Universidad de los Andes, MATE-4241
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Operator Theory

Problem Sheet 6

Hand in: 16th of September 2010

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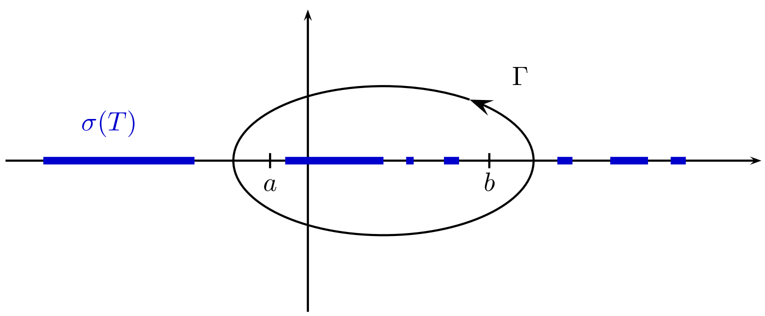
1. Let (Ω, Σ, μ) be a σ-finite measure space, X, Y Banach spaces and $f : \Omega \to X$ Bochner-integrable. Let $T \in L(H)$. Show that Tf is also Bochner-integrable and that

$$T \int_\Omega f \, \mathrm{d}\mu = \int_\Omega Tf \, \mathrm{d}\mu.$$

2. Let H be a complex Hilbert space and $T(H \to H)$ a selfadjoint linear operator. Let $a, b \in \varrho(T) \cap \mathbb{R}$ and Γ a positively oriented Jordan curve which encloses $(a, b) \cap \sigma(T)$ and the rest of the spectrum of T lies outside of Γ. Then

$$E(b) - E(a) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\lambda - T)^{-1} \, \mathrm{d}\lambda.$$



3. Let H be a complex Hilbert space, $T(H \to H)$ a selfadjoint linear operator with spectral resolution $(E_t)_{t \in \mathbb{R}}$ and $\lambda \in \mathbb{C}$. Show that the following is equivalent:

   (a) $\lambda \in \sigma_{\mathrm{d}}(T)$.

   (b) There exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(T)$ such that $x_n \not\to 0$ and $(T - \lambda)x_n \to 0$ for $n \to \mathbb{N}$ and every such sequence contains a convergent subsequence.

   (c) $0 \neq \dim(\operatorname{rg} E(\{\lambda\}) < \infty$ and there exists an $\varepsilon > 0$ such that $E((\lambda - \varepsilon, \lambda + \varepsilon)) = E(\{\lambda\})$.

4. Let H be a complex Hilbert space and $T(H \to H)$ a selfadjoint linear operator. Show that $\sigma(T) = \sigma_{\mathrm{ess}}(T) \cup \sigma_{\mathrm{d}}(T)$.