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Operator Theory

Problem Sheet 4

Cayley transform.

Hand in: 2nd of September 2010

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1. Let A be a bounded selfadjoint operator on a complex Hilbert space H with spectral resolution $(E_\lambda)_{\lambda\in\mathbb{R}}$. Show that A is compact if and only if for every $\varepsilon > 0$ the projection $E(\{|\lambda| > \varepsilon\})$ has finite rank.

2. Let R be the right shift operator on $\ell_2(\mathbb{N})$, that is $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$ for $x = (x_1, x_2, x_3, \dots) \in \ell_2(\mathbb{N})$. Is there an operator $A \in L(\ell(\mathbb{N}))$ such that $A^2 = S$?

3. Let H be a complex Hilbert space, A a selfadjoint operator such that A^{-1} exists and is densely defined. Let U be its Cayley transform. Show:
   (a) A^{-1} is symmetric.
   (b) The Cayley transform of A^{-1} is $-U^{-1}$.
   (c) A^{-1} is selfadjoint.

4. Let $(\mathrm{e}_n)_{n\in\mathbb{N}}$ be an orthonormal basis of a complex Hilbert space H and $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Define the operator A by
$$\mathcal{D} := \{x \in H : \sum_{n=1}^{\infty} |\alpha_n \langle x\,, \mathrm{e}_n\rangle|^2 < \infty\}, \qquad Ax := \sum_{n=1}^{\infty} \alpha_n \langle x\,, \mathrm{e}_n\rangle \mathrm{e}_n \quad \text{for } x \in \mathcal{D}.$$
   (a) Show that A is well-defined, closed and symmetric.
   (b) Find the Cayley transform of A.