)x\| + \|T_m(t)(y-x)\| + \|T_n(t)(y-x)\| \\ &\leq \varepsilon + (\|T_m(t)\| + \|T_n(t)\|)\,\|x - y\| \leq \left(1 + 2M e^{t_0\omega_1}\right)\varepsilon.\end{aligned}$$

Hence, for arbitrary $y \in X$, $(T_n(\cdot)y)_n$ is a Cauchy sequence in $C([0, t_0], X)$, and therefore it has a limit $T(\cdot)y \in C([0, t_0], X)$. Obviously, $T(t)y$ is independent of the choice of $t_0 > t$, so we obtain a function $T(\cdot)y$ which is well-defined on all of $\mathbb{R}_+$.
Step 4: *$\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup and $\|T(t)\| \leq M e^{t\omega}$, $t \geq 0$.*
Strong continuity of $\mathcal{T}$ was proved in Step 3. The semigroup property follows because on bounded intervals, $\mathcal{T}$ is the uniform strong limit of semigroups.

$$\|T(t)\| = \|\lim_{n\to\infty} T_n(t)\| \leq \lim_{n\to\infty} M e^{tn\omega/(n-\omega)} \leq M e^{t\omega}, \qquad t \geq 0.$$

Step 5: *A is the generator of $\mathcal{T} = (T(t))_{t\geq 0}$.*
Let B be the generator of $\mathcal{T}$. For $x \in \mathcal{D}(A)$ and $t_0 > 0$, $T_n(\cdot)x$ converges to $T(\cdot)x$ for $n \to \infty$, where the convergence is uniform on bounded intervals $[0, t_0]$. Since $A_n x \to Ax$ and $T_n \to T$ uniformly on $[0, t_0]$ for $n \to \infty$, also $\frac{d}{dt}T_n(\cdot)x = T_n(\cdot)A_n x$ converges to $T(\cdot)Ax$, uniformly on $[0, t_0]$. Hence $T(\cdot)x$ is differentiable and $\frac{d}{dt}T(t)x = T(t)Ax$, $t \in [0, t_0]$, implying that $x \in \mathcal{D}(B)$ and $Bx = \frac{d}{dt}T(0)x = T(0)Ax = Ax$. This shows $A \subseteq B$. For every $\lambda > \omega$ we have $\lambda \in \rho(A) \cap \rho(B)$, hence also $R(\lambda, A) \subseteq R(\lambda, B)$. From $\mathcal{D}(R(\lambda, A)) = X = \mathcal{D}(R(\lambda, B))$ it follows that $R(\lambda, A) = R(\lambda, B)$, so $A = B$. $\square$

We immediately obtain the following corollary for contractive groups.

Corollary 5.33 (Hille-Yosida). *For a Banach space X and a linear operator A on X the following is equivalent:*

(i) *A generates a strongly continuous contractive semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on X, that is,*

$$\|T(t)\| \leq 1, \qquad t \geq 0.$$

(ii) *A is densely defined and closed, $\{\lambda \in \mathbb{R} : \lambda > 0\} \subseteq \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}, \qquad \lambda > 0. \tag{5.27}$$

(iii) *A is densely defined and closed, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subseteq \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0. \tag{5.28}$$

Proof. The assertion follows with $M = 1$ and $\omega = 0$ from theorem 5.31 because $\|R(\lambda, A)^n\| \leq \|R(\lambda, A)\|^n \leq \frac{1}{\operatorname{Re}(\lambda)^n}$. □

Generator of strongly continuous groups

Definition 5.34. Let $\mathcal{S} = (S(t))_{t\in\mathbb{R}}$ strongly continuous group on a Banach space X. The operator A, defined by

$$\mathcal{D}(A) := \Big\{ x \in X : \lim_{h\to 0} \tfrac{1}{h}(S(h)x - x) \text{ exists } \Big\},$$
$$Ax := \lim_{h\to 0} \frac{1}{h}(S(h)x - x), \qquad x \in \mathcal{D}(A),$$

is called the (*infinitesimal*) *generator of $\mathcal{S}$*.

Obviously $\mathcal{T}_+ = (T_+(t))_{t\geq 0}$ and $\mathcal{T}_- = (T_-(t))_{t\geq 0}$ with $T_+(t) = S(t)$ and $T_-(t) = S(-t)$, $t \geq 0$ are strongly continuous semigroups on X with generator $\pm A$.

Theorem 5.35 (Generator theorem for strongly continuous groups). *Let X be a Banach space, A a linear operator on X, $M \geq 1$ and $\omega \in \mathbb{R}$. Then the following is equivalent:*

(i) *A generates a strongly continuous group $\mathcal{S} = (S(t))_{t\in\mathbb{R}}$ on X with*

$$\|S(t)\| \leq M\,\mathrm{e}^{|t|\omega}, \qquad t \in \mathbb{R}.$$

(ii) *A is densely defined and closed, $\{\lambda \in \mathbb{R} : |\lambda| > \omega\} \subseteq \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(|\lambda| - \omega)^n}, \qquad n \in \mathbb{N},\ |\lambda| > \omega. \tag{5.29}$$

(iii) *A is densely defined and closed, $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| > \omega\} \subseteq \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(|\operatorname{Re}\lambda| - \omega)^n}, \qquad n \in \mathbb{N},\ |\operatorname{Re}\lambda| > \omega. \tag{5.30}$$

(iv) *A and $-A$ generate strongly continuous semigroups $\mathcal{T}_\pm = (T_\pm(t))_{t\geq 0}$ with*

$$\|T_\pm(t)\| \leq M\,\mathrm{e}^{t\omega}, \qquad t \geq 0.$$

Proof. Exercise **??**. □

We saw in Theorem 5.31 that the generator A of a strongly continuous semigroup necessarily is densely defined. If this is not the case but all other assumptions of the Hille-Yosida-Phillips theorem (Theorem 5.31 (ii) and (iii) respectively) are satisfied, then the restriction of A to an appropriate subspace is generator of a strongly continuous semigroup. This semigroup is then defined only on a subspace of the original Banach space X.

Definition 5.36. Let X be a Banach space and $X_0 \subseteq X$ a subspace. For a linear operator A with domain $\mathcal{D}(A) \subseteq X$ (not necessarily dense in X) we define the *part of A in X_0* by

$$\mathcal{D}(A_|) = \{x \in \mathcal{D}(A) \cap X_0 : Ax \in X_0\}, \qquad A_| x = Ax, \quad x \in \mathcal{D}(A_|).$$

Lemma 5.37. *Let X be a Banach space, $A : \mathcal{D}(A) \subseteq X \to X$ a closed linear operator on X (not necessarily densely defined). Let $X_0 := \overline{\mathcal{D}(A)}$ and $A_|$ be the part of A in X_0. If there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\{\lambda \in \mathbb{R} : \lambda > \omega\} \subseteq \rho(A) \quad \text{and} \quad \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \quad n \in \mathbb{N},\ \lambda > \omega,$$

then $A_|$ is the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on X_0 with $\|T(t)\| \leq M\,\mathrm{e}^{t\omega}$, $t \geq 0$.

Proof. By assumption, X_0 is a Banach space. Note that $R(\lambda, A)(X_0) \subseteq \mathcal{D}(A_|)$ for $\lambda \in \rho(A)$ because $\mathcal{D}(A_|) = \{x \in \mathcal{D}(A) : Ax \in X_0\}$. Consequently, $\rho(A) \subseteq \rho(A_|)$ and $R(\lambda, A_|) \subseteq R(\lambda, A)$ for all $\lambda \in \rho(A)$. Hence, for $n \in \mathbb{N}$ and $\lambda > \omega$ we obtain $\|R(\lambda, A_|)^n\| \leq \|R(\lambda, A)^n\|$. Therefore, by the Hille-Yosida-Phillips theorem (Theorem 5.31), it suffices to prove that $\mathcal{D}(A_|)$ is dense in X_0. To show this, fix $x \in X_0$ and define $x_n = nR(n, A)x$ for $n \in \mathbb{N}$ and $n > \omega$. Observe that, because $x \in X_0$,

$$Ax_n = nAR(n, A)x = n(nR(n, A) - x) \in X_0,$$

hence $x_n \in \mathcal{D}(A_|)$. Lemma 5.32 shows that $x_n \to x$, $n \to \infty$, so the lemma is proved. □

Examples 5.38.

1. Translation semigroup on $BUC(\mathbb{R})$.

Let $X = BUC(\mathbb{R})$ and A be the linear operator on X be defined by

$$\mathcal{D}(A) = \{f \in X : f \in C^1(\mathbb{R}),\ f' \in X\}, \qquad Af = f'.$$

Then A generates the translation semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with $(T(t)f)(\xi) = f(t+\xi)$ for all $t \geq 0$, $f \in X$ and $\xi \in \mathbb{R}$.

Proof. (i) A is densely defined: Fix $f \in X$ and define (cf. proof of Theorem 5.16)

$$f_t(\xi) = \frac{1}{t}\int_0^t f(\xi+s)\,\mathrm{d}s, \qquad t>0,\ \xi \in \mathbb{R}.$$

Obviously, f_t is continuous and $\|f_t\| \leq \frac{1}{t}\int_0^t \|f\|\,\mathrm{d}s = \|f\| < \infty$. Moreover, f_t is uniformly continuous. To see this, fix $\varepsilon > 0$. Since f is uniformly continuous, there exits $\delta > 0$ such that $|f(\xi) - f(\eta)| < \varepsilon$ if $|\xi - \eta| < \delta$. Hence, for $\xi,\ \eta \in \mathbb{R}$ with $|\xi - \eta| < \delta$, it follows that

$$|\,f_t(\xi) - f_t(\eta)\,| \leq \frac{1}{t}\int_0^t \big|f(\xi+s) - f(\eta+s)\big|\,\mathrm{d}s \leq \varepsilon.$$

Clearly $f_t \in X,\ t>0$ and every f_t is continuously differentiable with derivative $f_t'(\xi) = \frac{1}{t}\big(f(t+\xi) - f(\xi)\big)$, hence we obtain $f_t \in \mathcal{D}(A),\ t>0$. Finally we show that $f_t \to f$ for $t \searrow 0$. Fix $\varepsilon > 0$ we choose $\delta > 0$ as above. Then, for all $t \in (0,\delta)$, we find

$$\|f_t - f\| \leq \sup_{\xi\in\mathbb{R}}\Big\{\frac{1}{t}\int_0^t \underbrace{|\,f(\xi+s) - f(\xi)\,|}_{<\varepsilon,\ \text{because } s\in(0,\delta)}\,\mathrm{d}s\Big\} < \varepsilon,$$

that is, $f_t \to f,\ t \to 0$.

(ii) A is closed and $\sigma(A) \subset \mathrm{i}\mathbb{R}$: For $\lambda \in \mathbb{C}\setminus \mathrm{i}\mathbb{R}$ define

$$g_\lambda(\xi) = \begin{cases} \int_\xi^\infty \mathrm{e}^{(\xi-s)\lambda} f(s)\,\mathrm{d}s, & \mathrm{Re}(\lambda)>0,\\ -\int_{-\infty}^\xi \mathrm{e}^{(\xi-s)\lambda} f(s)\,\mathrm{d}s, & \mathrm{Re}(\lambda)<0, \end{cases} \qquad \xi \in \mathbb{R}.$$

Obviously g_λ is continuous and we have $\|g_\lambda\| \leq \frac{\|f\|}{|\,\mathrm{Re}(\lambda)|}$. For instance, for $\mathrm{Re}(\lambda) > 0$ we have

$$\|g_\lambda\| = \sup_{\xi\in\mathbb{R}}\Big\{\Big|\int_\xi^\infty \mathrm{e}^{(\xi-s)\lambda}\ f(s)\,\mathrm{d}s\Big|\Big\} \leq \|f\| \sup_{\xi\in\mathbb{R}}\Big\{\int_\xi^\infty \mathrm{e}^{(\xi-s)\,\mathrm{Re}(\lambda)}\ \mathrm{d}s\Big\} = \frac{\|f\|}{\mathrm{Re}(\lambda)}. \tag{5.31}$$

The uniform continuity of g_λ follows for $\mathrm{Re}(\lambda) > 0$ from

$$\begin{aligned} |g_\lambda(\xi) - g_\lambda(\eta)| &= \Big|\int_\xi^\infty \mathrm{e}^{(\xi-s)\lambda} f(s)\,\mathrm{d}s - \int_\eta^\infty \mathrm{e}^{(\eta-s)\lambda} f(s)\,\mathrm{d}s\Big|\\ &= \Big|\int_\xi^\eta \mathrm{e}^{(\xi-s)\lambda} f(s)\,\mathrm{d}s + \int_\eta^\infty \mathrm{e}^{(\xi-s)\lambda} f(s) - \mathrm{e}^{(\eta-s)\lambda} f(s)\,\mathrm{d}s\Big|\\ &= \Big|\int_0^{\eta-\xi} \mathrm{e}^{s\lambda} f(s)\,\mathrm{d}s + \int_\eta^\infty \mathrm{e}^{(\eta-s)\lambda} f(s)\big[\mathrm{e}^{(\xi-\eta)\lambda} - 1\big]\,\mathrm{d}s\Big|\\ &\leq \|f\|\,\Big|\int_0^{\eta-\xi} \mathrm{e}^{-s\lambda}\ \mathrm{d}s\Big| + \Big|\big[\mathrm{e}^{(\xi-\eta)\lambda} - 1\big]\int_0^\infty \mathrm{e}^{-s\lambda} f(s)\,\mathrm{d}s\Big|, \end{aligned}$$

since the right side depends only of $\xi - \eta$ and converges to 0 if $\xi - \eta \to 0$. In summary, we showed $g_\lambda \in X$. Since obviously g_λ is continuously differentiable, it also follows that $g_\lambda \in \mathcal{D}(A)$ and an easy calculation shows $(A-\lambda)g_\lambda = f$. In particular, $\lambda - A$ is surjective. Injectivity of $\lambda - A$ follows because for $f \in C^1(\mathbb{R})$ we have

$$\lambda f - f' = 0 \qquad \Longleftrightarrow \qquad f(\xi) = c\,\mathrm{e}^{\xi\lambda}, \quad \xi \in \mathbb{R},$$

thus $f \in X$ if and only if $c = 0$. Because of (5.31), $\|(\lambda - A)^{-1} f\| = \|g_\lambda\| \leq \frac{\|f\|}{|\,\mathrm{Re}(\lambda)|}$ for all $f \in X$, i. e.,

$$\lambda \in \rho(A) \quad \text{and} \quad \|(A-\lambda)^{-1}\| \leq \frac{1}{|\,\mathrm{Re}\,\lambda|}, \qquad \lambda \in \mathbb{C}\setminus \mathrm{i}\mathbb{R}.$$

Hence $A - \lambda$ is closed by virtue of the closed graph theorem, hence also A is closed.

(iii) A is generator of a strongly continuous group $\mathcal{T}$: This follows from the Hille-Yosida theorem for contractive semigroups (Theorem 5.33) and the generator theorem for strongly continuous groups (Theorem 5.35).

(iv) Identify $\mathcal{T}$: For $f \in \mathcal{D}(A)$ define $u(t,\xi) = T(t)f(\xi)$ for $t,\ \xi \in \mathbb{R}$. Then $u \in C^1(\mathbb{R}\times\mathbb{R})$ and is a solution of

$$\begin{aligned} \frac{\partial}{\partial t}u(t,\xi) &= \frac{\partial}{\partial \xi}u(t,\xi), && \xi \in \mathbb{R},\ t>0,\\ u(0,\xi) &= f(\xi), && \xi \in \mathbb{R}. \end{aligned}$$

Let $v(t,\xi) = u(\xi - t,\ \xi + t)$ for $t,\ \xi \in \mathbb{R}$. Then $\frac{\partial}{\partial t}v(t,\xi) = 0$, hence $v(t,\xi) = v(\xi,\xi) = u(0, 2\xi)$ and therefore

$$T(t)f(\xi) = u(t,\xi) = v\Big(\frac{\xi+t}{2},\ \frac{\xi-t}{2}\Big) = u(0,\ \xi+t) = f(\xi+t), \qquad t,\ \xi \in \mathbb{R}. \ \square$$

2. Translation semigroup on $L_p(\mathbb{R})$.
Let $1 \le p < \infty$ and $X = L_p(\mathbb{R})$. Let $A(X \to X)$ be defined by

$$\mathcal{D}(A) = W^{1,p}(\mathbb{R}) = \{f \in X : f \text{ absolutely continuous}, \ f' \in X\}, \qquad Af = f'.$$

Then A generates the translation semigroup $\mathcal{T} = (T(t))_{t\ge 0}$ with

$$(T(t)f)(\xi) = f(t+\xi), \quad t \ge 0, \ f \in X, \ \xi \in \mathbb{R}.$$

Proof. See, e. g., [EN00, II.2.10, II.2.11]. □

3. Diffusion semigroup on $L_p(\mathbb{R}^n)$.
Let $1 < p < \infty$ and $X = L_p(\mathbb{R}^n)$. Then $\mathcal{T} = (T(t))_{t\ge 0}$, defined by $T(0) = \mathrm{id}$ and

$$(T(t)f)(\xi) \;=\; (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|\xi - s|^2}{4t}} f(s)\, \mathrm{d}s, \qquad \xi \in \mathbb{R}^n, \ f \in X, \ t > 0, \quad (5.32)$$

is the so-called *diffusion semigroup* (or *heat semigroup*).
$\mathcal{T}$ is a strongly continuous semigroup on X. Its generator A is

$$(Af)(\xi) = (\Delta f)(\xi) = \sum_{j=1}^{n} \frac{\partial^2}{\partial \xi_j^2} f(\xi), \qquad f \in \mathcal{D}(A), \ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

$$\mathcal{D}(A) = W^{2,p}(\mathbb{R}^n) = \{f \in L_p(\mathbb{R}^n) : f \text{ twice weakly diff'able and } \Delta f \in L_p(\mathbb{R}^n)\}.$$

Proof. See, e. g., [EN00, II.2.12, II.2.13]or [Wer00,]

(i) <u>$\mathcal{T}$ is a strongly continuous semigroup</u>:

Let $\gamma_t(s) := (4\pi t)^{-1} \mathrm{e}^{\frac{-|s|^2}{4t}}, \ t > 0, \ s \in \mathbb{R}^n$. It can be shown that

$$\gamma_t \in \mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \lim_{|\xi|\to\infty} |x|^k D^\alpha f(x) \to 0, \ k \in \mathbb{N}, \ \alpha \in \mathbb{N}^n\}$$

$\mathcal{S}(\mathbb{R}^n)$ is called the *Schwartz space*. It can be shown that $\mathcal{S}(\mathbb{R}^n) \subseteq L_p(\mathbb{R}^n)$ is dense for $p \ge 1$ and that $\mathcal{S}(\mathbb{R}^n)$ is invariant under Fourier transformation (see Section 4.4).
Observe that

$$T(t)f = \gamma_t * f, \qquad t > 0, \ f \in X,$$

hence Young's inequalities yields

$$\|T(t)f\|_p \le \|\gamma_t\|_1 \, \|f\|_p = \|f\|_p.$$

Hence we showed that $\|T(t)\| \le 1, \ t \ge 0$.
The semigroup properties of $\mathcal{T}$ follow from $\gamma_{t+s} = \gamma_s * \gamma_t$ (easy to verify) and the associativity of the convolution. Strong continuity of $\mathcal{T}$ can be shown using measure theory.

(ii) <u>Generator of $\mathcal{T}$</u>: We show the assertion only for $p = 2$.
Let A be the generator of $\mathcal{T}$. ....

□

5.6 Dissipative operators, contractive semigroups

Definition 5.39. Let X be a Banach space and A a (not necessarily densely defined) linear operator on X. A is called *dissipative* if

$$\|(\lambda - A)x\| \;\ge\; \lambda \|x\|, \qquad \lambda > 0, \ x \in \mathcal{D}(A).$$

Proposition 5.40. *If A is a dissipative operator on a Banach space X, then*

(i) *$\lambda - A$ is injective for $\lambda > 0$ and*

$$\|(\lambda - A)^{-1} y\| \le \frac{1}{\lambda}\, \|y\|, \qquad \lambda > 0, \ y \in \operatorname{rg}(\lambda - A).$$

(ii) *$\lambda - A$ is surjective for some $\lambda > 0$ $\iff$ $\lambda - A$ is surjective for all $\lambda > 0$.*
In this case, $(0, \infty) \subseteq \rho(A)$.

(iii) *A is closed $\iff$ $\operatorname{rg}(\lambda - A)$ is closed for some $\lambda > 0$,*
$\iff$ $\operatorname{rg}(\lambda - A)$ is closed for all $\lambda > 0$.

(iv) *If $\operatorname{rg}(\lambda - A) \subseteq \overline{\mathcal{D}(A)}$, then A <u>is closable</u>. In this case, also its closure $\overline{A}$ is dissipative and $\operatorname{rg}(\lambda - \overline{A}) = \overline{\operatorname{rg}(\lambda - A)}, \ \lambda > 0$.*

Proof. (i) is clear. (ii) Assume that $\lambda_0 - A$ is surjective for a $\lambda_0 > 0$. Then $\lambda_0 \in \rho(A)$ and $\|R(\lambda_0, A)\| \le \frac{1}{\lambda_0}$ by (i). For $\mu \in (0, 2\lambda_0)$ the operator

$$\mu - A = \mu - \lambda_0 + \lambda_0 - A = \Big((\mu - \lambda_0) R(\lambda_0, A) + \mathrm{id}\Big)(\lambda_0 - A)$$

is bijective by the theorem of von Neumann because $\|(\mu - \lambda_0)R(\lambda_0, A)\| < 1$, hence $(0, \, 2\lambda_0) \subseteq \rho(A)$. By induction, $(0, \infty) \subseteq \rho(A)$.
(iii) To show that A is closed, it suffices to show that $\lambda - A$ is closed for some (and then for all) $\lambda > 0$. This is equivalent to

$$(\lambda - A)^{-1} : \ \operatorname{rg}(\lambda - A) \to X$$

being closed for some (all) $\lambda > 0$. By the closed graph theorem, this is the case if and only if $\operatorname{rg}(\lambda - A)$ is close for some (all) $\lambda > 0$.
(iv) Assume that $\operatorname{rg}(\lambda - A) \subseteq \overline{\mathcal{D}(A)}$. Let $y \in X$ and $(x_n)_{n\in\mathbb{N}}$ with $x_n \to 0$ and $Ax_n \to y$ for $n \to \infty$. We have to show that $y = 0$. For all $w \in \mathcal{D}(A)$ and $\lambda > 0$ the following holds

$$\|\lambda(\lambda - A)x_n - (\lambda - A)w\| \ge \lambda \, \|\lambda x_n - w\|.$$

Taking the limit $n \to \infty$ we obtain

$$\begin{aligned} & \|\lambda y - (\lambda - A)w\| \ge \lambda \, \|w\|, \\ \Longrightarrow \quad & \|y - w - \lambda^{-1} A w\| \ge \|w\|, \\ \overset{\lambda\to\infty}{\Longrightarrow} \quad & \|y - w\| \ge \|w\|. \end{aligned}$$

Since $y \in \overline{\operatorname{rg}(A)} \subseteq \overline{\mathcal{D}(A)}$, there exists a sequence $(w_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ which converges to y. The inequality above yields $\|y\| = \lim_{n\to\infty} \|w_n\| \le \lim_{n\to\infty} \|y - w_n\| = 0$.

For the proof of the dissipativity of $\overline{A}$ fix $x \in \mathcal{D}(\overline{A})$. By assumption there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $Ax_n \to \overline{A}x$ for $n \to \infty$. Since $\|\cdot\|$ is continuous, it follows that

$$\|(\lambda - A)x\| \;=\; \lim_{n\to\infty} \|(\lambda - A)x_n\| \;\ge\; \lambda \lim_{n\to\infty} \|x_n\| \;=\; \lambda\|x\|.$$

Using that $\operatorname{rg}(\lambda - A)$ is dense in $\operatorname{rg}(\lambda - \overline{A})$, we find $\overline{\operatorname{rg}(\lambda - A)} = \overline{\operatorname{rg}(\lambda - \overline{A})} = \operatorname{rg}(\lambda - \overline{A})$. The last equality follows from (iii) because $\overline{A}$ is closed. ☐

In the special case of Hilbert spaces we have the following lemma.

Lemma 5.41. *Let H be a Hilbert space and A a linear operator on H. Then*

$$A \text{ dissipative} \qquad \Longleftrightarrow \qquad \operatorname{Re}\langle Ax\,, x\rangle \le 0, \quad x \in \mathcal{D}(A).$$

Proof. "$\Longleftarrow$" Fix $x \in \mathcal{D}(A)$, without restriction we assume $\|x\| = 1$. Then, for $\lambda > 0$,

$$\begin{aligned}\|(\lambda - A)x\| &= \|(\lambda - A)x\|\,\|x\| \ge |\langle(\lambda - A)x\,, x\rangle| \ge \operatorname{Re}\big(\lambda - \langle Ax\,, x\rangle\big)\\ &= \lambda - \operatorname{Re}\big(\langle Ax\,, x\rangle\big) \ge \lambda.\end{aligned}$$

"$\Longrightarrow$" Fix $x \in \mathcal{D}(A)$, without restriction we assume $\|x\| = 1$. For $\lambda > 0$ define $x_\lambda = \|(\lambda - A)x\|^{-1}(\lambda - A)x$. Then $\lim_{\lambda\to\infty} x_\lambda = \lim_{\lambda\to\infty} \|x - \lambda^{-1} Ax\|^{-1} (x - \lambda^{-1}A)x = x$ and, by hypothesis,

$$\begin{aligned}\lambda \le \|(\lambda - A)x\| &= \langle(\lambda - A)x\,, x_\lambda\rangle = \operatorname{Re}\langle\lambda x\,, x_\lambda\rangle - \operatorname{Re}\langle Ax\,, x_\lambda\rangle\\ &\le \lambda\|x\|\,\|x_\lambda\| - \operatorname{Re}\langle Ax\,, x_\lambda\rangle = \lambda - \operatorname{Re}\langle Ax\,, x_\lambda\rangle.\end{aligned}$$

Hence it follows that $\operatorname{Re}\langle Ax\,, x_\lambda\rangle \le 0$. ☐

Lemma 5.42. *Let H be a Hilbert space and A a dissipative operator on H. If $\lambda - A$ is surjective for some $\lambda > 0$, then A is densely defined.*

Proof. By Proposition 5.40 (ii) we know that $\lambda \in \rho(A)$. We have to show that $\operatorname{rg}(\lambda - A)^{-1}$ is dense in H. Choose $v \in \operatorname{rg}(\lambda - A)^{\perp}$. Hence $\langle v\,, (\lambda - A)^{-1}u\rangle = 0$, $u \in H$. In particular, taking $u = v$ yields

$$\begin{aligned}0 &= \langle v\,, (\lambda - A)^{-1}v\rangle = \langle(\lambda - A)(\lambda - A)^{-1}v\,, (\lambda - A)^{-1}v\rangle\\ &= \lambda\|(\lambda - A)^{-1}v\|^2 - \operatorname{Re}\langle A(\lambda - A)^{-1}v\,, (\lambda - A)^{-1}v\rangle \ge \lambda\|(\lambda - A)^{-1}v\|^2 \ge 0,\end{aligned}$$

hence $\|(\lambda - A)^{-1}v\| = 0$. Since $(\lambda - A)^{-1}$ is injective, it follows that $v = 0$, as we wanted to show. ☐

Lemma 5.41 and Lemma 5.42 are special cases of the following lemmas:

Dissipative operators in Banach spaces

Definition 5.43. Let X be a Banach space with dual space X'. For every $x \in X$ we call

$$\mathcal{J}(x) := \big\{\, x' \in X' \,:\, \langle x\,, x'\rangle = \|x\|^2 = \|x'\|^2 \,\big\}. \tag{5.33}$$

the *duality set* of x.

By the Hahn-Banach theorem $\mathcal{J}(x) \neq \{0\}$. The elements $x' \in \mathcal{J}(x)$ are called *normalised tangent functionals to x*. If X is a Hilbert space, then $\mathcal{J}(x)$ consists of exactly one element.

In analogy to Lemma 5.41 we have:

Lemma. *Let X be a Banach space and A a linear operator on X. Then*

$$A \text{ dissipative} \quad \Longleftrightarrow \quad \forall\, x \in \mathcal{D}(A) \;\; \exists\, j(x) \in \mathcal{J}(x) \;:\; \operatorname{Re}\langle Ax\,, j(x)\rangle \le 0.$$

If X is a reflexive Banach space, then in analogy to Lemma 5.42 we have:

Lemma. *Let X be a reflexive Banach space and A a dissipative operator on X. If $\lambda - A$ is surjective for some $\lambda > 0$, then A is densely defined.*

Theorem 5.44 (Lumer-Phillips). *Let X be a Banach space and A a densely defined dissipative linear operator on X. Then the following is equivalent:*

(i) *$\overline{A}$ generates a contractive semigroup.*

(ii) *There exists some $\lambda > 0$ such that $\operatorname{rg}(\lambda - A)$ is dense in X.*

Proof. (i) ⇒ (ii) By the Hille-Yosida theorem (Corollary 5.33) we know that $\operatorname{rg}(\lambda - \overline{A}) = X$, consequently by Proposition 5.40 $\overline{\operatorname{rg}(\lambda - A)} = \operatorname{rg}(\lambda - \overline{A}) = X$.
(ii) ⇒ (i) Since $\mathcal{D}(A)$ is dense in X, Proposition 5.40 (iv) shows that A is closable and that $\overline{A} - \lambda$ is surjective for every $\lambda > 0$. Proposition 5.40 (i) yields that $\lambda \in \rho(A)$ and $\|R(\lambda, \overline{A})\| \le \frac{1}{\lambda}$. Therefore, by the Hille-Yosida theorem (Corollary 5.33) $\overline{A}$ the generator of a contractive semigroup. ☐

Remark. Let H be a Hilbert space and A a linear operator on H. If

(i) $\langle Ax\,, x\rangle \le 0$, $x \in \mathcal{D}(A)$

(ii) $\overline{\operatorname{rg}(\lambda - A)} = H$ for some $\lambda > 0$,

then $\overline{A}$ generates a contractive semigroup on H. The hypothesis (i) shows that A is dissipative, together with condition (ii) it follows that A is densely defined (Proposition 5.42). The Lumer-Phillips theorem implies then that $\overline{A}$ generates a strongly continuous semigroup.

In particular for spaces of functions, the conditions (i) and (ii) are often easier to check the hypothesis in the Hille-Yosida theorem.

Example 5.45. Let $X = C([0,1])$ and the linear operator A on X be defined by

$$Af = f', \quad f \in \mathcal{D}(A) = \{ f \in C^1([0,1]) \,:\, f(0) = 0,\ f' \in C([0,1]) \}.$$

Then A is closed, $\lambda - A$ is bijective for every $\lambda \in \mathbb{C}$ and

$$R(\lambda, A)f(\xi) \,=\, \int_0^\xi \mathrm{e}^{-(\xi - s)\lambda} f(s)\, \mathrm{d}s, \qquad \xi \in [0,1],\ \lambda \in \mathbb{C},\ f \in X.$$

For $\lambda \neq 0$, the estimate

$$\|R(\lambda, A)f\| \,\leq\, \|f\| \sup_{\xi \in [0,1]} \int_0^\xi \mathrm{e}^{-(\xi - s)\operatorname{Re}\lambda}\, \mathrm{d}s \,=\, \frac{1}{\lambda}\|f\|\,(1 - \mathrm{e}^{-\operatorname{Re}\lambda}) \,\leq\, \frac{1}{\lambda}\|f\|,$$

shows that A is dissipative.

However, A is not densely defined and therefore does not generate a strongly continuous semigroup on X. By Lemma 5.37, A induces a strongly continuous semigroup on the subspace

$$X_0 = \overline{\mathcal{D}(A)} = \{ f \in X \,:\, f(0) = 0 \}.$$

Let $A_|$ be the part of A in X, that is,

$$A_| f = f', \qquad f \in \mathcal{D}(A_|) = \{ f \in X \,:\, f \in C^1([0,1]),\ f(0) = f'(0) = 0 \}.$$

Then $A_|$ is densely defined in X_0 (Lemma 5.37), dissipative and $\lambda - A_| : X_0 \to X_0$ is surjective, hence $A_|$ generates a strongly continuous semigroup by the Lumer-Phillips theorem (Theorem 5.44).

Definition 5.46. A (strongly continuous) semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on a Banach space X is called a (*strongly continuous*) *unitary semigroup*, if every $T(t)$, $t \geq 0$, is unitary. Analogously, (*strongly continuous*) *unitary groups* are defined.

Theorem 5.47 (Stone). *Let H be a Hilbert space and A a densely defined linear operator on H. Then the following is equivalent:*

(i) *A generates a unitary group $\mathcal{T} = (T(t))_{t \in \mathbb{R}}$ on H.*

(ii) *A is skew-selfadjoint, that is, $A^* = -A$.*

Proof. (i) $\Rightarrow$ (ii) Observe that $T(t)^* = T(t)^{-1} = T(-t)$ for all $t \in \mathbb{R}$ by assumption. Hence $\mathcal{T}^* = (T(t)^*)_{t \in \mathbb{R}}$ is a strongly continuous group with generator $-A$.
If $x \in \mathcal{D}(A)$, then

$$\langle x\,, Ay \rangle = \lim_{t \searrow 0} \Big\langle x\,,\ \frac{1}{t}(T(t) - \mathrm{id})\, y \Big\rangle = \lim_{t \searrow 0} \Big\langle \frac{1}{t}(T(t)^* - \mathrm{id})\, x\,,\ y \Big\rangle = \langle -Ax\,,\, y \rangle,$$

so $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $A^*x = -Ax$ for $x \in \mathcal{D}(A)$.
It remains to show that $A^* \subseteq -A$. Note that this is equivalent to show that $(\mathrm{i}A)^* \subseteq \mathrm{i}A$. By what we already showed, we know that $\mathrm{i}A$ is symmetric. It is closed because it is the generator of a strongly continuous semigroup. Hence it suffices to show that $\pm\mathrm{i}$ belong to the resolvent set of $\mathrm{i}A$ (see Corollary 3.12). Note that A generates a contractive semigroups, so $\mathbb{R} \setminus \{0\} \subseteq \rho(A)$ (Theorem 5.35). Hence $\mathrm{i}\mathbb{R} \setminus \{0\} \subseteq \rho(\mathrm{i}A)$ which completes the proof.
Alternative proof of "$A^* \subseteq -A$":
Let $x \in \mathcal{D}(A^*)$. Since $-A$ is the generator of $\mathcal{T}^*$ ist, Proposition 5.25 (iv) shows that

$$\frac{1}{t}\,(T(t)^*x - x) = \frac{1}{t}\,(-A)\int_0^t T(s)^*x\, \mathrm{d}s.$$

Using that $-A \subseteq A^*$ and $T(s)^*x \in \mathcal{D}(-A) \subseteq \mathcal{D}(A^*)$ for all $s \in [0,t]$, we conclude

$$\frac{1}{t}\,(T(t)^*x - x) = \frac{1}{t}\,A^*\int_0^t T(s)^*x\, \mathrm{d}s = \frac{1}{t}\int_0^t A^*\,T(s)^*x\, \mathrm{d}s.$$

Note that $\langle T(s)^*x\,, Ay \rangle = \langle x\,, T(s)Ay \rangle = \langle x\,, AT(s)y \rangle = \langle A^*x\,, T(s)y \rangle$, $y \in \mathcal{D}(A)$, so that $T(s)^*x \in \mathcal{D}(A^*)$. Since $T(s)$ is bounded, it follows that $A^*T(s)^* = (T(s)A)^*$. Note that A and $T(s)$ commute and, because of $(AT(s))^* \supseteq T(s)^*A^*$, it follows that

$$\frac{1}{t}\,(T(t)^*x - x) = \frac{1}{t}\int_0^t T(s)^*A^*x\, \mathrm{d}s \ \xrightarrow{t \to 0}\ T(0)^*A^*x = A^*x.$$

The last equality holds because $s \to T(s)^*A^*x$ in continuous in 0. Consequently, $x \in \mathcal{D}(-A)$ (because $-A$ is the generator or $\mathcal{T}^*$) and we have $-Ax = A^*x$.

(ii) $\Rightarrow$ (i) By assumption, A and $-A$ are densely defined and closed and

$$\langle Ax\,, x \rangle = \langle x\,, A^*x \rangle = -\langle x\,, Ax \rangle = -\overline{\langle Ax\,, x \rangle}, \quad x \in \mathcal{D}(A),$$

hence A and $-A$ are dissipative. By the Lumer-Phillips theorem (Theorem 5.44), both A and $-A$ generate contractive semigroups, hence A generates a contractive group $\mathcal{T} = (T(t))_{t \in \mathbb{R}}$ on H (see Theorem 5.35). It remains to be proved that $T(t)^* = T(t)^{-1}$, $t \in \mathbb{R}$.
For every $s \in \mathbb{R}$, $T(s)$ is surjective (because it is even invertible) and isometric because

$$\|x\| = \|T(s)^{-1}T(s)x\| \leq \|T(s)^{-1}\|\, \|T(s)x\| \leq \|T(-s)\|\, \|T(s)\|\, \|x\|, \quad x \in H.$$

Since $\|T(s)\| \leq 1$, $s \in \mathbb{R}$, (recall that $\mathcal{T}$ is a contractive semigroup) the above inequality shows that $\|x\| = \|T(s)x\|$, $x \in H$. Therefore $T(s)$ is unitary (see, e.g., [Kat95, V § 2.2]). ☐

Remark 5.48. By scaling we can always convert a strongly continuous semigroup on a Banach space X in a bounded strongly continuous semigroup. The spectrum of the generator is then shifted to the left (Lemma 5.27). But we do not necessarily obtain a contractive semigroup.

The next lemma shows that there exists a norm on X, equivalent to the original norm, such that the semigroup is a contractive semigroup. Therefore the Lumer-Phillips theorem is true for arbitrary strongly continuous semigroups.

Lemma 5.49. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a bounded strongly continuous semigroup on X. Then*

$$\|x\|_{\mathcal{T}} := \sup\{\, \|T(s)x\| : s \geq 0 \,\}, \quad x \in X,$$

defines a norm which is equivalent to $\|\cdot\|$, and $\mathcal{T}$ is a contractive semigroup on $(X, \|\cdot\|_{\mathcal{T}})$.

Proof. Since $\mathcal{T}$ is a bounded semigroup, there exists $M \geq 1$ such that $\|T(s)\| \leq M$ for all $s \geq 0$. It is easy to check that $\|\cdot\|_{\mathcal{T}}$ has all properties of a norm. Moreover,

$$\|x\| = \|T(0)x\| \leq \|x\|_{\mathcal{T}} = \sup\{\, \|T(s)x\| : s \geq 0 \,\} \leq M\|x\|, \quad x \in X,$$

therefore $\|\cdot\|$ and $\|\cdot\|_{\mathcal{T}}$ are equivalent. If $x \in X$ and $t \geq 0$, then

$$\begin{aligned} \|T(t)x\|_{\mathcal{T}} &= \sup\{\, \|T(t)T(s)x\| : s \geq 0 \,\} = \sup\{\, \|T(t+s)x\| : s \geq 0 \,\} \\ &\leq \sup\{\, \|T(s)x\| : s \geq 0 \,\} = \|x\|_{\mathcal{T}}, \end{aligned}$$

hence $\|T(t)\| \leq 1$, $t \geq 0$. $\square$

Chapter 6

Analytic semigroups

In Proposition 5.40 it was shown that the spectrum of a dissipative operator A lies in a left semiplane in $\mathbb{C}$ and that the resolvent on the right semiaxis satisfies the estimate $\|R(\lambda, A)\| \le \lambda^{-1}$ if $\lambda - A$ is surjective for $\lambda > 0$.

In this chapter we deal with linear operators whose spectrum lies in a sector and whose resolvent satisfies a certain estimate outside of the sector.

Let us recall:

Cauchy's integral formula. Let $\Omega \subset \mathbb{C}$ be a domain, $z_0 \in \Omega$, $r > 0$ such that the closed disk $K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \le r\}$ belongs to Ω. If $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial K_r(z_0)} \frac{f(\zeta)}{\zeta - z_0}\, \mathrm{d}\zeta,$$

where $\partial K_r(z_0)$ is the positively oriented boundary of $K_r(z_0)$.
More generally, if γ is a closed path in $\Omega \setminus \{z_0\}$ and $\nu(z_0, \gamma)$ is the winding number of γ around z_0, then

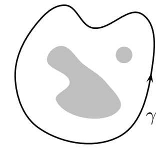
$$f^{(n)}(z_0)\nu(z_0,\gamma) = \frac{n!}{2\pi \mathrm{i}} \int_{\partial K_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta.$$

Dunford functional calculus. Let X be a Banach space and A a densely defined linear operator on X. Then the map $\rho(A) \to L(X),\ \ \lambda \to R(\lambda, A)$, is holomorphic. If A is a everywhere defined bounded operator, then $\sigma(A)$ bounded. Let $\Omega \subseteq \mathbb{C}$ be a domain with $\sigma(A) \subseteq \Omega$ and γ a closed path which lies in Ω and goes around every point in $\sigma(A)$ exactly once positively oriented. Then we define for holomorphic $f : \Omega \to \mathbb{C}$

$$f(A) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(\zeta) R(\zeta, A)\, \mathrm{d}\zeta. \tag{6.1}$$

This definition does not depend on the choice of γ.

If A is selfadjoint, then $f(A)$ defined in the definition (6.1) coincides with the definition with the help of the spectral family (Definition **??**).



Examples:

(i) $f : \mathbb{C} \to \mathbb{C},\ \ f(z) = 1$, then $f(A) = \mathrm{id}$.

Proof. For arbitrary $y \in X'$ the map $z \mapsto \langle (z - A)^{-1}x\,, y\rangle$ is holomorphic in $\rho(A)$. Without restriction, we can assume that $\gamma = K_r(0)$ for large enough r. Then

$$\begin{aligned} f(A)x &= \frac{1}{2\pi \mathrm{i}} \int_\gamma R(\zeta, A)x\, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\zeta}\Big(1 - \frac{1}{\zeta}R(\zeta, A)\Big)^{-1} x\, \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\zeta} \underbrace{\sum_{n=0}^{\infty} \zeta^{-n} A^n x}_{\text{converges unif. for } \zeta \in \gamma}\, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{\infty} \underbrace{\int_\gamma \zeta^{-n-1} A^n x\, \mathrm{d}\zeta}_{\substack{=0, \text{ falls } n \ge 1 \\ =2\pi\mathrm{i}, \text{ falls } n=0}} \\ &= x. \end{aligned}$$

(ii) $f : \mathbb{C} \to \mathbb{C},\ \ f(z) = z$, then $f(A) = A$.

(iii) For the exponential function $\exp(tA)$ as in Definition **??**

$$\exp(tA) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{t\zeta}\, R(\zeta, A)\, \mathrm{d}\zeta.$$

For unbounded operators, the spectrum is in general unbounded. Therefore, the functional calculus described above cannot be applied to unbounded operators without additional assumptions. For sectorial operators there is an integral representation of the generated semigroup.

For $\varphi \in (0, \pi]$ we define the (open) sector

$$\Sigma_\varphi := \{z \in \mathbb{C} : |\arg z| < \varphi\} \setminus \{0\}.$$

Definition 6.1. Let X be a Banach space. A densely defined linear operator $A(X \to X)$ is called *sectorial with angle δ* if there exists a $\delta \in (0, \pi/2]$ such that

$$\Sigma_{\pi/2+\delta} \subseteq \rho(A),$$

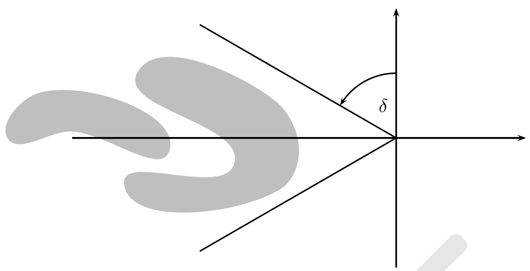


FIGURE 6.1: Spectrum of a sectorial operator.

and if for every $\varepsilon \in (0, \delta)$ there exists an $C_\varepsilon \geq 0$ such that

$$\|R(\lambda, A)\| \leq \frac{C_\varepsilon}{|\lambda|}, \qquad \lambda \in \overline{\Sigma_{\pi/2+\delta-\varepsilon}} \setminus \{0\}. \tag{6.2}$$

Definition 6.2. Let X be a Banach space and A a sectorial operator on X with angle $\delta \in (0, \pi/2]$. We define $T(0) := \mathrm{id}$ and for $z \in \Sigma_\delta$ we define $T(z)$ as follows. Choose an arbitrary $\delta' \in (|\arg(z)|, \delta)$ and define

$$T(z) := \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\mu z} R(\mu, A)\, \mathrm{d}\mu, \tag{6.3}$$

where γ is an arbitrary piecewise smooth contour in $\Sigma_{\pi/2+\delta}$ from $\infty\, \mathrm{e}^{-\mathrm{i}(\delta'+\pi/2)}$ to $\infty\, \mathrm{e}^{\mathrm{i}(\delta'+\pi/2)}$, see Figure 6.2.

The condition $z \in \Sigma_\delta$ guarantees that $\arg(\mu z) \in (\pi/2 + \varepsilon,\ 3\pi/2 - \varepsilon)$ for sufficiently small $\varepsilon > 0$, so that $\mathrm{Re}(\mu z) \sim -C|\mu|$ for a positive constant C for $|\mu|$ large enough. Consequently, the norm of the integrand decays exponentially and the integral is well-defined. More precisely:

Proposition 6.3. *Let X be a Banach space and A a sectorial operator on X with angle $\delta \in (0, \pi/2]$. Then (6.2) defines a bounded linear operator and*

(i) $\|T(z)\|$ *is uniformly bounded for $z \in \Sigma_{\delta'}$ for every $\delta' \in (0, \delta)$.*

(ii) *The map $z \mapsto T(z)$ is analytic.*

(iii) $T(z_1 + z_2) = T(z_1)T(z_2), \quad z_1, z_2 \in \Sigma_\delta$.

(iv) *The map $z \mapsto T(z)$ is strongly continuous in $\Sigma_{\delta'} \cup \{0\}$ for every $\delta' \in (0, \delta)$.*

(v) $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup with generator A.*

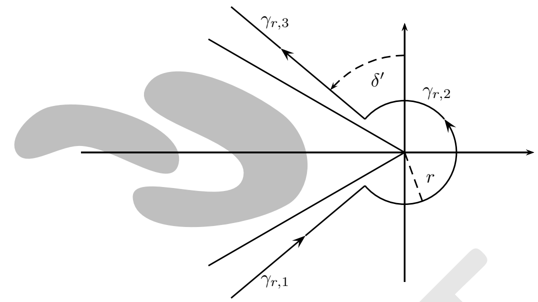


FIGURE 6.2: Path of integration $\gamma_{r,\delta'}$.

Definition 6.4. Let $\delta \in (0, \pi/2]$. A family $\mathcal{T} = (T(z))_{z\in\Sigma_\delta} \subseteq L(X)$ is called a *bounded analytic semigroup with angle δ* if

(i) $T(0) = \mathrm{id}$ and $T(z_1 + z_2) = T(z_1)T(z_2),\ z_1, z_2 \in \Sigma_\delta$;

(ii) $z \mapsto T(z)$ is analytic in Σ_δ;

(iii) $\lim\limits_{\substack{z\to 0\\ z\in\Sigma_{\delta'}}} T(z)x = x,\ \delta' \in (0, \delta),\ x \in X$ (strong continuity in sectors $\Sigma_{\delta'}$).

If in addition the following holds,

(iv) for every $\delta' \in (0, \delta)$ there exits an $M_{\delta'}$ such that $\|T(z)\| \leq M_{\delta'}$ for all $z \in \Sigma_{\delta'}$,

then $\mathcal{T}$ is called an analytic semigroup.

Remark. For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with generator A the maps $[0, \infty) \to X,\ t \mapsto T(t)x$ are differentiable for every $x \in \mathcal{D}(A)$. If $\mathcal{T}$ is an analytic semigroup with angle δ, then $T(\cdot)$ is norm differentiable in every sector $\Sigma_{\delta'}$ with $0 < \delta' < \delta$.

Remark. If $\mathcal{T}$ is an analytic semigroup and its restriction to real t is a bounded strongly continuous semigroup, then $\mathcal{T}$ is not necessarily a bounded analytic semigroup. For instance, the multiplication semigroup $(\mathrm{e}^{iz})_{z\in\mathbb{C}}$ on $X = \mathbb{C}$ is a non-bounded analytic semigroup whose restriction $(\mathrm{e}^{it})_{t\geq 0}$ to $\mathbb{R}_+$ is a bounded semigroup.

Proof of Proposition 6.3. Proof that $T(z)$ is well-defined and of (i): Fix $\delta' \in (0, \delta)$ and $z \in \Sigma_{\delta'}$. Since the integrand in (6.2) is analytic, the integral does not depend on the path γ if the integral exists. Let $r = |z|^{-1}$, $\varepsilon = (\delta - \delta')/2$ and choose a contour $\gamma = \gamma_{r,\,\delta-\varepsilon} = \gamma^1_{r,\,\delta-\varepsilon} \cup \gamma^2_{r,\,\delta-\varepsilon} \cup \gamma^3_{r,\,\delta-\varepsilon}$ (see Figure 6.2) with

$$\begin{aligned}
\gamma^1_{r,\,\delta-\varepsilon} &= \big\{ s\,\mathrm{e}^{-\mathrm{i}(\pi/2+\delta-\varepsilon)} \,:\, s \in (\infty,\, r) \big\},\\
\gamma^3_{r,\,\delta-\varepsilon} &= \big\{ s\,\mathrm{e}^{\mathrm{i}(\pi/2+\delta-\varepsilon)} \,:\, s \in (r,\, \infty) \big\},\\
\gamma^2_{r,\,\delta-\varepsilon} &= \big\{ r\,\mathrm{e}^{\mathrm{i}s} \,:\, s \in (-(\pi/2+\delta-\varepsilon),\, (\pi/2+\delta-\varepsilon)) \big\}.
\end{aligned}$$

For $\mu \in \gamma^3_{r,\,\delta-\varepsilon}$ we have $\arg(\mu z) = \arg(\mu) + \arg(z) \in (\pi/2+\varepsilon,\ 3\pi/2-\varepsilon)$. Since $\cos(\varphi) \le \cos(\pi/2+\varepsilon) = -\sin\varepsilon < 0$, $\varphi \in (\pi/2+\varepsilon,\ 3\pi/2-\varepsilon)$, it follows that

$$\operatorname{Re}(\mu z) = |\mu z| \cos(\arg(\mu z)) \le -|\mu z| \sin\varepsilon. \tag{6.4}$$

It is easy to check that (6.4) holds also for $\mu \in \gamma^1_{r,\,\delta-\varepsilon}$. For $\mu \in \gamma^2_{r,\,\delta-\varepsilon}$ we obtain

$$\operatorname{Re}(\mu z) \le |\mu z| = 1.$$

Since A is sectorial, we obtain, using estimate (6.2),

$$\begin{aligned}
\|\,\mathrm{e}^{\mu z} R(\mu, A)\| &\le \mathrm{e}^{\operatorname{Re}(\mu z)} \|R(\mu, A)\| \le \frac{C_\varepsilon}{|\mu|}\, \mathrm{e}^{-|\mu z| \sin\varepsilon}, && \mu \in \gamma^1_{r,\,\delta-\varepsilon} \cup \gamma^3_{r\,\delta-\varepsilon},\\
\|\,\mathrm{e}^{\mu z} R(\mu, A)\| &\le \mathrm{e}\,\frac{C_\varepsilon}{|\mu|} \le \mathrm{e}\,|z| C, && \mu \in \gamma^2_{r\,\delta-\varepsilon}.
\end{aligned}$$

For the integral this yields

$$\begin{aligned}
\left\| \int_\gamma \mathrm{e}^{\mu z} R(\mu, A)\, \mathrm{d}\mu \right\| &\le \int_\gamma \|\,\mathrm{e}^{\mu z} R(\mu, A)\,\|\ \mathrm{d}\mu\\
&\le 2 \int_r^\infty \left\| \mathrm{e}^{-s|z|\sin\varepsilon} \frac{C_\varepsilon}{s} \right\| \mathrm{d}s + \int_{-(\pi/2+\delta-\varepsilon)}^{\pi/2+\delta-\varepsilon} \mathrm{e}\,|z| C_\varepsilon |\mathrm{i}r\,\mathrm{e}^{\mathrm{i}s}|\ \mathrm{d}s\\
&= 2 \int_1^\infty \left\| \mathrm{e}^{-s\sin\varepsilon} \frac{C_\varepsilon}{s} \right\| \mathrm{d}s + 2(\pi/2+\delta-\varepsilon)\,\mathrm{e}\,C_\varepsilon < \infty.
\end{aligned}$$

Hence $T(z)$ is well-defined and uniformly bounded in the sector $\Sigma_{\delta'}$ because the right-hand-side does not depend on $z \in \Sigma_{\delta'}$.

(ii) The integrand in (6.2) is analytic and, as in (i), it can be shown that the integrals of the derivatives exist.

(iii) Let $z_1, z_2 \in \Sigma_\delta$ and choose $\delta' \in (0, \delta)$ such that $z_1, z_2 \in \Sigma_{\delta'}$. Choose $\gamma = \gamma_{1,\delta-\varepsilon}$ with $\varepsilon = (\delta - \delta')/2$ as before and let $\gamma' = \gamma + c$ with $c > 0$ large enough such that $\gamma \cap \gamma' = \emptyset$. Then, using the resolvent identity $R(\mu, A)R(\lambda, A) =$

$(\lambda - \mu)^{-1}[R(\mu, A) - R(\lambda, A)]$:

$$\begin{aligned}
T(z_1)T(z_2) &= \frac{1}{(2\pi\mathrm{i})^2} \int_\gamma \int_{\gamma'} \mathrm{e}^{\mu z_1}\, \mathrm{e}^{\lambda z_2} R(\mu, A) R(\lambda, A)\ \mathrm{d}\mu\ \mathrm{d}\lambda\\
&= \frac{1}{(2\pi\mathrm{i})^2} \int_\gamma \int_{\gamma'} \mathrm{e}^{\mu z_1}\, \mathrm{e}^{\lambda z_2} (\lambda - \mu)^{-1} [R(\mu, A) - R(\lambda, A)]\ \mathrm{d}\lambda\ \mathrm{d}\mu\\
&= \frac{1}{(2\pi\mathrm{i})^2} \int_\gamma \mathrm{e}^{\mu z_1} R(\mu, A) \int_{\gamma'} (\lambda - \mu)^{-1}\, \mathrm{e}^{\lambda z_2}\ \mathrm{d}\lambda\ \mathrm{d}\mu\\
&\quad - \frac{1}{(2\pi\mathrm{i})^2} \int_{\gamma'} \mathrm{e}^{\lambda z_2} R(\lambda, A) \int_\gamma (\lambda - \mu)^{-1}\, \mathrm{e}^{\mu z_1}\ \mathrm{d}\mu\ \mathrm{d}\lambda\\
&= \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{\mu z_2} R(\mu, A)\, \mathrm{e}^{\mu z_1}\ \mathrm{d}\mu\ = T(z_1 + z_2),
\end{aligned}$$

because $\int_{\gamma'} (\mu - \lambda)^{-1}\, \mathrm{e}^{\lambda z_2}\ \mathrm{d}\lambda = 2\pi\mathrm{i}\,\mathrm{e}^{\mu z_2}$ and $\int_\gamma (\mu - \lambda)^{-1}\, \mathrm{e}^{\mu z_1}\ \mathrm{d}\mu = 0$ by Cauchy's integral formula (if the contours are closed "to the left at infinity" with a piece of circle).

(iv) Again, let $\delta' \in (0, \delta)$ and $\varepsilon = (\delta - \delta')/2$. Because of (i), (ii) and because A is densely defined, it suffices to show

$$\lim_{\substack{z \to 0\\ z \in \Sigma_{\delta'}}} T(z)x - x = 0, \qquad x \in \mathcal{D}(A).$$

Choose again $\gamma = \gamma_{1,\delta-\varepsilon}$ as before. Cauchy's integral formula yields

$$\int_\gamma \frac{\mathrm{e}^{\mu z}}{\mu}\ \mathrm{d}\mu = \mathrm{e}^0 = 1,$$

hence

$$T(z)x - x = \int_\gamma \mathrm{e}^{\mu z} \left(R(\mu, A) - \frac{1}{\mu} \right) x\ \mathrm{d}\mu = \int_\gamma \mathrm{e}^{\mu z}\, \mu^{-1} R(\mu, A) A x\ \mathrm{d}\mu.$$

The norm of the integrand can be estimated as follows:

$$\|\mu^{-1}\, \mathrm{e}^{\mu z} R(A, \mu) A x\| \le \begin{cases} \|Ax\|\, |\mu|^{-2} C_\varepsilon\, \mathrm{e}^{-|\mu z| \sin\varepsilon} & \text{for } \ \mu \in \gamma^1_{1,\,\delta-\varepsilon} \cup \gamma^3_{1,\,\delta-\varepsilon},\\ \|Ax\|\, |\mu|^{-2}\, \mathrm{e}\, C_\varepsilon, & \text{for } \ \mu \in \gamma^2_{1,\,\delta-\varepsilon}. \end{cases}$$

Hence the integrand can be bounded by an integrable function, hence, by Lebesgue's theorem of dominated convergence,

$$\lim_{\substack{z \to 0\\ z \in \Sigma_{\delta'}}} T(z)x - x = \int_\gamma \lim_{\substack{z \to 0\\ z \in \Sigma_{\delta'}}} \mathrm{e}^{\mu z}\, \mu^{-1} R(\mu, A) A x\ \mathrm{d}\mu = \int_\gamma \mu^{-1} R(\mu, A) A x\ \mathrm{d}\mu = 0.$$

The last equality, again, is a consequence of Cauchy's integral theorem if the contour γ is closed on the right side.

(v) From (iv) we obtain that $(T(t))_{t \ge 0}$ is a strongly continuous semigroup. Let B be the generator of $(T(t))_{t \ge 0}$. If λ is large enough, then $\lambda \in \rho(A) \cap \rho(B)$ (for instance choose $\lambda = |\omega_0| + 1$ where ω_0 is the growth bound of $(T(t))_{t\geq 0})$. For the proof of $A = B$ we show $R(\lambda, A) = R(\lambda, B)$. By Theorem 5.28

$$R(B,\lambda)x = \lim_{t_0\to\infty} \int_0^{t_0} \mathrm{e}^{-\lambda s}\, T(s)x \,\mathrm{d}s.$$

For $t_0 > 0$ and the contour $\gamma = \gamma_1$ as above, Fubini's theorem shows that

$$\begin{aligned}\int_0^{t_0} \mathrm{e}^{-\lambda s}\, T(s)x \,\mathrm{d}s &= \frac{1}{2\pi \mathrm{i}} \int_0^{t_0} \int_\gamma \mathrm{e}^{-\lambda s}\, \mathrm{e}^{\mu s}\, R(\mu, A)x \,\mathrm{d}\mu \,\mathrm{d}s \\ &= \int_\gamma \frac{1}{2\pi \mathrm{i}} \int_0^{t_0} \mathrm{e}^{-\lambda s}\, \mathrm{e}^{\mu s}\, R(\mu, A)x \,\mathrm{d}s \,\mathrm{d}\mu \\ &= \frac{1}{2\pi \mathrm{i}} \int_\gamma (\mu-\lambda)^{-1} (\mathrm{e}^{(\mu-\lambda)t_0} - 1) R(\mu, A)x \,\mathrm{d}\mu \xrightarrow{t_0\to\infty} R(\lambda, A)x,\end{aligned}$$

because (again by Cauchy's integral theorem, close right)

$$\int_\gamma (\mu-\lambda)^{-1} R(\mu, A)x \,\mathrm{d}\mu = R(\lambda, A),$$

and because for $\mathrm{Re}(\mu - \lambda) < 0$

$$\left\| \int_\gamma (\mu-\lambda)^{-1}\, \mathrm{e}^{(\mu-\lambda)t_0}\, R(\mu, A)x \,\mathrm{d}\mu \right\| \leq \mathrm{e}^{-t_0}\, \|x\| \int_\gamma |\mu-\lambda|^{-1} \frac{C_\varepsilon}{|\mu|}\, |\,\mathrm{d}\mu| \longrightarrow 0$$

for $t_0 \to \infty$. □

Note that the proposition shows that the generator of an analytic continuous semigroup is unique because it is the unique generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$.

Example 6.5. If H is a Hilbert space and A is selfadjoint and dissipative linear operator on H, then A is sectorial with arbitrary angle $\delta \in (0, \pi/2)$. In particular, A generates an analytic semigroup with angle $\delta \in (0, \pi/2)$.

Proof. By assumption, $W(A) \subset (-\infty, 0]$ (because A is sectorial and selfadjoint), hence $\mathbb{C} \setminus (-\infty, 0] \subset \rho(A)$ (because A is selfadjoint and the defect index of A is constant in connected components of $\mathbb{C} \setminus \overline{W(A)}$). Fix $\delta \in (0, \pi/2)$ arbitrary. It remains to prove the resolvent estimate (6.2) for $\lambda \in \Sigma_{\pi/2+\delta}$. Since $\lambda \in \Sigma_{\pi/2+\delta}$, there exist $\rho > 0$ and $\vartheta \in (-\pi/2 - \delta, \pi/2 + \delta)$ such that $\lambda = \rho\, \mathrm{e}^{\mathrm{i}\vartheta}$. For $x \in H$ let $u = R(\lambda, A)x$, hence $\rho\, \mathrm{e}^{\mathrm{i}\vartheta}\, u - Au = x$. Multiplication by $\mathrm{e}^{-\mathrm{i}\vartheta/2}$ and scalar multiplication by u yields

$$\rho\, \mathrm{e}^{\mathrm{i}\vartheta/2}\, \|u\|^2 - \mathrm{e}^{-\mathrm{i}\vartheta/2} \langle Au\,, u\rangle = \mathrm{e}^{-\mathrm{i}\vartheta/2} \langle x\,, u\rangle.$$

Taking the real part on both sides, leads to

$$\begin{aligned}& \rho\|u\|^2 \underbrace{\cos(\vartheta/2)}_{\in(\cos(\delta/2),1)} - \underbrace{\langle Au\,, u\rangle \cos(\vartheta/2)}_{\leq 0} = \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\vartheta/2} \langle x\,, u\rangle) \leq \|x\|\,\|u\| \\ \Longrightarrow \quad & \|R(\lambda, A)x\| = \|u\| \leq \frac{\|x\|}{\rho\cos(\delta/2)} = \frac{\|x\|}{|\lambda|\cos(\delta/2)}.\end{aligned}$$

□

Example 6.6.

- Consider the differential operator A defined by $Af = f''$, $f \in \mathcal{D}(A) = W^{2,2}(\mathbb{R})$ on $X = L_2(\mathbb{R})$. Then A generates an analytic semigroup on $L_2(\mathbb{R})$.
- Translation semigroup: $X = L_p(\mathbb{R})$, $\mathcal{T} = (T(t))_{t\geq 0}$ with $T(t)f = f(t+\cdot)$ is not an analytic semigroup because its generator A is $Af = f'$, $f \in \mathcal{D}(A) = W^{1,p}(\mathbb{R})$. Since $\sigma(A) = \mathrm{i}\mathbb{R}$, A is not sectorial (see Proposition **??**).

Lemma 6.7. *If X is a Banach space and $\mathcal{T} = (T(z))_{z\in\Sigma_\delta}$ is an analytic semigroup on X with generator A, then*

(i) $t > 0,\ k \in \mathbb{N},\ x \in X \quad \Longrightarrow \quad T(t)x \in \mathcal{D}(A^k)$ *and* $A^k T(t)x = (AT(t/k))^k x$,

$t > 0,\ k \in \mathbb{N},\ x \in \mathcal{D}(A^k) \quad \Longrightarrow \quad A^k T(t)x = T(t)A^k$.

(ii) *For every $x \in X$ the map $(0, \infty) \to X$, $t \mapsto T(t)x$ is infinitely differentiable with derivatives*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\, T(t)x = A^k T(t)x, \qquad k \in \mathbb{N}.$$

Note that the assertions are true in the case of a strongly continuous semigroup only for $x \in \mathcal{D}(A)$.

Proof. (i) Let $t > 0$ and $\delta' \in (0, \delta)$. By assumption, T is norm-differentiable in the sector $\Sigma_{\delta'}$, so the limit for $h \to 0$ of

$$\frac{1}{h}\,(T(t+h) - T(t))\,x = \frac{1}{h}\,(T(h) - \mathrm{id})\,T(t)x.$$

Hence $T(t)x \in \mathcal{D}(A)$. We already saw in Proposition 5.25 that $AT(t)x = T(t)Ax$ for $x \in \mathcal{D}(A)$. Because of

$$AT(t)x = AT(t/2)T(t/2)x = T(t/2)AT(t/2)x \in \mathcal{D}(A)$$

it follows that $T(t)x \in \mathcal{D}(A^2)$ and $A^2T(t)x = (AT(t/2))^2 x$, $x \in X$, $t > 0$. Now the assertion follows by induction.
(ii) Let $\varepsilon \in (0,\, t/(2k))$. Then, by (i),

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\, T(t)x &= \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\, AT(t)x = \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\, T(t-\varepsilon)A \underbrace{T(\varepsilon)x}_{\mathcal{D}(A)} \\ &= \ldots = T(t-k\varepsilon)\big(AT(\varepsilon)\big)^k x = T(t-k\varepsilon)A^k T(k\varepsilon)x = A^k T(t)x.\end{aligned}$$

□

Proposition 6.8 (Characterisation of analytic semigroups). *Let X be a Banach space and A a linear operator on X. Then the following is equivalent:*

(i) *A is sectorial.*

(ii) *A generates a bounded analytic semigroup* $\mathcal{T} = (T(z))_{z\in\Sigma_\delta\cup\{0\}}$ *on* X.

(iii) *A generates a bounded strongly continuous semigroup* $\mathcal{T} = (T(t))_{t\geq 0}$ *on* X, $\operatorname{rg}(T(t)) \subseteq \mathcal{D}(A)$ *for all* $t > 0$, *and*

$$C := \sup\{\, \|tAT(t)\| \,:\, t > 0\,\} < \infty.$$

Proof. (i) $\Rightarrow$ (ii) Proposition 6.3.

(ii) $\Rightarrow$ (i) Let $\delta \in (0, \pi/2]$ be the angle of $\mathcal{T}$. By assumption, $(T(t))_{t\geq 0}$ is a strongly continuous semigroup with generator A. We have to show that A is sectorial with angle δ.
Choose $\alpha \in (-\delta, \delta)$ and define

$$T_\alpha(t) := T(\mathrm{e}^{\mathrm{i}\alpha}\, t), \qquad t \geq 0.$$

Clearly, $\mathcal{T}_\alpha = (T_\alpha(t))_{t\geq 0}$ is a strongly continuous semigroup on X. Let A_α be the generator of $\mathcal{T}_\alpha$. We show that $A_\alpha = \mathrm{e}^{\mathrm{i}\alpha} A$. Let $\gamma_\alpha = \mathrm{e}^{\mathrm{i}\alpha}\,\gamma$. For $x \in X$, Theorem **??** and Cauchy's integral theorem show that

$$\begin{aligned} R(1,A)x &= \int_0^\infty \mathrm{e}^{-t}\, T(t)x \,\mathrm{d}t = \int_{\gamma_\alpha} \mathrm{e}^{-\mu}\, T(\mu)x \,\mathrm{d}\mu = \int_0^\infty \mathrm{e}^{-t\,\mathrm{e}^{\mathrm{i}\alpha}}\, T(\mathrm{e}^{\mathrm{i}\alpha}\, t)x \,\mathrm{d}t \\ &= \mathrm{e}^{\mathrm{i}\alpha} \int_0^\infty \mathrm{e}^{-t\,\mathrm{e}^{\mathrm{i}\alpha}}\, T_\alpha(t)x \,\mathrm{d}t = \mathrm{e}^{\mathrm{i}\alpha}\, R(\mathrm{e}^{\mathrm{i}\alpha}, A)x, \end{aligned}$$

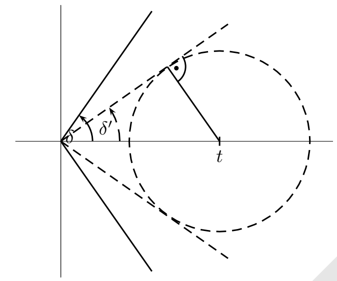
hence $x \in \mathcal{D}(A_\alpha)$ if and only if $x \in \mathcal{D}(A)$, and in this case$A_\alpha x = \mathrm{e}^{\mathrm{i}\alpha}\, Ax$.
Since A_α is the generator of a strongly continuous semigroup, it follows that $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\} \subseteq \rho(A_\alpha) = \rho(\mathrm{e}^{\mathrm{i}\alpha}\, A) = \mathrm{e}^{\mathrm{i}\alpha}\, \rho(A)$. Hence also

$$\rho(A) \;\supset \bigcup_{\alpha\in(-\delta,\delta)} \mathrm{e}^{\mathrm{i}\alpha}\, \{\, \lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\,\} \;=\; \{\, \lambda \in \mathbb{C} : |\arg(\lambda)| < \pi/2 + \delta\,\} \;=\; \Sigma_{\pi/2+\delta}.$$

It remains to show the resolvent estimate (6.2). Choose $\delta' \in (0, \delta)$ and $\varepsilon > 0$ such that $\delta - \delta' > \varepsilon$. Since $\mathcal{T}$ is a bounded semigroup, there exists an $M \geq 1$ such that $\|T(z)\| \leq M$ for all $z \in \Sigma_{\delta'+\varepsilon}$. Now fix $\lambda \in \Sigma_{\pi/2+\delta'}$ and choose $\alpha \in (-\delta' - \varepsilon, \delta' + \varepsilon)$ such that $\mathrm{e}^{\mathrm{i}\alpha}\,\lambda \in \Sigma_{\pi/2-\varepsilon}$. It follows that

$$\|R(\lambda, A)\| = \|R(\mathrm{e}^{\mathrm{i}\alpha}\,\lambda, \mathrm{e}^{\mathrm{i}\alpha}\, A)\| = \|R(\mathrm{e}^{\mathrm{i}\alpha}\,\lambda, A_\alpha)\| \leq \frac{M}{\operatorname{Re}(\mathrm{e}^{\mathrm{i}\alpha}\,\lambda)} \leq \frac{M}{|\lambda|\cos(\pi - \varepsilon)}.$$

In the second to last inequality we applied the Hille-Phillips-Yosida theorem to A_α (note that $\|T_\alpha(t)\| \leq M$ for all $t \geq 0$ and that $\operatorname{Re}(\mathrm{e}^{\mathrm{i}\alpha})\lambda > 0$).



(ii) $\Rightarrow$ (iii) By assumption, $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on X with generator A. Since $\mathcal{T}$ is norm-differentiable in every sector $\Sigma_{\delta'}$ with $\delta' \in (0, \delta)$, for every $t > 0$ the limit

$$\lim_{h\to 0}\; h^{-1}\big(T(t+h)x - T(t)x\big) \;=\; \lim_{h\to 0}\; h^{-1}\big(T(h) - \mathrm{id}\,\big)T(t)x$$

exists, therefore $T(t)x \in \mathcal{D}(A)$. Define the contour $\gamma_{r,\delta'}$ as in the proof of Proposition 6.3. Since A is closed and, as we will show, $\int_{\gamma_{r,\delta'}} A\,\mathrm{e}^{t\mu}\, R(\mu, A)\,\mathrm{d}\mu$ exists, we obtain as in Proposition 6.3:

$$\begin{aligned} \|AT(t)\| \;&=\; \left\| \int_{\gamma_{t^{-1},\delta'}} A\,\mathrm{e}^{t\mu}\, R(\mu, A)\,\mathrm{d}\mu \right\| \;=\; \frac{1}{2\pi} \left\| \int_{\gamma_{t^{-1},\delta'}} \mathrm{e}^{t\mu} \Big(\mu R(\mu, A) - 1\Big)\,\mathrm{d}\mu \right\| \\ &= \frac{1}{2\pi} \left\| \int_{t^{-1}}^{\infty} \mathrm{e}^{ts\,\mathrm{e}^{\mathrm{i}\delta'}} \Big(\mathrm{e}^{\mathrm{i}\delta'}\, sR(\mathrm{e}^{\mathrm{i}\delta'}\, s,\, A) - 1\Big)\, \mathrm{e}^{\mathrm{i}\delta'}\;\mathrm{d}s \right. \\ &\qquad + \int_{\infty}^{t^{-1}} \mathrm{e}^{ts\,\mathrm{e}^{-\mathrm{i}\delta'}} \Big(\mathrm{e}^{-\mathrm{i}\delta'}\, sR(\mathrm{e}^{-\mathrm{i}\delta'}\, s,\, A) - 1\Big)\, \mathrm{e}^{-\mathrm{i}\delta'}\;\mathrm{d}s \\ &\qquad \left. + \int_{-\delta'}^{\delta'} \mathrm{e}^{\mathrm{e}^{\mathrm{i}s}} \Big(t^{-1}\,\mathrm{e}^{\mathrm{i}s}\, R(t^{-1}\,\mathrm{e}^{\mathrm{i}s},\, A) - 1\Big)\, \frac{\mathrm{i}}{t}\, \mathrm{e}^{\mathrm{i}s}\;\mathrm{d}s \right\| \\ &\leq \frac{1}{\pi} \left\| \int_{t^{-1}}^{\infty} \mathrm{e}^{ts\cos\delta'} \Big(s\frac{M}{s} + 1\Big)\,\mathrm{d}s + \frac{1}{2\pi} \int_{-\delta'}^{\delta'} \mathrm{e}\,\Big(t^{-1}\frac{M}{t^{-1}} + 1\Big) t^{-1}\,\mathrm{d}s \right\| \\ &= \frac{1}{t}\,\frac{1}{\pi} \int_1^\infty \mathrm{e}^{s\cos\delta'} \Big(M + 1\Big)\,\mathrm{d}s + \frac{1}{t}\,\frac{1}{2\pi} \int_{-\delta'}^{\delta'} \mathrm{e}\,\Big(M + 1\Big)\,\mathrm{d}s \;=\; \frac{C}{t}, \end{aligned}$$

with a constant $C < \infty$ that does not depend on t.
(iii) $\Rightarrow$ (ii) Let $x \in X$. By Lemma 6.7, the map $(0, \infty) \to X,\; s \mapsto T(s)x$ is arbitarily differentiable and $\operatorname{rg}(T(s)) \subseteq \mathcal{D}(A^\infty) = \cap_{k=1}^\infty \mathcal{D}(A^k)$ for all $s > 0$. Moreover, Lemma 6.7 and the inequality $k^k \leq \mathrm{e}^k\, k!$ show that

$$\frac{1}{k!} \left\| \frac{\mathrm{d}^k}{\mathrm{d}s^k} T(s) \right\| = \frac{1}{k!}\, \big\|A^k T(s)\big\| = \frac{1}{k!}\, \big\|(AT(s/k))^k\big\| \leq \frac{k^k}{s^k k!}\, \|s/k(AT(s/k))\|^k \leq \frac{C^k\,\mathrm{e}^k}{s^k}.$$

For $t > 0$ and $|h| \in (0, t)$ the Taylor expansion shows that

$$T(t+h)x = \sum_{k=0}^{n} \frac{h^k}{k!} T^{(k)}(t)x + \frac{1}{n!} \int_t^{t+h} (t+h-s)^n T^{(n+1)}(s)x \, \mathrm{d}s. =: \sum_{k=0}^{n} \frac{h^k}{k!} T^{(k)}(t)x + R_{n+1}(h)$$

The integral term $R_{n+1}(h)$ can be estimated as follows:

$$\|R_{n+1}(h)\| \le \frac{\|x\|}{n!} \int_t^{t+h} |t+h-s|^n (n+1)! \left(\frac{C\,\mathrm{e}}{s}\right)^k \mathrm{d}s \le (n+1) \left(\frac{|h|\, C\,\mathrm{e}}{t-|h|}\right)^{n+1}.$$

For $q \in (0, 1)$ and $|h| < \frac{qt}{C\mathrm{e}+1}$, we have that

$$|h| \frac{C\,\mathrm{e}}{t-|h|} \le \frac{qtC\,\mathrm{e}}{(C\,\mathrm{e}+1)(t - \frac{qt}{C\,\mathrm{e}+1})} = \frac{qC\,\mathrm{e}}{Ce+1-q} \le q,$$

so

$$\|R_{n+1}(h)\| \le (n+1)q^{n+1} \longrightarrow 0, \quad n \to \infty.$$

This leads to the Taylor expansion for $T(\cdot)$

$$T(t+h)x = \sum_{k=0}^{\infty} \frac{h^k}{k!} T^{(k)}(t)x, \quad |h| < \frac{qt}{C\,\mathrm{e}+1}.$$

The series converges also for $h \in \mathbb{C}$ with $|h| < \frac{qt}{C\,\mathrm{e}+1}$, hence T has an analytic extension to Σ_δ with $\delta = \arctan \frac{1}{C\,\mathrm{e}+1}$.
It remains to be shown that the extension to every sector $\Sigma_{\delta'}$ with $\delta' \in (0, \delta)$ is bounded. If $z \in \Sigma_{\delta'}$, then $|\operatorname{Im} z| \le t \tan \delta' \le \frac{tq}{C\,\mathrm{e}+1}$, and consequently

$$\begin{aligned}\|T(z)\| &= \|T(\operatorname{Re} z + \mathrm{i} \operatorname{Im} z)\| \le \sum_{k=0}^{\infty} \frac{1}{k!} \|T^{(k)}(\operatorname{Re} z)\| \, |\operatorname{Im} z|^k \\ &\le \sum_{k=0}^{\infty} \left(\frac{C\,\mathrm{e}}{t}\right)^k \left(\frac{qt}{C\,\mathrm{e}+1}\right)^k \le \sum_{k=0}^{\infty} q^k = (1-q)^{-1}.\end{aligned}$$ □

Not densely defined operators

In Proposition 6.3, we used that A is densely defined only to prove that the generated semigroup $\mathcal{T}$ is strongly continuous. If we do not assume that A is densely defined, then in Proposition 6.3, instead of (iv), the following:

(iv') For all $x \in \overline{\mathcal{D}(A)}$ the map $z \mapsto T(z)x$ is continuous in $\Sigma_{\delta'} \cup \{0\}$ for every $\delta' \in (0, \delta)$.

More precisely:

Proposition 6.9. *Let X be a Banach space, A a linear operator on X and $\delta \in (0, \pi/2]$ with $\Sigma_{\pi/2+\delta} \subseteq \rho(A)$ and assume that for every $\varepsilon \in (0, \delta)$ there exists a constant C_ε such that*

$$\|R(\lambda, A)\| \le \frac{C_\varepsilon}{|\lambda|}, \qquad \lambda \in \overline{\Sigma_{\pi/2+\delta-\varepsilon}} \setminus \{0\}.$$

Then the claims (i)–(iii) from Proposition 6.3 hold. In addition:

(i) (a) $x \in \overline{\mathcal{D}(A)} \implies \lim_{t\to 0} T(t)x = x$,
 (b) *If the limit $y = \lim_{t\to 0} T(t)x$ exists, then $x \in \overline{\mathcal{D}(A)}$ and $y = x$.*

(ii) (a) $x \in X,\ t \ge 0 \implies \int_0^t T(s)x \,\mathrm{d}s \in \mathcal{D}(A)$ *and* $A\int_0^t T(s)x \,\mathrm{d}s = T(t)x - x$.
 (b) *If the fuction $s \mapsto AT(s)x$ in $(0, \varepsilon)$ is integrable for some $\varepsilon > 0$, then*

$$A \int_0^t T(s)x \,\mathrm{d}s = \int_0^t AT(s)x \,\mathrm{d}s.$$

(iii) (a) $x \in \mathcal{D}(A),\ Ax \in \overline{\mathcal{D}(A)} \implies \lim_{t\to 0} t^{-1}\Big(T(t)x - x\Big) = Ax$,
 (b) *If the limit $y = \lim_{t\to 0} t^{-1}\Big(T(t)x - x\Big)$ exists, then $x \in \mathcal{D}(A),\ Ax \in \overline{\mathcal{D}(A)}$ and $y = Ax$.*

(iv) $x \in \mathcal{D}(A),\ Ax \in \overline{\mathcal{D}(A)} \implies \lim_{t\to 0} AT(t)x = Ax$.

Proof. (i) (a) was shown in Proposition 6.3 (iv). Assume that x, y satisfy (b). Since $T(t)x \in \mathcal{D}(A)$ for all $t > 0$ and $y = \lim_{t\searrow 0} T(t)x$, it follows that $y \in \overline{\mathcal{D}(A)}$. Now let $\lambda \in \rho(A)$. By (a), we obtain

$$R(\lambda, A)y = \lim_{t\searrow 0} R(\lambda, A)T(t)x = \lim_{t\searrow 0} T(t) \underbrace{R(\lambda, A)x}_{\in \mathcal{D}(A)} = R(\lambda, A)x.$$

(ii) (a) Let $\lambda \in \rho(A)$, $x \in X$ and $t > 0$. For $\varepsilon \in (0, t)$ it follows that

$$\begin{aligned}\int_\varepsilon^t T(s)x \,\mathrm{d}s &= \int_\varepsilon^t (\lambda - A)R(\lambda, A)T(s)x \,\mathrm{d}s = \lambda \int_\varepsilon^t R(\lambda, A)T(s)x \,\mathrm{d}s - \int_\varepsilon^t AR(\lambda, A)T(s)x \,\mathrm{d} \\ &= \lambda \int_\varepsilon^t R(\lambda, A)T(s)x \,\mathrm{d}s - \int_\varepsilon^t \frac{d}{ds} T(s)R(\lambda, A)x \,\mathrm{d}s \\ &= \lambda \int_\varepsilon^t T(s)R(\lambda, A)x \,\mathrm{d}s - T(t)R(\lambda, A)x + T(\varepsilon)R(\lambda, A)x.\end{aligned}$$

Hence the limit for $\varepsilon \to 0$ exists and

$$\begin{aligned}\int_0^t T(s)x \,\mathrm{d}s &= \lambda \int_0^t T(s)R(\lambda, A)x \,\mathrm{d}s - R(\lambda, A)T(t)x + R(\lambda, A)T(0)x \\ &= \lambda R(\lambda, A) \int_0^t T(s)x \,\mathrm{d}s - R(\lambda, A)\Big(T(t)x - x\Big) \in \mathcal{D}(A).\end{aligned}$$

The claim follows from

$$R(\lambda,A)A\int_0^t T(s)x\,\mathrm{d}s = \big(\lambda R(\lambda,A)-1\big)\int_0^t T(s)x\,\mathrm{d}s = R(\lambda,A)\big(T(t)x-x\big)$$

(b) Let $x \in X$ and $\varepsilon > 0$. Suppose that $s \mapsto T(s)x$ is integrable in $(0,\varepsilon)$. Then also $s \mapsto \|T(s)x\|$ is integrable in $(0,\varepsilon)$ and therefore the improper integral of $s \mapsto T(s)x$ in $(0,t)$ exists. So the claim follows from Theorem **??**.

(iii) (a) Shows that (ii) that

$$\begin{aligned} t^{-1}\Big(T(t)x-x\Big) &= t^{-1}A\int_0^t T(s)x\,\mathrm{d}s = t^{-1}\int_0^t AT(s)x\,\mathrm{d}s \\ &= t^{-1}\int_0^t T(s)Ax\,\mathrm{d}s \xrightarrow{t\to 0} T(0)Ax = Ax, \end{aligned}$$

because the integrand $T(\cdot)x$ is continuous in $[0,t]$ by (i) because $Ax \in \overline{\mathcal{D}(A)}$.

(b) Let $x \in X$ such that the limit $y = \lim_{t\to 0} t^{-1}\Big(T(t)x - x\Big)$ exists. Then, for $\lambda \in \rho(A)$:

$$\begin{aligned} R(\lambda,A)y &= \lim_{t\to 0} t^{-1}R(\lambda,A)\big(T(t)x-x\big) = \lim_{t\to 0} t^{-1}R(\lambda,A)A\int_0^t T(s)x\,\mathrm{d}s \\ &= \lim_{t\to 0} t^{-1}\big(\lambda R(\lambda,A)-1\big)\int_0^t T(s)x\,\mathrm{d}s = \big(\lambda R(\lambda,A)-1\big)\lim_{t\to 0} t^{-1}\int_0^t T(s)x\,\mathrm{d}s \\ &\overset{(*)}{=} \big(\lambda R(\lambda,A)-1\big)x \end{aligned}$$

so $x \in \mathcal{D}(A)$ and $R(\lambda,A)y = R(\lambda,A)Ax$. In $(*)$ we used

$$\begin{aligned} \lim_{t\searrow 0} t^{-1}\big(T(t)x-x\big) \text{ exists} &\implies \lim_{t\searrow} T(t)x = x \implies x \in \overline{\mathcal{D}(A)} \\ &\implies s \mapsto T(s)x \text{ continuous in } [0,t]. \end{aligned}$$

□

Connection with the Cauchy problem.

Let A as in the proposition above, $\mathcal{T} = (T(z))_{z\in\Sigma_\delta}$ se analytic semigroup generated by A and $x_0 \in X$. Consider the initial value problem

$$x'(t) = Ax(t), \quad t > 0, \qquad x(0) = x_0. \tag{6.5}$$

- $x_0 \in X$ arbitrary Then $z \mapsto T(z)x_0$ is an analytic solution of $x' = Ax$ in the open sector Σ_δ and $T(z)x_0 \in \mathcal{D}(A)$ for all $z \in \Sigma_\delta$.
- $x_0 \in \overline{\mathcal{D}(A)}$: The solution $T(\cdot)x_0$ is continuous in 0, hence it solves the initial value problem (6.5) for $t > 0$.
- $x_0 \in \mathcal{D}(A)$: The solution $T(\cdot)x_0$ is differentiable in 0, hence it solves the initial value problem (6.5) for $t > 0$.

Remark. If $x_0 \in X$, then by definition $T(0)x_0 = x_0$, but $\lim_{t\searrow 0} T(t)x_0 = x_0$ holds only if $x_0 \in \overline{\mathcal{D}(A)}$. But it is always true that

$$\lim_{t\searrow 0} R(\lambda,A)T(t)x_0 = R(\lambda,A)x_0, \quad \lambda \in \rho(A).$$

Chapter 7

Exercises

Exercises for Chapter 1

1. Sea H un espacio de Hilbert. Si $(x_n)_{n\in\mathbb{N}}$ es una sucesión de vectores ortogonales dos a dos en H, entonces lo siguiente es equivalente:

   (a) $\displaystyle\sum_{n=1}^{\infty} x_n$ converge en norma en H.

   (b) $\displaystyle\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \langle x_n, y\rangle$ converge para cada $y \in H$.

2. Sean P_1 y P_2 proyecciones ortogonales en el espacio de Hilbert H. Entonces tenemos que

$$\|P_1 - P_2\| = \max\{\rho_{12},\ \rho_{21}\}$$

   donde

$$\rho_{jk} := \sup\Big\{\|P_j x\| \ :\ x \in \operatorname{rg}(P_k)^{\perp},\ \|x\| \le 1\Big\}.$$

3. Si P y Q son proyecciones ortogonales en el espacio de Hilbert H tales que $\|P - Q\| < 1$, entonces

$$\dim(\operatorname{rg} P) = \dim(\operatorname{rg} Q), \quad \dim(\operatorname{rg}(I-P)) = \dim(\operatorname{rg}(I-Q)).$$

4. Sea S el right shift en $\ell_2(\mathbb{Z})$ definido por

$$(Sx)_k = x_{k-1}, \quad k \in \mathbb{Z},$$

   donde $x = (x_k)_{k=-\infty}^{\infty}$ pertenece a $\ell_2(\mathbb{Z})$. Determine $\sigma_{\mathrm{p}}(S)$, $\sigma_{\mathrm{c}}(S)$, $\sigma_{\mathrm{r}}(S)$.

5. (a) Muestre que el espectro de un operador acotado en un espacio de Banach nunca es vacío.

   (b) También se tiene para operadores no acotados? (Prueba o contraejemplo!)

6. Sean H_1, H_2 y H_3 espacios de Hilbert y $S(H_1 \to H_2)$ y $T(H_2 \to H_3)$ operadores lineales densamente definidos.

   (a) Si $T \in L(H_2, H_3)$ entonces TS es densamente definido y $(TS)^* = S^*T^*$.

   (b) Si S es inyectivo y $S^{-1} \in L(H_2, H_1)$ entonces TS es densamente definido y $(TS)^* = S^*T^*$.

   (c) Si S es inyectivo y $S^{-1} \in L(H_2, H_1)$ entonces S^* es inyectivo y $(S^*)^{-1} = (S^{-1})^*$

Sea X un espacio de Banach, $A \subseteq X$, $B \subseteq X'$. Se definen los conjuntos

$$A^\circ := \{\varphi \in X' : \varphi(a) = 0 \ \text{ for all } a \in A\} =: \ \text{annihilator of } A,$$
$$^\circ B := \{x \in X : b(x) = 0 \ \text{ for all } b \in B\} =: \ \text{annihilator of } B.$$

7. Sea X un espacio de Banach, $A \subseteq X$, $B \subseteq X'$.

   (a) Muestre que A° y $^\circ B$ son subespacios cerrados y que

$$A^\circ = \big(\overline{\operatorname{span} A}\big)^\circ \qquad \text{y} \qquad {}^\circ B = {}^\circ\big(\overline{\operatorname{span} B}\big).$$

   (b) Muestre ${}^\circ\big(A^\circ\big) = \overline{\operatorname{span} A}$ y $\big({}^\circ B\big)^\circ \supseteq \overline{\operatorname{span} B}$

8. (a) Sean X, Y espacios de Banach, $Y \neq \{0\}$ y $T(X \to Y)$ un operador lineal cerrado con dominio denso. Muestre que $\mathcal{D}(T') \neq \{0\}$.
   Hint. Muestre que para todo $y \in \mathcal{D}(T')$, $y \neq 0$, existe un $\varphi \in \mathcal{D}(T')$ tal que $\varphi(y) \neq 0$.

   (b) Muestre por lo menos dos puntos de lo siguiente:

   (i) $\big(\operatorname{rg} T\big)^\circ = \big(\overline{\operatorname{rg} T}\big)^\circ = \ker T'$,

   (ii) $\overline{\operatorname{rg} T} = {}^\circ\big(\ker T'\big)$,

   (iii) $\overline{\operatorname{rg} T} = Y \iff T'$ es inyectivo,

   (iv) ${}^\circ\overline{\operatorname{rg} T'} \cap \mathcal{D}(T) = \ker T$,

   (v) $\overline{\operatorname{rg} T'} \subseteq \big(\ker T\big)^\circ$.

Exercises for Chapter 2

1. Sea $\alpha \in \mathrm{BV}[a,b]$, $f \in I[a,b]$ y defina $K : [a,b] \to \mathbb{K}$ por $K(x) := \int_a^x f(t)\, \mathrm{d}\alpha(t)$ para $x \in (a,b]$ y $K(a) := 0$. Muestre:

   (a) $K \in \mathrm{BV}[a,b]$.

   (b) Si α es continua por la derecha en $s \in [a,b)$, entonces K también lo es.

   (c) $\int_a^b g(t)\, \mathrm{d}K(t) = \int_a^b (fg)(t)\, \mathrm{d}\alpha(t)$ para todo $g \in I[a,b]$.

2. Sea H un espacio de Hilbert y $T \in L(H)$ un operador compacto autoadjunto con autovalores distintos μ_j. Sea P_j la proyección ortogonal sobre el espacio propio de T respecto a λ_j. Muestre que $(E_\lambda)_{\lambda\in\mathbb{R}}$ es una resoluión de la identidad donde

$$E_\lambda x := \begin{cases} \sum_{\lambda_j \le \lambda} P_j x, & \lambda < 0, \\ x - \sum_{\lambda_j > \lambda} P_j x, & \lambda \ge 0, \end{cases} \qquad \lambda \in \mathbb{R},\ x \in H.$$

3. Sea H un espacio de Hilbert, $(E_\lambda)_{\lambda\in\mathbb{R}}$ una resoluión de la identidad en H y $\varphi : \mathbb{R} \to (a,b)$ una biyección continua no decreciente. Suponga que $E_a = 0$ y $E_{b-0} = E_b = I$. Muestre que $(F(\lambda))_{\lambda\in\mathbb{R}}$ es una resoluión de la identidad en H donde

$$F_\lambda := E_{\varphi(\lambda)}, \quad \lambda \in \mathbb{R}.$$

4. (a) Sea H un espacio de Hilbert, sean $\varphi : \mathbb{R} \to (a,b)$, $(E_\lambda)_{\lambda\in\mathbb{R}}$ y $(F_\lambda)_{\lambda\in\mathbb{R}}$ como en Exercise 2.4. Sea $f : (a,b) \to \mathbb{R}$ tal que $f|_{[a_0,b_0]} \subseteq I[a_0,b_0]$ para cada subintervalo compacto $[a_0,\, b_0]$ de (a,b). Muestre:

   (i) $\int_{\varphi(\alpha)}^{\varphi(\beta)} f(\lambda)\, \mathrm{d}E_\lambda = \int_\alpha^\beta (f\circ\varphi)(\lambda)\, \mathrm{d}F_\lambda$ para todo $[\alpha,\beta] \subseteq \mathbb{R}$.

   (ii) Sea $x \in H$. Entonces $\int_{a+0}^{b-0} f(\lambda)\, \mathrm{d}E_\lambda x$ existe si y solo si $\int_{-\infty}^{\infty} (f \circ \varphi)(\lambda)\, \mathrm{d}F_\lambda x$ existe.[1]

   (b) Sea H un espacio de Hilbert y $\varphi : \mathbb{R} \to \mathbb{R}$ creciente y continua y $f : \mathbb{R} \to \mathbb{R}$ una función continua. Sea $A \in L(H)$ un operador autoadjunto y $B := \varphi(A)$. Muestre $(f\circ\varphi)(A) = f(B)$.

5. Muestre Lemma **??**: Sea H un espacio de Hilbert, $(E_\lambda)_{\lambda\in\mathbb{R}}$ una resoluión de la identidad en H y $f,g \in I[a,b]$. Muestre:

[1] $\int_{a+0}^{b-0} f(\lambda)\, \mathrm{d}E_\lambda x := \lim_{\substack{\lambda_1 \searrow a \\ \lambda_2 \nearrow b}} \int_{\lambda_1}^{\lambda_2} f(\lambda)\, \mathrm{d}E_\lambda$, $\quad \int_{-\infty}^{\infty} (f\circ\varphi)(\lambda)\, \mathrm{d}F_\lambda x := \lim_{\substack{\lambda_1 \searrow -\infty \\ \lambda_2 \nearrow \infty}} \int_{\lambda_1}^{\lambda_2} (f \circ \varphi)(\lambda)\, \mathrm{d}F_\lambda\, x$

   (a) $\left\langle \left(\int_a^b f(\lambda)\, \mathrm{d}E_\lambda \right) x\,, y \right\rangle = \int_a^b f(\lambda)\, \langle E_\lambda x\,, y\rangle, \qquad x,y \in H;$

   (b) $\int_a^b f(\lambda)\, \mathrm{d}E_\lambda = 0$ para $f \equiv 0$, $\qquad \int_a^b f(\lambda)\, \mathrm{d}E_\lambda = \int_a^b \mathrm{d}E_\lambda = E_b - E_a$ para $f \equiv 1$;

   (c) $E_\mu \int_a^b f(\lambda)\, \mathrm{d}E_\lambda = \int_a^\mu f(\lambda)\, \mathrm{d}E_\lambda, \quad a \le \mu \le b;$

   (d) $\left(\int_a^b f(\lambda)\, \mathrm{d}E_\lambda \right) \left(\int_a^b g(\lambda)\, \mathrm{d}E_\lambda \right) = \int_a^b f(\lambda) g(\lambda)\, \mathrm{d}E_\lambda;$

   (e) $\left(\int_a^b f(\lambda)\, \mathrm{d}E_\lambda \right)^* = \int_a^b \overline{f(\lambda)}\, \mathrm{d}E_\lambda;$

   (f) $\left\| \int_a^b f(\lambda)\, \mathrm{d}E_\lambda x \right\|^2 = \int_a^b |f(\lambda)|^2\, \mathrm{d}\|E_\lambda x\|^2, \quad x \in H.$

6. Sea $a : [0,1] \to \mathbb{R}$ continua y sea $A : L_2(0,1) \to L_2(0,1)$ definido por

$$(Ax)(t) := a(t)x(t), \qquad t \in (0,1), \quad x \in L_2(0,1).$$

   (a) Muestre que A es autoadjunto.

   (b) Encuentre $m := \inf_{x\in H, \|x\|=1} \langle Ax,\, x\rangle$ y $M := \sup_{x\in H,\|x\|=1} \langle Ax,\, x\rangle$.

   (c) Encuentre la resolución espectral de A.

7. Sean A y B operatores acotadas autoadjuntas en un espacio de Hilbert H con resoluciones espectrales $(E_A(\lambda))_{\lambda\in\mathbb{R}}$ y $(E_B(\lambda))_{\lambda\in\mathbb{R}}$. Si $A \ge B$, entonces[2] $\dim E_A(\lambda) \le \dim E_B(\lambda)$ para cada $\lambda \in \mathbb{R}$.

8. Sea H un espacio e Hilbert y $A \in L(H)$.

   (a) Muestre que $\mathrm{Exp}(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ converge en norma. Muestre que $\big(\mathrm{Exp}(A)\big)^* = \mathrm{Exp}(A^*)$. En particular, $\mathrm{Exp}(A)$ es autoadjunto y $\big(\mathrm{Exp}(\mathrm{i}A)\big)^* = \mathrm{Exp}(-\mathrm{i}A)$ si A es autoadjunto.

   (b) Muestre que $\mathrm{Exp}(A) = \exp(A)$ si A es autoadjunto y $\exp(A)$ es definido a traves del cálculo funcional.

9. Sea H un espacio complejo de Hilbert, A un operador autoadjunto tal que A^{-1} existe y es densamente definido. Sea U la transformada de Cayley de A. Muestre:

   (a) A^{-1} es simétrico.

   (b) La transformada de Cayley de A^{-1} es $-U^{-1}$.

[2] usando la notación $\dim P := \dim(\operatorname{rg} P)$ para una proyección ortogonal P.

(c) A^{-1} es autoadjunto.

10. Sea $(\mathrm{e}_n)_{n\in\mathbb{N}}$ una base ortonormal en un espacio complejo de Hilbert H y $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Define el operador $A(H \to H)$ por

$$\mathcal{D} := \{x \in H : \sum_{n=1}^{\infty} |\alpha_n \langle x\,, \mathrm{e}_n\rangle|^2 < \infty\}, \qquad Ax := \sum_{n=1}^{\infty} \alpha_n \langle x\,, \mathrm{e}_n\rangle\, \mathrm{e}_n \quad \text{for } x \in \mathcal{D}.$$

(a) Muestre que A es bien definido, cerrado y simétrico.

(b) Encuentre la transformada de Cayley de A.

11. Sea H un espacio complejo de Hilbert, $A \in L(H)$ un operador autoadjunto y $f \in C(\sigma(A))$. Muestre que $f(\sigma(A)) = \sigma(f(A))$. Muestre que $f(A)x = f(\lambda)x$ si $Ax = \lambda x$.

12. Sea $(E_\lambda)_{\lambda\in\mathbb{R}}$ una resolución de la identidad y $f : \mathbb{R} \to \mathbb{R}$ continua. Define $A(H \to H)$ por

$$Ax := \int_{\infty}^{\infty} f(\lambda)\, \mathrm{d}E_\lambda x \quad \text{para} \quad x \in \mathcal{D}(A) := \{x \in H : \int_{-\infty}^{\infty} |f(\lambda)|^2\, \mathrm{d}\langle E_\lambda x\,, x\rangle < \infty\}.$$

(a) $\operatorname{rg}(E_\lambda - E_\mu) \subseteq \mathcal{D}(A)$ para todo $\mu < \lambda \in \mathbb{R}$.

(b) $\mathcal{D}(A)$ es un subespacio denso de H y A es bien definido.

(c) A es autoadjunto.

(d) $E_\lambda A \subseteq AE_\lambda$.

13. (a) El left shift en $\ell_2(\mathbb{N})$ es la transformada de Cayley de un operador simétrico A? Si es así, determine A y sus indices de defecto $\dim(\operatorname{rg}(A \pm \mathrm{i})^\perp)$.

(b) El right shift en $\ell_2(\mathbb{N})$ es la transformada de Cayley de un operador simétrico B? Si es así, determine B y sus indices de defecto $\dim(\operatorname{rg}(B \pm \mathrm{i})^\perp)$.

14. Sea A un operador autoadjunto y $z \in \rho(A)$. Muestre que $\|(A - z)^{-1}\|^{-1} = \operatorname{dist}(z, \sigma(A))$.

15. Sea $P : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$, $(Pf)(t) = f(-t)$. Muestre que P es autoadjunto, calcule su espectro y su resolución espectral.

16. Sea A un operador autoadjunto en un espacio de Hilbert complejo H con resolución espectral $(E_\lambda)_{\lambda\in\mathbb{R}}$. Muestre

$$\operatorname*{s\text{-}lim}_{\varepsilon\searrow 0} \frac{1}{2\pi \mathrm{i}} \int_a^b \left[(A - \lambda - \mathrm{i}\varepsilon)^{-1} - (A - \lambda + \mathrm{i}\varepsilon)^{-1}\right] \mathrm{d}\lambda = \frac{1}{2}\big(E([a, b]) + E((a, b))\big).$$

17. Use la fórmula de Stone para encontrar la resolución espectral de al menos uno de los operadores siguientes:

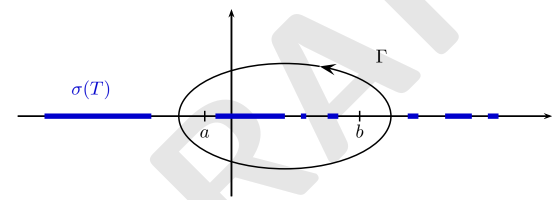
(a) $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ on $\mathbb{C}^2$.

(b) Sea (X, μ) un espacio de medida y $g : X \to \mathbb{R}$ una función μ-medible. Define el operador maximal de multiplicación T_g en $L_2(X)$ por

$$\mathcal{D}(T_g) := \big\{f \in L_2(X) : fg \in L_2(X)\big\}, \qquad T_g f := gf \quad \text{for } x \in \mathcal{D}(T_g).$$

18. Sea H un espacio de Hilbert complejo y $T(H \to H)$ un operador lineal autoadjunto. Sea $a, b \in \rho(T) \cap \mathbb{R}$ y Γ una curva de Jordan simple rectificable y positivamente orientada tal que encierra a $(a, b) \cap \sigma(T)$ y que el resto del espectro está fuera de Γ. Muestre

$$E(b) - E(a) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\lambda - T)^{-1}\, \mathrm{d}\lambda.$$



19. Sea X un espacio de Banach complejo y $T \in L(X)$ un operador acotado. Sea Γ una curva de Jordan simple rectificable y positivamente orientada tal que encierra $\sigma(T)$. Muestre:

$$T^n = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \lambda^n (\lambda - T)^{-1}\, \mathrm{d}\lambda, \qquad \lambda \in \mathbb{N}_0.$$

20. Sea X un espacio de Banach y $T(X \to X)$ un operador lineal cerrado. Un *conjunto espectral* (*spectral set*) es un subconjunto Σ de $\sigma(T)$ tal que Σ y $\sigma(T) \setminus \Sigma$ con cerrados en el plano complejo extendido. Sea Σ un conjunto espectral de T acotado y Γ una curva de Jordan rectificable en $\rho(T)$ tal que encierre Σ y $\sigma(T) \setminus \Sigma$ queda fuera de Γ. Muestre que $\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\lambda - T)^{-1}\, \mathrm{d}\lambda$ es una proyección que conmuta con T.

21. Sea H un espacio de Hilbert complejo y S, $T(H \to H)$ operadores autoadjuntos.

(a) Sea $z \in \rho(T)$ y $\lambda \in \mathbb{C} \setminus \{z\}$. Muestre que $\lambda \in \sigma_{\mathrm{ess}}(T)$ si y solo si existe una sucesión $(x_n)_{n\in\mathbb{N}} \subseteq H$ tal que

$$x_n \not\to 0, \quad x_n \xrightarrow{w} 0 \quad \text{and} \quad \big((T-z)^{-1} - (\lambda - z)^{-1}\big)x_n \to 0 \qquad \text{for } n \to \infty.$$

(b) Suponga que existe $z \in \rho(S) \cap \rho(T)$ tal que $(S-z)^{-1} - (T-z)^{-1}$ es compacto. Muestre que $\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(T)$.

Exercises for Chapter 3

1. Sea $\mathbb{R}^+ := (0,\infty)$ y $\mathcal{D}(T) := C_c^\infty(\mathbb{R}^+) = \{f \in \mathbb{C}^\infty \ : \ \operatorname{supp}(f) \text{ es compacto}\}$. Define

$$T : \mathcal{D}(T) \subseteq L_2(\mathbb{R}^+) \to L_2(\mathbb{R}^+), \quad Tx = \mathrm{i}x'.$$

Se puede mostrar que

$$\mathcal{D}(T^*) = \left\{ x \in L_2(\mathbb{R}^+) \ : \ \begin{array}{l} x|_I \text{ es abs. continua para cada intervalo compacto } I \subseteq \mathbb{R}^+ \\ \text{y } \ x' \in L_2(\mathbb{R}^+) \end{array} \right\}$$

y $T^*x = \mathrm{i}x'$ para $x \in \mathcal{D}(T^*)$.

Calcule los indices de defecto de T. Tiene extensiones autoadjuntos? Si es así, determine todas las extensiones autoadjuntas.

2. Sea H un espacio de Hilbert complejo y $S(H \to H)$ un operador lineal densamente definido y clausurable. Muestre que $\Gamma(S) = \Gamma(\overline{S})$ y que $n(S,\lambda) = n(\overline{S},\lambda)$ para todo $\lambda \in \Gamma(S)$. Concluya que $n(S,\cdot)$ es constante en componentes conexas de $\Gamma(S)$.

3. Sea H un espacio de Hilbert complejo, $S(H \to H)$ un operador simétrico con indices de defecto $n_+(S) = n_-(S) = m < \infty$.

(i) Sean T_1, T_2 extensiones autoadjuntas de S y $\lambda \in \mathbb{C}$ tal que $\operatorname{rg}(T_1 - \lambda)$ no es cerrado. Muestre que $\operatorname{rg}(T_2 - \lambda)$ tampoco lo es. Concluya que $\sigma_c(T_1) \subseteq \sigma(T_2)$ y $\sigma_c(T_2) \subseteq \sigma(T_1)$.

(ii) Sea $\lambda \in \Gamma(S) \cap \mathbb{R}$. Muestre que existe una extensión autoadjunta T de S tal que λ es un autovalor de T de dimensión finita.

Hint. $\dim(\ker(S^* - \lambda)) = ?$

4. Sea H un espacio de Hilbert complejo y $T(H \to H)$ autoadjunto. Suponga que existe $\lambda \in \mathbb{C}$ tal que $(T-\lambda)^{-1}$ es compacto. Muestre:

(i) $(T-\mu)^{-1}$ es compacto para todo $\mu \in \rho(T)$.

(ii) $\sigma(T) = \sigma_p(T)$, los autovalores no tienen un punto de acumulación y cada autovalor tiene multiplicidad finita.

5. Sea H un espacio de Hilbert complejo y $S(H \to H)$, $T(H \to H)$ operadores autoadjuntos. Suponga que existe un $z \in \rho(S) \cap \rho(T)$ tal que $(S-z)^{-1} - (T-z)^{-1}$ es compacto.

(a) Muestre que $(S-\lambda)^{-1} - (T-\lambda)^{-1}$ es compacto para todo $\lambda \in \rho(S) \cap \rho(T)$.

(b) Muestre que $\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(T)$.

Exercises for Chapter 4

1. Sean X, Y, Z espacios de Banach y $T(X \to Y)$, $S(X \to Z)$ operadores lineales. Muestre que S es T-acotado si y solo si $\mathcal{D}(S) \supseteq \mathcal{D}(T)$ y existe $\alpha, \beta \geq 0$ tal que

$$\|Sx\|^2 \leq \alpha^2\|x\|^2 + \beta^2\|Tx\|^2, \qquad x \in \mathcal{D}(T). \tag{$*$}$$

Muestre que el ínfimo de todo los $\beta > 0$ que satisfacen $(*)$ para un $\alpha \geq 0$ es igual a la T-cota de S.

Hint. Muestre que $2xy \leq c^2x^2 + c^{-2}y^2$ for $c, x, y \in \mathbb{R}$, $c \neq 0$.

2. Sea X un espacio de Banach y $T(X \to X)$ un operador lineal cerrado. Sea $S(X \to X)$ con $\mathcal{D}(S) \supseteq \mathcal{D}(T)$ y $z \in \rho(T)$. Muestre que S es T-compacto si y solo si $S(T-z)^{-1}$ es compacto.

3. Sea H un espacio de Hilbert complejo, $T(H \to H)$ un operador autoadjunto y semiacotado por abajo con cota inferior γ (es decir, $\langle Tx\,, x\rangle \geq \gamma\|x\|$ para todo $x \in \mathcal{D}(T)$). Sea $S(H \to H)$ un operador simétrico y T-acotado con T-cota < 1. Muestre que $T + S$ es semiacotado por abajo.

4. Sean X, Y, Z espacios de Banach, $T(X \to Y)$, $S(X \to Z)$ operadores lineales tal que S es clausurable y T-compacto. Muestre que S es T-acotado con T-cota 0.

5. Muestre que existen espacios de Hilbert H_1, H_2, y un operador lineal $T(H_1 \to H_2)$ y un operador S tal que S es T-compacto con T-cota 1.

Hint. Considere un funcional lineal no acotado en H_1.

6. Sea X un espacio de Banach, $T(X \to X)$ un operador cerrado con $\rho(T) \neq \emptyset$ y $\mathcal{D}_0 \subseteq \mathcal{D}(T)$. Muestre que $\mathcal{D}_0$ es un *core*[3] of T si y solo si $(T-\lambda)\mathcal{D}_0$ es denso en X para un (para todo) $\lambda \in \rho(T)$.

[3] Un subespacion $\mathcal{D}_0 \subseteq \mathcal{D}(T)$ es un *core* del operador cerrado T si la clausura de la restricción $T|_{\mathcal{D}_0}$ es igual a T.

7. Sea H un espacio de Hilbert complejo, $T, S(H \to H)$ operadores simétricos. Suponga que S es T-acotado con T-cota < 1. Muestre que $n_+(T+S) = n_+(T)$ y $n_-(T+S) = n_-(T)$.

   Hints. Basta mostrar que $\dim(\mathrm{rg}(T \pm \mathrm{i}\lambda)^\perp) = \dim(\mathrm{rg}(T + S \pm \mathrm{i}\lambda)^\perp)$ para un/todo $\lambda > 0$. Para mostrar $\dim(\mathrm{rg}(T \pm \mathrm{i}\lambda)^\perp) \geq \dim(\mathrm{rg}(T + S \pm \mathrm{i}\lambda)^\perp)$, se puede escoger, por ejemplo, $\lambda = a/b$ con a, b los constantes de la acotación relativa.

   Para mostrar que $\dim(\mathrm{rg}(T \pm \mathrm{i}\lambda)^\perp) \leq \dim(\mathrm{rg}(T + S \pm \mathrm{i}\lambda)^\perp)$ muestre que existe un $n \in \mathbb{N}$ tal que $\frac{1}{n}S$ tiene $(T + \mu S)$-cota < 1 para todo $\mu \in [0, 1]$. Concluye que
   $\dim\Big(\mathrm{rg}\,(T+S\pm\mathrm{i})^\perp\Big) = \dim\Big(\mathrm{rg}\,\big((T+\frac{n-1}{n}S)+\frac{1}{n}S\pm\mathrm{i}\big)^\perp\Big) \leq \dim\Big(\mathrm{rg}\,\big((T+\frac{n-1}{n}S)\pm\mathrm{i}\big)^\perp\Big) \leq \dots.$

Exercises for Chapter 5

1. Muestre que cada solución continua $f : \mathbb{R} \to \mathbb{R}$ de

$$f(s+t) = f(s)f(t)$$

   es diferenciable y por tanto es de la forma $f(t) = c\,\mathrm{e}^{ta}$.

2. Sea X un espacio de Banach complejo y $A \in L(X)$ un operador lineal acotado y $t \in \mathbb{R}$. Sea Γ una curva de Jordan rectificable y positivamente orientada tal que encierra el espectro de A. Muestre que

$$\sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \mathrm{e}^{\lambda t} (\lambda - A)^{-1}\, \mathrm{d}\lambda. \qquad (*)$$

   Si X es un espacio de Hilbert y A es autoadjunto, el operador en $(*)$ coincide con $\exp(tA)$ definido a través del cálculo funcional.

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DRAFT

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