Universidad de los Andes, MATE-4241
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Operator Theory

Problem Sheet 8

Relative boundedness; relative compactness.

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1. Let X, Y, Z be Banach spaces and $T(X \to Y)$, $S(X \to Z)$ linear operators. Show that S is T-bounded if and only if $\mathcal{D}(S) \supseteq \mathcal{D}(T)$ and there exist $\alpha, \beta \geq 0$ such that

$$\|Sx\|^2 \leq \alpha^2 \|x\|^2 + \beta^2 \|Tx\|^2, \qquad x \in \mathcal{D}(T). \tag{$*$}$$

   Show that the infimum over all $\beta > 0$ such that $(*)$ holds for some $\alpha e 0$ is equal to the T-bound of S.

   Hint. Show that $2xy \leq c^2x^2 + c^{-2}y^2$ for $c, x, y \in \mathbb{R}$, $c \neq 0$.

2. Let X be a Banach spaces and $T(X \to X)$ a closed linear operator. Let $S(X \to X)$ with $\mathcal{D}(S) \supseteq \mathcal{D}(T)$ and $z \in \varrho(T)$. Show that S is T-compact if and only if $S(T - z)^{-1}$ is compact.

3. Let S and T be closed operators on a Banach space X. Show that $(S - z)^{-1} - (T - z)^{-1}$ is compact for some $z \in \varrho(S) \cap \varrho(T)$ if and only if it is compact for all $z \in \varrho(S) \cap \varrho(T)$.

4. Recall: If T is a closed operator between Hilbert spaces H_1 and H_2 and S is T-compact, then S has T-bound 0.

   Show that there exist Hilbert spaces H_1, H_2, a linear operator $T(H_1 \to H_2)$ and a T-compact operator S with T-bound 1.

   Hint. Consider an unbounded linear functional on H_1.