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Operator Theory

Problem Sheet 10

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1. Let Ω be a domain in $\mathbb{C}$ and $q \in C(\Omega)$ an unbounded function with $\sup_{\xi\in\Omega}\{\operatorname{Re} q(\xi)\} < \infty$. Let $X = C_0(\Omega)$ together with the supremum norm and $M(X \to X)$ the maximal multiplication operator corresponding to q and define $\mathcal{T} = (T(t))_{t\geq 0}$ by

$$\big(T(t)f\big)(\xi) = \mathrm{e}^{tq(\xi)} f(\xi), \qquad f \in X,\ \xi \in \Omega.$$

   (a) Show that $\mathcal{T}$ is a strongly continuous semigroup.
   (b) Show that $\mathcal{T}$ is not uniformly continuous.
   (c) Show that M is the generator of $\mathcal{T}$.

A semigroup is called *uniformly exponentially stable* if there exist $\omega > 0$ and $M \geq 1$ such that $\|T(t)\| \leq M\mathrm{e}^{-\omega t}$ for all $t \geq 0$.

2. Let $X = C_0(\mathbb{R})$ and $q(s) = -\frac{1}{1+|s|} + \mathrm{i}s$. Show that the corresponding multiplication semigroup is not uniformly exponentially stable but converges strongly to 0.

3. If $\mathcal{T} = (T(t))_{t\geq 0}$ is a uniformly continuous semigroup, then the following is equivalent:

   (a) $\mathcal{T}$ is uniformly exponentially stable.
   (b) $\lim_{t\to\infty} \|T(t)\| = 0$.
   (c) There exists a $t_0 > 0$ such that $\|T(t_0)\| < 1$.
   (d) There exists a $t_1 > 0$ such that $r(T(t_1)) < 1$ where $r(T(t_1))$ denotes the spectral radius of $T(t_1)$.

4. Let X be a Banach space and $K \subseteq \mathbb{R}$ a compact set. For a function $F : K \to L(X)$ the following is equivalent:

   (a) F is strongly continuous.
   (b) F is uniformly bounded on K and there exists a dense subset $D \subseteq X$ such that for every $x \in D$ the following map is continuous:

$$K \to X, \quad t \mapsto F(t)x.$$

   (c) For every compact subset $C \subseteq X$ the following map is uniformly continuous:

$$K \times C \to X, \quad (t, x) \mapsto F(t)x.$$