Universidad de los Andes, MATE-4241
Monika Winklmeier, Fabián Latorre

Operator Theory

Problem Sheet 2

Functions of bounded variation; spectral resolution.

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1. Let $\alpha \in \mathrm{BV}[a,b]$, $f \in I[a,b]$ and define $K : [a,b] \to \mathbb{K}$ by $K(x) := \int_a^x f(t)\,\mathrm{d}\alpha(t)$ for $x \in (a,b]$ and $K(a) := 0$. Show:
   (a) $K \in \mathrm{BV}[a,b]$.
   (b) If α is right continuous, then so is K.
   (c) $\int_a^b g(t)\,\mathrm{d}K(t) = \int_a^b (fg)(t)\,\mathrm{d}\alpha(t)$ for all $g \in I[a,b]$.

2. Let H be a Hilbert space and $T \in L(H)$ a compact selfadjoint operator with pairwise distinct eigenvalues μ_j and let P_j be the orthogonal projections on the corresponding eigenspaces. Show that $(E_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral resolution where
$$E_\lambda x := \begin{cases} \sum_{\lambda_j \le \lambda} P_j x, & \lambda < 0, \\ x - \sum_{\lambda_j > \lambda} P_j x, & \lambda \ge 0, \end{cases} \qquad \lambda \in \mathbb{R},\ x \in H.$$

3. Let H be a Hilbert space, $(E_\lambda)_{\lambda\in\mathbb{R}}$ a spectral resolution on H and $\varphi : \mathbb{R} \to (a,b)$ a continuous monotonically increasing bijection. Moreover assume that $E_a = 0$ and $E_{b-0} = E_b = I$. Show that $(F(\lambda))_{\lambda\in\mathbb{R}}$ is a spectral resolution on H where
$$F_\lambda := E_{\varphi(\lambda)}, \quad \lambda \in \mathbb{R}.$$

4. Let H be a Hilbert space, $(E_\lambda)_{\lambda\in\mathbb{R}}$ a spectral resolution on H and $f, g \in I[a,b]$. Show:
   (a) $\Big\langle \Big(\int_a^b f(\lambda)\,\mathrm{d}E_\lambda\Big) x\,, y \Big\rangle = \int_a^b f(\lambda)\,\langle E_\lambda x\,, y\rangle, \qquad x, y \in H;$
   (b) $\int_a^b f(\lambda)\,\mathrm{d}E_\lambda = 0$ for $f \equiv 0$, $\qquad \int_a^b f(\lambda)\,\mathrm{d}E_\lambda = \int_a^b \mathrm{d}E_\lambda = E_b - E_a$ for $f \equiv 1$;
   (c) $E_\mu \int_a^b f(\lambda)\,\mathrm{d}E_\lambda = \int_a^\mu f(\lambda)\,\mathrm{d}E_\lambda, \quad a \le \mu \le b;$
   (d) $\Big(\int_a^b f(\lambda)\,\mathrm{d}E_\lambda\Big)\Big(\int_a^b g(\lambda)\,\mathrm{d}E_\lambda\Big) = \int_a^b f(\lambda)g(\lambda)\,\mathrm{d}E_\lambda;$
   (e) $\Big(\int_a^b f(\lambda)\,\mathrm{d}E_\lambda\Big)^* = \int_a^b \overline{f(\lambda)}\,\mathrm{d}E_\lambda;$
   (f) $\Big\| \int_a^b f(\lambda)\,\mathrm{d}E_\lambda x \Big\|^2 = \int_a^b |f(\lambda)|^2\,\mathrm{d}\|E_\lambda x\|^2, \quad x \in H.$