Universidad de los Andes, MATE-4241
Monika Winklmeier, Fabián Latorre

Operator Theory

Problem Sheet 5

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1. (a) Is the right shift on $\ell_2(\mathbb{N})$ the Cayley transform of a closed symmetric operator A? If so, find A and its deficiency indices $\dim(\operatorname{rg}(A \pm \mathrm{i})^\perp)$.

   (b) Is the left shift on $\ell_2(\mathbb{N})$ the Cayley transform of a closed symmetric operator B? If so, find B and its deficiency indices $\dim(\operatorname{rg}(B \pm \mathrm{i})^\perp)$.

2. Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a spectral resolution on a complex Hilbert space H. Let $x \in H$ and $f \in C(\mathbb{R}, \mathbb{C})$. Then the following is equivalent:

   (a) $\displaystyle\int_{-\infty}^{\infty} f(\lambda)\,\mathrm{d}E_\lambda x$ exists.

   (b) $\displaystyle\int_{-\infty}^{\infty} |f(\lambda)|^2\,\mathrm{d}\langle E_\lambda x\,, x\rangle$ exists
   (that is, $f \in L_2(\mathbb{R}, \mathrm{d}\alpha_x)$ where $\alpha_x(\lambda) = \langle E_\lambda x\,, x\rangle$ for $\lambda \in \mathbb{R}$).

   (c) The map $\varphi : H \to \mathbb{C}$, $\varphi(y) = \displaystyle\int_{-\infty}^{\infty} f(\lambda)\,\mathrm{d}\langle E_\lambda x\,, y\rangle$ is a bounded anti-linear functional.

3. Let A be a selfadjoint operator on a complex Hilbert space H with spectral resolution $(E_\lambda)_{\lambda\in\mathbb{R}}$. Then

$$s\text{-}\lim_{\varepsilon\searrow 0} \frac{1}{2\pi\mathrm{i}} \int_a^b \big[(A-\lambda-\mathrm{i}\varepsilon)^{-1} - (A-\lambda+\mathrm{i}\varepsilon)^{-1}\big]\mathrm{d}\lambda = \frac{1}{2}\big(E([a,b]) - E((a,b))\big).$$

4. Use Stone's formula to find the spectral resolution of at least one of the following operators:

   (a) $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ on $\mathbb{C}^2$.

   (b) Let (X, μ) be a measure space. For a μ-measurable function $g : X \to \mathbb{R}$ define the maximal multiplication operator T_g on $L_2(X)$ by

$$\mathcal{D}(T_g) := \big\{f \in L_2(X) : fg \in L_2(X)\big\}, \qquad T_g f := gf \quad \text{for } x \in \mathcal{D}(T_g).$$