Universidad de los Andes, MATE-4241
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Operator Theory

Problem Sheet 1

Hand in: 13th of August 2010

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1. **1.** Let H be a Hilbert space. If $(x_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal vectors in H, then the following are equivalent:

   (a) $\sum_{n=1}^{\infty} x_n$ converges in the norm topology of H.

   (b) $\sum_{n=1}^{\infty} ||x_n||^2 < \infty$.

   (c) $\sum_{n=1}^{\infty} \langle x_n, y\rangle$ converges for every $y \in H$.

2. Let P_1 and P_2 be orthogonal projections acting on the Hilbert space H. Then we have

$$\|P_1 - P_2\| = \max\{\varrho_{12},\ \varrho_{21}\}$$

   where

$$\varrho_{jk} := \sup\Big\{\|P_j x\| \ :\ x \in \operatorname{rg}(P_k)^{\perp},\ \|x\| \le 1\Big\}.$$

3. If P and Q are orthogonal projection on the Hilbert space H such that $||P - Q|| < 1$, then we have

$$\dim(\operatorname{rg} P) = \dim(\operatorname{rg} Q), \quad \dim(\operatorname{rg}(I-P)) = \dim(\operatorname{rg}(I-Q)).$$

4. Define the right shift operator S on $\ell_2(\mathbb{Z})$ by

$$(Sx)_k = x_{k-1}, \quad k \in \mathbb{Z},$$

   where $x = (x_k)_{k=-\infty}^{\infty}$ is in $\ell_2(\mathbb{Z})$. Find $\sigma_{\rm p}(S)$, $\sigma_{\rm c}(S)$, $\sigma_{\rm r}(S)$.