Universidad de los Andes, MATE-4241
Monika Winklmeier, Fabián Latorre

Operator Theory

Problem Sheet 11

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Let X be a Banach space and $X_0 \subseteq X$ a subspace. For a linear operator A with not necessarily dense domain $\mathcal{D}(A) \subseteq X$ we define the *part of A in X_0* by

$$\mathcal{D}(A_|) = \{x \in \mathcal{D}(A) \cap X_0 \,:\, Ax \in X_0\}, \qquad A_| x = Ax, \quad x \in \mathcal{D}(A_|).$$

1. Let X be a Banach space, $A : \mathcal{D}(A) \subseteq X \to X$ a closed linear operator on X (not necessarily densely defined). Let $X_0 := \overline{\mathcal{D}(A)}$ and $A_|$ be the part of A in X_0. If there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

   $$\{\lambda \in \mathbb{R} \,:\, \lambda > \omega\} \subseteq \varrho(A) \quad \text{and} \quad \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \quad n \in \mathbb{N},\ \lambda > \omega,$$

   then $A_|$ is the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on X_0 with $\|T(t)\| \leq M \mathrm{e}^{t\omega}$, $t \geq 0$.

2. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a bounded strongly continuous semigroup on X. Then

   $$\|x\|_{\mathcal{T}} := \sup\{\, \|T(s)x\| : s \geq 0\,\}, \quad x \in X,$$

   defines a norm which is equivalent to $\|\cdot\|$.

3. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a bounded strongly continuous semigroup on X. Show that there exists an equivalent norm on X such that $\mathcal{T}$ is a contraction semigroup with respect to the new norm.

4. Let $X = \{f \in C[0,1] : f(1) = 0\}$ and

   $$T(t) : X \to X, \quad T(t)f(\xi) = \begin{cases} f(\xi + t), & \text{if } 0 \leq x + t \leq 1, \\ 0 & \text{else.} \end{cases}$$

   Show that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on X. Find its generator and its growth bound.