Universidad de los Andes, MATE-4241
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Operator Theory

Problem Sheet 3

Spectral theorem.

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1. Let H, $\varphi : \mathbb{R} \to (a, b)$, $(E_\lambda)_{\lambda\in\mathbb{R}}$ and $(F_\lambda)_{\lambda\in\mathbb{R}}$ be as in Problem Sheet 2, Exercise 3. Moreover let $f : (a, b) \to \mathbb{R}$ such that $f|_{[a_0,b_0]} \in I[a_0, b_0]$ for every compact subinterval $[a_0, b_0]$ of (a, b). Show:

   (a) $\displaystyle\int_{\varphi(\alpha)}^{\varphi(\beta)} f(\lambda)\,\mathrm{d}E_\lambda = \int_\alpha^\beta (f\circ\varphi)(\lambda)\,\mathrm{d}F_\lambda$ for all $[\alpha, \beta] \subseteq \mathbb{R}$.

   (b) Let $x \in H$. Then
   $$\int_{a+0}^{b-0} f(\lambda)\,\mathrm{d}E_\lambda x := \lim_{\substack{\lambda\searrow a\\ \lambda_2\nearrow b}} \int_{\lambda_1}^{\lambda_2} f(\lambda)\,\mathrm{d}E_\lambda$$
   exists if and only if
   $$\int_{-\infty}^{\infty} (f\circ\varphi)(\lambda)\,\mathrm{d}F_\lambda x := \lim_{\substack{\lambda\searrow -\infty\\ \lambda_2\nearrow\infty}} \int_{\lambda_1}^{\lambda_2} (f\circ\varphi)(\lambda)\,\mathrm{d}F_\lambda\, x$$
   exists.

2. Let $a : [0, 1] \to \mathbb{R}$ be continuous and $A : L_2(0, 1) \to L_2(0, 1)$ be defined by
   $$(Ax)(t) := a(t)x(t), \qquad t \in (0, 1), \quad x \in L_2(0, 1).$$

   (a) Show that A is selfadjoint.

   (b) Find $m := \inf_{x\in H,\|x\|=1}(Ax, x)$ and $M := \sup_{x\in H,\|x\|=1}(Ax, x)$.

   (c) Find the spectral resolution of A.

3. Let A and B be bounded selfadjoint operators on a Hilbert space H with spectral resolutions $(E_A(\lambda))_{\lambda\in\mathbb{R}}$ and $(E_B(\lambda))_{\lambda\in\mathbb{R}}$. Show that $\dim E_A(\lambda) \le \dim E_B(\lambda)$ [1] for every $\lambda \in \mathbb{R}$ if $A \ge B$.

4. Let H be a Hilbert space and $A \in L(H)$.

   (a) Show that $\mathrm{Exp}(A) := \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} A^n$ converges in the operator norm. Show that $\big(\mathrm{Exp}(A)\big)^* = \mathrm{Exp}(A^*)$. In particular, $\mathrm{Exp}(A)$ is selfadjoint and $\big(\mathrm{Exp}(\mathrm{i}A)\big)^* = \mathrm{Exp}(-\mathrm{i}A)$ if A is selfadjoint.

   (b) Show that $\mathrm{Exp}(A) = \exp(A)$ for selfadjoint A where $\exp(A)$ is defined via the continuous functional calculus.

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[1] using the notation $\dim P := \dim(\mathrm{rg}\, P)$ for an orthogonal projection P.