Universidad de los Andes, MATE-4241
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Operator Theory

Problem Sheet 9

Core of a linear operator; exp; semigroups. Hand in: 28th of October 2010

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1. Let X be a Banach space, $T(X \to X)$ a closed operator and $\mathcal{D}_0 \subseteq \mathcal{D}(T)$. Show that $\mathcal{D}_0$ is a core of T if and only of $(T - \lambda)\mathcal{D}_0$ is dense in X for one (for all) $\lambda \in \varrho(T)$.

2. Let X be a Banach space and $A(X \to X)$. Let Γ be a positively oriented Jordan curve which does not intersect $\sigma(A)$.

   (a) Show that $\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\lambda - A)^{-1} \,\mathrm{d}\lambda$ is a projection.

   (b) Show that, if A is bounded and Γ encloses $\sigma(A)$, then
   $$\exp(tA) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \mathrm{e}^{\lambda t} (\lambda - A)^{-1} \,\mathrm{d}\lambda.$$

3. Show that every continuous solution $f : \mathbb{R} \to \mathbb{R}$ of
   $$f(s + t) = f(s)f(t)$$
   is differentiable and consequently of the form $f(t) = c\mathrm{e}^{ta}$.

4. Define the semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ by
   $$(T(t)f)(\xi) = f(\xi + t), \qquad f \in X,\ \xi \in \mathbb{R}.$$

   (a) Show that $\mathcal{T}$ is a strongly continuous, but not uniformly continuous semigroup if $X = BUC(\mathbb{R})$ or $X = L_p(\mathbb{R})$ with $1 \leq p < \infty$.

   (b) Show that $\mathcal{T}$ is not strongly continuous if $X = L_\infty(\mathbb{R})$.