Contents

These lecture notes are work in progress. They may be abandoned or changed radically at any moment. If you find mistakes or have suggestions how to improve them, please let me know.

Nothing in this lecture notes is new. My main sources are excellent books by H. Bauer, *Measure and integration theory* [Bau01], J. Elstrodt, *Maß- und Integrationstheorie* [Els05], G. Folland, *Real analysis* [Fol84], P.!Halmos, *Measure theory* [Hal50], W. Rudin, *Real and complex analysis* [Rud87], H. Widom, *Lectures on measure and integration* [Wid69].

An important part of any mathematics lecture are exercises. For each week there is a problem sheet with exercises (stolen from various books) which hopefully help to understand the material presented in the lecture.

Corrections, comments and remarks on the text are most welcome!

Bogotá, August 2026,
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Chapter 1

Rings, Algebras, Contents, Measures

1.1 Rings and Algebras

Throughout this section, let X be a set and let $A, B \subseteq X$.
The *characterstic function of A* is

$$\chi_A : X \to \mathbb{C}, \qquad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Of course, we can view χ_A also as function from X to $\mathbb{R}$.

Definition 1.1. The *symmetric difference* of A and B is

$$A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Clearly,

$$A \triangle \varnothing = A, \qquad A \triangle X = X \setminus A, \qquad A \triangle A = \varnothing.$$

Remark 1.2. Let $A, B \subseteq X$.

(i) $A \triangle B = B \triangle A$.

(ii) If $A \cap B = \varnothing$, then $A \triangle B = A \cup B$.

(iii) If $A \subseteq B$, then $A \triangle B = B \setminus A$.

Theorem 1.3. *The triple $(\mathbb{P}(X), \triangle, \cap)$ is a commutative ring with identity, where $\triangle$ is addition, $\cap$ is multiplication, $\varnothing$ is zero, and X is one.*

Proof. The claim follows from straightforward calculations.

An alternative, shorter proof is achieved by identifying sets with their characteristic functions.

Let $\mathbb{K} = \{0, 1\}$ be the field with two elements and let

$$\mathcal{R} := \{f : X \to \mathbb{K}\}$$

be the set of all functions from X to $\mathbb{K}$. It is easy to check that $\mathcal{R}$ is a commutative ring with addition and multiplication defined pointwise. Its zero element is $f \equiv 0$ and its one is $f \equiv 1$.
We define the map

$$\Phi : \mathbb{P}(X) \longrightarrow \mathcal{R}, \qquad A \longmapsto \chi_A.$$

Clearly, Ψ is a bijetion and

$$\Psi(A \cap B) = \Psi(A)\Psi(B), \qquad \Psi(A \triangle B) = \Psi(A) + \Psi(B). \qquad A, B \subseteq X. \tag{1.1}$$

The first equality in (1.1) is clear, the second equality follows from

$$\Psi(A \triangle B) = \chi_{(A\setminus B)\dot\cup(B\setminus A)} = \chi_{A\setminus B} + \chi_{B\setminus A} = \chi_{A\setminus B} + \chi_{B\setminus A} + \underbrace{\chi_{A\cap B} + \chi_{A\cap B}}_{=0} = \chi_A + \chi_B = \Psi(A) + \Psi(B).$$

Hence $(\mathbb{P}X, \cap, \triangle)$ is a commutative ring with

$$0 = \Psi^{-1}(0) = \Psi^{-1}(\chi_\varnothing) = \varnothing, \qquad 1 = \Psi^{-1}(1) = \Psi^{-1}(\chi_X) = X. \qquad \square$$

Definition 1.4. Let X be a set.

(a) A family $\mathcal{R} \subseteq \mathbb{P}(X)$ is called a *ring over* X if it is a subring of $(\mathbb{P}(X), \triangle, \cap)$.

(b) A family $\mathcal{A} \subseteq \mathbb{P}(X)$ is called an *algebra over* X if it is a ring over X and $X \in \mathcal{A}$.

Theorem 1.5. *Let X be a set and let $\mathcal{R} \subseteq \mathbb{P}(X)$. The following are equivalent:*

(i) *$\mathcal{R}$ is a ring over X.*

(ii) *$\varnothing \in \mathcal{R}$ and, for all $A, B \in \mathcal{R}$, one has $A \triangle B \in \mathcal{R}$ and $A \cap B \in \mathcal{R}$.*

(iii) *$\varnothing \in \mathcal{R}$ and, for all $A, B \in \mathcal{R}$, one has $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$.*

(iv) *$\varnothing \in \mathcal{R}$ and, for all $A, B \in \mathcal{R}$, one has $A \triangle B \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$.*

(v) *$\varnothing \in \mathcal{R}$ and, for all $A, B \in \mathcal{R}$, one has $A \triangle B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$.*

Proof. Let us start with a remark on the condition $\varnothing \in \mathcal{R}$. Note that a ring is never empty because it contains at leat the empty set. Therefore, in (ii)–(v) we need explicitely the condition $\varnothing \in \mathcal{R}$ because $\mathcal{R} = \varnothing$ would satisfy closedness under unions, intersections, etc., but it would not be a ring.
Alternatively, we could could replace the assumption $\mathcal{R} \neq \varnothing$ by $\varnothing \in \mathcal{R}$. Then, under each of the assumptions in (ii)–(v) it follows that $\varnothing \in \mathcal{R}$ because $\varnothing = A \triangle A = A \setminus A$ for every $A \in \mathcal{R}$.

(i) $\iff$ (ii) is just the definition of a ring.

(ii) $\Longrightarrow$ (iii): Let $A, B \in \mathcal{R}$. Then

$$A \cup B = (A \triangle B) \triangle (A \cap B) \in \mathcal{R}, \qquad A \setminus B = (A \triangle B) \cap A \in \mathcal{R}.$$

Note that for the second equality we could also use for instance that $A \setminus B = (A \cap B) \triangle A \in \mathcal{R}$.
(iii) $\Longrightarrow$ (iv): Let $A, B \in \mathcal{R}$. Then

$$A \triangle B = (A \setminus B) \cup (A \setminus B) \in \mathcal{R}.$$

(iv) $\Longrightarrow$ (ii): Let $A, B \in \mathcal{R}$. Then

$$A \cap B = (A \cup B) \triangle (A \triangle B) \in \mathcal{R}.$$

(ii) $\Longrightarrow$ (v) is clear because (ii) implies (iii).
(v) $\Longrightarrow$ (ii): Let $A, B \in \mathcal{R}$. Then

$$A \cap B = A \setminus (A \triangle B) \in \mathcal{R}.$$

□

Just to practice a bit algebra with sets, we note that (iv) $\Longrightarrow$ (iii) follows from

$$A \setminus B = A \triangle (A \cup B) \in \mathcal{R},$$

and (ii) $\Longrightarrow$ (iv) follows from

$$A \cup B = (A \triangle (A \cup B)) \triangle B \in \mathcal{R}.$$

Remark. You should also show directly that (v) is equivalent to each of the other assumptions.

Theorem 1.6. *Let X be a set and let $\mathcal{A} \subseteq \mathbb{P}(X)$. The following are equivalent:*

(i) *$\mathcal{A}$ is an algebra over X.*

(ii) *$X \in \mathcal{A}$ and, for all $A, B \in \mathcal{A}$, one has $X \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.*

(iii) *$X \in \mathcal{A}$ and, for all $A, B \in \mathcal{A}$, one has $X \setminus A \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$.*

Proof. (i) $\Longrightarrow$ (ii): Follows from the definition of an algebra and Theoreem 1.5, (i) $\Longrightarrow$ (iii).
(ii) $\Longrightarrow$ (iii): Let $A, B \in \mathcal{A}$. Then

$$A \cap B = X \setminus \Big[(X \setminus A) \cup (X \setminus B)\Big] \in \mathcal{A}.$$

(iii) $\Longrightarrow$ (i): Let $A, B \in \mathcal{A}$. Then

$$\begin{aligned} A \triangle B &= (A \cup B) \setminus (A \cap B) = (A \cup B) \cap \Big[X \setminus (A \cap B)\Big] \\ &= \Big(X \setminus [(X \setminus A) \cap (X \setminus B)]\Big) \cap \Big[X \setminus (A \cap B)\Big] \in \mathcal{A}. \end{aligned}$$

Therefore, $\mathcal{A}$ is a ring. Since it contains X, it is even an algebra. □

Examples 1.7. Let X be a set.

(i) For every $A \subseteq X$,

- $\{\varnothing,\, A\}$ is a ring over X; it is an algebra if and only if $A = X$.
- $\{\varnothing,\, X,\, A,\, X \setminus A,\}$ is an algebra over X.

(ii) $\mathbb{P}(X)$ is an algebra over X.

(iii) If $X \neq \varnothing$, then

$$E := \{A \subseteq X : A \text{ is finite }\}$$

is a ring. It is an algebra if and only if X is finite.

(iv) If $X \neq \varnothing$, then

$$C := \{A \subseteq X : A \text{ is countable }\}$$

is a ring. It is an algebra if and only if X is countable[1].

(v) If X is uncountable, then

$$\{A \subseteq X : A \text{ is finite or } X \setminus A \text{ is finite}\}$$

is an algebra over X.

Definition 1.8. Let X be a set.

(a) A family $\mathcal{R} \subseteq \mathbb{P}(X)$ is called a *σ-ring over X* if $\mathcal{R}$ is a ring over X and for every sequence $(A_j)_{j\in\mathbb{N}}$ in $\mathcal{R}$ one has

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{R}.$$

(b) A family $\mathcal{A} \subseteq \mathbb{P}(X)$ is called a *σ-algebra over X* if $X \in \mathcal{A}$ and $\mathcal{A}$ is a σ-ring over X.

Theorem 1.9. *Let X be a set, $\mathcal{R} \subseteq \mathbb{P}(X)$, and $\mathcal{A} \subseteq \mathbb{P}(X)$.*

(a) *$\mathcal{R}$ is a σ-ring over X* $\iff$ $\begin{cases} \varnothing \in \mathcal{R}, \\ A, B \in \mathcal{R} \implies A \setminus B \in \mathcal{R}, \\ (A_j)_{j\in\mathbb{N}} \subseteq \mathcal{R} \implies \bigcup_{j=1}^{\infty} A_j \in \mathcal{R}. \end{cases}$

(b) *$\mathcal{R}$ is a σ-algebra over X* $\iff$ $\begin{cases} X \in \mathcal{A}, \\ A \in \mathcal{A} \implies X \setminus A \in \mathcal{A}, \\ (A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}. \end{cases}$

(c) *$\mathcal{R}$ is a σ-algebra over X* $\iff$ $\begin{cases} X \in \mathcal{A}, \\ A \in \mathcal{A} \implies X \setminus A \in \mathcal{A}, \\ (A_j)_{j\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcap_{j=1}^{\infty} A_j \in \mathcal{A}. \end{cases}$

[1] A set is *countable* if it is finite or countably infinite.

Proof. (a): "$\Longrightarrow$" is clear. For the direction "$\Longleftarrow$", fix $A, B \in \mathcal{R}$ and set $A_1 := A$, $A_2 := B$, $A_j := \varnothing$ for $j \geq 3$. Then $A \cup B = \bigcup_{j=1}^{\infty} A_j \in \mathcal{R}$. Therefore, $\mathcal{R}$ is a ring by Theorem 1.5.

(b) "$\Longrightarrow$" is true by defnition of a σ-algebra. "$\Longleftarrow$" follows as above using Theorem 1.6.

The right hand side of (c) is equivalent to the right hand side of (b) because

$$\bigcup_{j=1}^{\infty} A_j = X \setminus \Big(\bigcap_{j=1}^{\infty} (X \setminus A_j) \Big) \quad \text{and} \quad \bigcap_{j=1}^{\infty} A_j = X \setminus \Big(\bigcup_{j=1}^{\infty} (X \setminus A_j) \Big). \qquad \square$$

Examples 1.10. Let X, Y be sets.

(i) $\mathbb{P}(X)$ is a σ-algebra over X.

(ii) Every finite ring is a σ-ring; every finite algebra is a σ-algebra.

(iii) $\{A \subseteq X : A \text{ is countable } \}$ is a σ-ring over X.

(iv) If $f : X \to Y$ and $\mathcal{R}$ is a σ-ring over Y, then

$$f^{-1}(\mathcal{R}) := \{f^{-1}(B) : B \in \mathcal{R}\}$$

is a σ-ring over X.

(v) If $\mathcal{A}$ is a σ-algebra over Y, then $f^{-1}(\mathcal{A})$ is a σ-algebra over X.

(vi) If $X \subseteq Y$ and $\mathcal{C}$ is a σ-ring/σ-algebra over Y, then

$$\mathcal{C}|_X := \{B \cap X : B \in \mathcal{C}\}$$

is a σ-ring/σ-algebra over X. it is called the *trace* of $\mathcal{C}$ on X.

Note that this is a special case of (iv) and (v).

Remark 1.11. (i) The intersection of a family of rings is a ring.
The intersection of a family of σ-rings is a σ-ring.

(ii) The intersection of a family of algebras is an algebra.
The intersection of a family of σ-algebras is a σ-algebra.

Let X be a set and $\mathcal{Y} \subseteq \mathbb{P}X$. Since $\mathbb{P}X$ is a σ-algebra (in particular also an algebra and a (σ-)ring), the intersections

$$\mathcal{R}(\mathcal{Y}) := \bigcap_{\substack{\mathcal{Y} \subseteq \mathcal{R} \\ \mathcal{R} \text{ ring over } X}} \mathcal{R}, \qquad \sigma\text{-}\mathcal{R}(\mathcal{Y}) := \bigcap_{\substack{\mathcal{Y} \subseteq \mathcal{R} \\ \mathcal{R}\ \sigma\text{-ring over } X}} \mathcal{R},$$

$$\mathcal{A}(\mathcal{Y}) := \bigcap_{\substack{\mathcal{Y} \subseteq \mathcal{A} \\ \mathcal{A} \text{ algebra over } X}} \mathcal{A}, \qquad \sigma\text{-}\mathcal{A}(\mathcal{Y}) := \bigcap_{\substack{\mathcal{Y} \subseteq \mathcal{A} \\ \mathcal{R}\ \sigma\text{-algebra over } X}} \mathcal{A}$$

contain $\mathcal{Y}$. By Remark 1.11, $\mathcal{R}(\mathcal{Y})$, σ-$\mathcal{R}(\mathcal{Y})$, $\mathcal{A}(\mathcal{Y})$, σ-$\mathcal{A}(\mathcal{Y})$ are a ring, σ-ring, algebra, σ-algebra, respectively. Clearly, they are the smallest ring/σ-ring/algebra/σ-algebra that contains $\mathcal{Y}$. This justifies the following definition.

Definition 1.12. Let X be a set and $\mathcal{Y} \subseteq \mathbb{P}X$. Then the *ring/$\sigma$-ring/algebra/$\sigma$-algebra generated by* $\mathcal{Y}$ is the smallest ring/σ-ring/algebra/σ-algebra over X that contains $\mathcal{Y}$.

A very important example are the Borel σ-algebras.

Example 1.13 (Borel σ-algebra). Let X be a topological space and let τ be the topology on X (that is, τ is the set of all open sets in X). Then

$$\mathfrak{B}(X) := \sigma(\tau) := \sigma\text{-algebra generated by } \tau$$

is called the *Borel σ-algebra on X*.

Definition 1.14. Let X be a set. A family $H \subseteq \mathbb{P}(X)$ is called a *semiring over X* if

(i) $\varnothing \in H$;

(ii) $A, B \in H$ implies $A \cap B \in H$;

(iii) for all $A, B \in H$, there exist pairwise disjoint $C_1, \dots, C_n \in H$ such that

$$A \setminus B = \dot{\bigcup_{j=1}^{n}} C_j.$$

Examples 1.15. (i) The family $\{\varnothing\} \cup \{\{a\} : a \in X\}$ is a semiring over X. It is a ring only in the trivial cases where $|X| \leq 1$.

(ii) The families

$$\mathcal{J} := \{[a,b) : a \leq b,\ a, b \in \mathbb{R}\}, \qquad \{(a,b] : a \leq b,\ a, b \in \mathbb{R}\} \tag{1.2}$$

are semirings over $\mathbb{R}$.

In contrast to Remark 1.11, we have the follwing.

Remark 1.16. The intersection of two semirings is *not* necessarily a semiring.
For example, let $H_1 := \{\varnothing,\ \{a,b\},\ \{c\},\ \{a,b,c\}\}$ $H_1 := \{\varnothing,\ \{a\},\ \{b\},\ \{c\},\ \{a,b,c\}\}$. Then H_1 and H_2 are smeirings but $H_1 \cap H_2$ is not.

Lemma 1.17. *Let H and K be semirings over X and Y, respectively. Then*

$$H \times K := \{A \times B : A \in H,\ B \in K\}$$

is a semiring over $X \times Y$.

Proof. Clearly $\varnothing = \varnothing \times \varnothing \in H \times K$. Moreover, let $C_1 = A_1 \times B_1,\ C_2 = A_2 \times B_2 \in H \times K$. Then

$$C_1 \cap C_2 = (A_1 \cap A_2) \times (B_1 \cap B_2) \in H \times K.$$

In order to see that $C_1 \setminus C_2$ is the disjoint union of elements in $H \times K$, we write $A_1 \setminus A_2 = \dot\bigcup_{j=1}^{n} C_j$ and $B_1 \setminus B_2 = \dot\bigcup_{k=1}^{m} D_k$ as disjoint unions of elements in H and K. Then

$$\begin{aligned} C_1 \setminus C_2 &= (A_1 \times B_1) \setminus (A_2 \times B_2) = ((A_1 \setminus A_2) \times B_1) \dot\cup \Big[(A_1 \cap A_2) \times (B_1 \setminus B_2) \Big] \\ &= \bigcup_{j=1}^{n} (C_j \times B_1) \ \dot\cup \ \bigcup_{k=1}^{m} ((A_1 \cap A_2) \times D_k). \end{aligned}$$

$\square$

Corollary 1.18. *Let $n \in \mathbb{N}$ and let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ with $a_j \le b_j$ for all j. Define*

$$[a, b) := [a_1, b_1) \times \cdots \times [a_n, b_n), \qquad (a, b] := (a_1, b_1] \times \cdots \times (a_n, b_n].$$

Then

$$\mathcal{J}^n := \{[a, b) : a, b \in \mathbb{R}^n,\ a_j \le b_j \text{ for all } j\}$$

and

$$\mathcal{K}^n := \{(a, b] : a, b \in \mathbb{R}^n,\ a_j \le b_j \text{ for all } j\}$$

are semirings over $\mathbb{R}^n$.

Theorem 1.19. *Let X be a set and let $H \subseteq \mathbb{P}(X)$ be a semiring over X. Then*

$$\mathcal{R}(H) = \left\{ \dot{\bigcup}_{j=1}^{n} A_j \ :\ n \in \mathbb{N},\ A_1, \dots, A_n \in H \right\}. \tag{1.3}$$

Equivalently, every element of the generated ring is a finite union of sets from the semiring. Such a union may be chosen pairwise disjoint.

Proof. Let $\mathcal{R}$ be the set on the right hand side of (1.4). Clearly $H \subseteq \mathcal{R}$ and if every ring that contains H, must also contain $\mathcal{F}$. So it remains to show that $\mathcal{R}$ is a ring, because in that case, it is the smallest ring containing H. Note that $\varnothing \in H \in \mathcal{R}$. Hence, Theorem 1.5 we have to show that $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$ for all $A, B \in \mathcal{R}$.
Let $A, B \in \mathcal{R}$. Then we write them as $A = \bigcup_{j=1}^{n} N_j$ and $B = \bigcup_{k=1}^{m} M_j$ with $N_j, M_k \in H$. Consequentrly, $A \cup B = \left(\bigcup_{j=1}^{n} N_j\right) \cup \left(\bigcup_{k=1}^{m} M_k\right) \in \mathcal{R}$.
That $A \setminus B \in \mathcal{R}$, is shonw by induction. If $m = 1$, then

$$A \setminus B = A \setminus M_1 = \bigcup_{j=1}^{n} (N_j \setminus M_1),$$

Since each $N_j \setminus M_1$ is the (disjoint) union of sets elements in H, it follows that $A \setminus B \in \mathcal{R}$.
Now assume that the claim is true for some m and let $B = \bigcup_{k=1}^{m+1} N_k$ with $M_1, \dots, M_{m+1} \in H$. Set $\widetilde{B} = \bigcup_{k=1}^{m} N_k$. By the hypothesis of induction, $A \setminus \widetilde{B} \in \mathcal{R}$, hence, by the base case, also

$$A \setminus B = (A \setminus \widetilde{B}) \setminus M_{m+1} \in \mathcal{R}.$$

□

Remark 1.20. It is easy to see that also

$$\mathcal{R}(H) = \left\{ \dot{\bigcup}_{j=1}^{n} A_j \ :\ n \in \mathbb{N},\ A_1, \dots, A_n \in H \right\}. \tag{1.4}$$

Definition 1.21. Let X be a set. Then $\mathfrak{M} \subseteq \mathbb{P}X$ is called a *monotone class* if for every sequence $(A_j)_{j \in \mathbb{N}} \subseteq \mathfrak{M}$ the following holds:

(i) $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \quad \Longrightarrow \quad \displaystyle\bigcup_{j=1}^{\infty} A_j \in \mathfrak{M}$,

(ii) $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \quad \Longrightarrow \quad \bigcap_{j=1}^{\infty} A_j \in \mathfrak{M}$.

Remark 1.22. (i) Every σ-ring is a monotone class. (Use that $\bigcap_{j=1}^{\infty} = A_1 \setminus \bigcup_{j=1}^{\infty}(A_1 \setminus A_j)$ for decreasing sequences $(A_j)_{j\in\mathbb{N}} \subseteq \mathfrak{M}$.)

(ii) Every monotone ring ia s σ-ring.

(iii) The intersection of monotone classes is a monotone class.

(iv) $\mathbb{P}X$ is a monotone class.

(v) A monotone class does not need to be a ring. For instance, $\mathfrak{M} := \{[0, x) : x \in \mathbb{R}\}$ is a monotone class, but not a ring.

Let $\mathcal{Y} \subseteq X$. As in Definition 1.12, we call the smallest monotone class containing $\mathcal{Y}$ the *monotone class generated by* $\mathcal{Y}$. It is given by

$$\mathfrak{M}(\mathcal{Y}) := \bigcap_{\substack{\mathfrak{M}\subseteq\mathcal{R} \\ \mathcal{R} \text{ mon. class in } X}} \mathfrak{M}.$$

Theorem 1.23. *Let $\mathcal{R}$ be a ring over a set X and let*

$$S := \sigma - \textit{ring generated by } \ \mathcal{R}, \qquad \mathfrak{M} := \textit{monotone class generated by } \ \mathcal{R}.$$

Then $S = \mathfrak{M}$.

Proof. $\underline{\mathfrak{M} \subseteq S}$: By Remark 1.22, S is a monotone class, hence $\mathfrak{M} \subseteq S$.
$\underline{S \subseteq \mathfrak{M}}$: It is sufficient to show that $\mathfrak{M}$ is a σ-ring. Let us fix $A \in \mathfrak{M}$ and define

$$Q(A) = \{B \subseteq X \ : \ A \setminus B,\ B \setminus A,\ A \cup B \in \mathfrak{M}\}.$$

This is the set of all sets B so that the "ring operations with A are good".
Clearly, for each fixed $A \in \mathfrak{M}$, the set $Q(A)$ is a montone class Moreover, for $A, B \in \mathfrak{M}$, we have

$$A \in Q(B) \quad \Longleftrightarrow \quad B \in Q(A). \tag{1.5}$$

We note that

(a) $A \in R \implies \mathfrak{M} \subseteq Q(A)$ because clearly $R \subseteq Q(A)$ and $Q(A)$ is a monotone class;

(b) $B \in \mathfrak{M} \implies \mathfrak{M} \subseteq Q(B)$ because for all $A \in R$, we have $B \in Q(A)$ by (a), hence also $A \in Q(B)$ by (1.5). Therefore $R \subseteq Q(B)$, so also $\mathfrak{M} \subseteq Q(B)$ because $Q(B)$ is a monotone class containing R.

For all $\widetilde{A}, \widetilde{B} \in \mathfrak{M}$ we have

- $\widetilde{A} \cup \widetilde{B}$, $\widetilde{A} \setminus \widetilde{B} \in \mathfrak{M}$ by definition of $Q(\widetilde{B}$ since $\widetilde{A} \in Q(\widetilde{B})$ by (b).
- $\varnothing \in \mathfrak{M}$ is clear because $\varnothing \in R \subseteq \mathfrak{M}$.

Therefore, $\mathfrak{M}$ is a ring by Theorem 1.5. It is even a σ-ring by Remark 1.22. $\square$

Corollary 1.24. *Let $\mathcal{A}$ be an algebra over a set X. Then*

$$\sigma\text{-algebra generated by } \mathcal{A} = \text{monotone class generated by } \mathcal{A}.$$

Definition 1.25. A family $D \subseteq \mathbb{P}X$ is called a *Dynkin system* over X if

(i) $X \in D$;
(ii) $A \in D \implies X \setminus A \in D$;
(iii) if $(A_j)_{j\in\mathbb{N}} \subseteq D$ are pairwise disjoint, then

$$\dot{\bigcup}_{j=1}^{\infty} A_j \in D.$$

Clearly, $\mathbb{P}X$ is a Dynkin system, and arbitrary intersections of Dynkin systems are Dynkin systems, so again there exists a smallest Dynkin system containg a given subset $\mathcal{Y} \subset X$, and it is the intersetcion of all Dynkin systems containing $\mathcal{Y}$.

Theorem 1.26. *Let $D \subseteq \mathbb{P}X$. Then*

$$D \ \text{ is a Dynkin system} \iff \begin{cases} X \in D, \\ A, B \in D \text{ with } A \supseteq B \implies A \setminus B \in D, \\ D \ \text{ is a monotone class.} \end{cases}$$

Proof. Assume that D ia a Dynkin system. Then $\varnothing = X \setminus X \in D$. Now let $A, B \in D$ with $B \subseteq A$. Then

$$A \setminus B = X \setminus \Big[(X \setminus A) \,\dot\cup\, B \,\dot\cup\, \varnothing \,\dot\cup\, \varnothing \dots \Big] \in D.$$

Now let $(A_j)_{j\in\mathbb{N}} \subseteq D$. If $A_1 \subseteq A_2 \subseteq \cdots$ we define $B_1 = A_1$ and $B_j = A_j \setminus A_{j-1}$ for $j \geq 2$. Then $B_j \in D$ for all $j \in \mathbb{N}$, they are pairwise disjoint and

$$\dot{\bigcup}_{j=1}^{\infty} A_j = \dot{\bigcup}_{j=1}^{\infty} B_j \in D.$$

If $A_1 \supseteq A_2 \supseteq \cdots$, then

$$\bigcap_{j=1}^{\infty} A_j = A_1 \setminus \bigcup_{j=1}^{\infty} (A_1 \setminus A_j) \in D.$$

Thus D is a monotone class.
Conversely, suppose that the right hand side in the claim holds. We have to show that $\dot{\bigcup}_{j=1}^{\infty} A_j \in D$ if $(A_j)_{j\in\mathbb{N}}$ is a sequence of pairwise disjoint elements in D. We first prove the following claim: if $A, B \in D$ with $A \cap B = \varnothing$, then $A \cup B \in D$. Note that by hypothesis $C \in D$ if and only if $X \setminus C \in D$. Therefore, it suffices to show that $X \setminus (A \cup B) \in D$. This follows from

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) = (X \setminus A) \setminus B.$$

Since $A \in D$, it follows that $X \setminus A \in D$. Moreover, $A \cap B = \varnothing$ implies that $B \subseteq X \setminus A$, hence also $(X \setminus A) \setminus B \in D$.

The claim now shows that $B_k := \dot{\bigcup}_{j=1}^{k} A_j \in D$ for every $k \in \mathbb{N}$, but then also $\dot{\bigcup}_{j=1}^{\infty} A_j = \bigcup_{k=1}^{\infty} B_k \in D$ because D is a monotone class and $B_1 \subseteq B_2 \subseteq \cdots$. □

Corollary 1.27. *Let $D \subseteq \mathbb{P}X$ be a Dynkin system. Then*

$$D \text{ is stable under intersections} \quad \Longleftrightarrow \quad D \text{ is a } \sigma\text{-algebra.}$$

Proof. "$\Longleftarrow$" is clear because every σ-algebra is stable under intersections.

"$\Longrightarrow$"" If D is stable under intersections, then it is an algebra. By Theorem 1.26 it is also a monotone class. Therefore, by Corollary 1.22, D is a σ-algebra. □

Theorem 1.28. *Let $\mathcal{Y} \subseteq \mathbb{P}X$ be stable under intersections. Let*

$$\begin{aligned} D &:= \text{Dynkin system generated by } \mathcal{Y}, \\ \mathcal{A} &:= \sigma\text{-algebra generated by } \mathcal{Y}. \end{aligned}$$

Then $D = \mathcal{A}$.

Proof. $\underline{D \subseteq \mathcal{A}}$: This is clear because $\mathcal{A}$ is a Dynkin system and $\mathcal{Y} \subseteq \mathcal{A}$.

$\underline{\mathcal{A} \subseteq D}$: By Corollary 1.27 it suffices to prove that D is stable under intersections. This will be done similar as in the proof of Theorem 1.23. Let $A \in D$ and define $Q(A) := \{B \subseteq X : A \cap B \in D\}$. Then $Q(A)$ is a Dynkin system. Moreover,

- $A \in \mathcal{Y} \quad \Longrightarrow \quad \mathcal{Y} \subseteq Q(A)$ because $\mathcal{Y}$ is stable under intersections,
  $\phantom{A \in \mathcal{Y}} \quad \Longrightarrow \quad D \subseteq Q(A)$ because D is the smallest Dynkin system that contains $\mathcal{Y}$
- $D \subseteq Q(B)$ for all $B \in D$ because if $A \in \mathcal{Y}$, then $B \in D \subseteq Q(A)$ by the above, hence also $A \in Q(B)$ by definition. Since D is the smallest Dynkin system containing $\mathcal{Y}$, it follows that $D \subseteq Q(B)$.

Therefore, for all $A, B \in D$ we have that $A \in Q(B)$, hence $A \cap B \in D$, showing that D is stable under intersections. □

1.2 Contents and Measures

We use the following convention: For $a \in \mathbb{R}$ and $b > 0$,

$$a + \infty = \infty, \qquad \infty + \infty = \infty, \qquad b \cdot \infty = \infty.$$

Definition 1.29. Let H be a semiring over X. A map

$$\mu : H \longrightarrow \mathbb{R} \cup \{\infty\}$$

is called a *content* on H if

(a) $\mu(\varnothing) = 0$;
(b) $\mu(A) \geq 0$ for every $A \in H$;
(c) if $A_1, \dots, A_n \in H$ are pairwise disjoint and $\dot{\bigcup}_{j=1}^{n} A_j \in H$, then

$$\mu\Big(\dot{\bigcup}_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \mu(A_j).$$

A content is a *premeasure* on H if the finite additivity in (c) is replaced by *σ-additivity*:

((c)') if $(A_j)_{j\in\mathbb{N}} \subseteq H$ is a sequence of pairwise disjoint sets and $\dot{\bigcup}_{j=1}^{\infty} A_j \in H$, then

$$\mu\Big(\dot{\bigcup}_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j).$$

If H is a σ-algebra and μ is a premeasure on H, then μ is called a *measure* on H, and (X, H, μ) is called a *measure space*. A content, premeasure, or measure is called *finite* if $\mu(A) < \infty$ for every $A \in H$.

If H is a ring, then the condition $\dot{\bigcup}_{j=1}^{n} \in H$ in (c) is automatically fullfilled. Similarly, if H is a σ-ring, then the condition $\bigcup_{j=1}^{\infty} A_j \in H$ in ((c)') is automatically fullfilled.

Remark 1.30. (i) If at least one summand in (c) or ((c)') is infinite, then the sum/series is infinite.
(ii) The order of summation in ((c)') is irrelevant because all terms are nonnegative and therefore the series can be reordered arbitrarily without changing its value. The series either converges in $\mathbb{R}$ or is ∞.
(iii) The condition $\mu(\varnothing) = 0$ excludes the somewhat trivia case $\mu(A) = \infty$ for all $A \in H$.
(iv) For a finite content, $\mu(\varnothing) = 0$ follows from positivity and finite additivity, since

$$\mu(\varnothing) = \mu(\varnothing \mathbin{\dot\cup} \varnothing) = \mu(\varnothing) + \mu(\varnothing).$$

Examples 1.31. (i) For $n \in \mathbb{N}$ let

$$\mathcal{J} = \{[a,\, b) : a, b \in R^n,\ a \leq b\}$$

the semiring from Corollary 1.18. Then

$$\mu\left([a, b)\right) = \prod_{j=1}^{n} (b_j - a_j)$$

defines a content on $\mathcal{J}^n$.

(ii) Let H be a semiring over a set X.

$$\mu(A) = \begin{cases} 0, & A = \varnothing, \\ \infty, & A \neq \varnothing \end{cases}$$

defines a content on H.

(iii) The *counting measure* on a set X is

$$\mu : \mathbb{P}X \longrightarrow \mathbb{R} \cup \{\infty\}, \qquad \mu(A) = \begin{cases} |A|, & A \text{ finite}, \\ \infty, & \text{else}. \end{cases}$$

It is easy to verify that μ is indeed a measure.

(iv) Let X be a set and fix $a \in X$. The *Dirac measure* at a is

$$\delta_a(B) = \mathbf{1}_B(a) = \begin{cases} 1, & a \in B, \\ 0, & a \notin B. \end{cases}$$

It is easy to verify that $(X, \mathbb{P}X, \delta_a)$ is a measure space. More generally, if $(a_j)_{j\in\mathbb{N}} \subseteq X$ are pairwise distinct and $\alpha_j \geq 0$, then $\sum_{j=1}^{\infty} \alpha_j \delta_{a_j}$ defines a measure on X. We say that μ is *concentrated on* $\{a_j : j \in \mathbb{N}\}$.

(v) Let X be countably infinite. Then

$$\mathcal{A} := \{A \subseteq X : A \text{ or } X \setminus A \text{ is finite}\}$$

is an algebra, but not a σ-algebra, and

$$\mu(A) = \begin{cases} 0, & A \text{ finite}, \\ 1, & X \setminus A \text{ finite} \end{cases}$$

defines a content on $\mathcal{A}$ which is not a premeasure.

Proof. Homework. □

Lemma 1.32. *Let μ be a content on a semiring H, and let $A, B \in H$ with $A \subseteq B$. Then*

$$\mu(A) \leq \mu(B).$$

Proof. Since H is a semiring, we can choose pairwise disjoint $C_1, \dots, C_n \in H$ such that $B \setminus A = \bigcup_{j=1}^{n} C_j$. Then

$$\mu(B) = \mu\Big(A \,\dot\cup\, \dot{\bigcup_{j=1}^{n}} C_j\Big) = \mu(A) + \sum_{j=1}^{n} \mu(C_j) \geq \mu(A). \qquad \square$$

Proposition 1.33 (Extension of a content). *Let H be a semiring over X and let $\mu : H \to \mathbb{R} \cup \{\infty\}$ be a content. Let $\mathcal{R}(H)$ be the ring generated by H (see Theorem 1.19). For $A \in \mathcal{R}(H)$ choose pairwise disjoint $A_1, \dots, A_n \in H$ such that $A = \bigcup_{j=1}^{n} A_j$, and define*

$$\widetilde{\mu}(A) = \sum_{j=1}^{n} \mu(A_j).$$

Then

(i) *$\widetilde{\mu}$ is a content on $\mathcal{R}(H)$;*
(ii) *$\widetilde{\mu}$ is a premeasure on $\mathcal{R}(H)$ if μ is σ-additive;*
(iii) *$\widetilde{\mu}|_H = \mu$;*
(iv) *$\widetilde{\mu}$ is uniquely determined by (i) and (iii).*

Proof. First we show that $\widetilde{\mu}$ is well defined. Let $M_1, \dots, M_m$ and $N_1, \dots, N_n$ in H with $M_j \cap M_k = \varnothing$ and $N_j \cap N_k = \varnothing$ for $j \neq k$ so that $\dot{\bigcup}_{j=1}^m M_i = \bigcup_{j=1}^n N_j =: A \in \mathcal{R}(H)$. Note that $M_j \cap N_k \in H$ for all j, k and that $M_j = \dot{\bigcup}_{k=1}^n M_j \cap N_k$ and $N_j = \dot{\bigcup}_{j=1}^m N_j \cap M_k$. Therefore we obtain

$$\sum_{j=1}^m \mu(M_j) = \sum_{j=1}^m \sum_{k=1}^n \mu(M_j \cap N_k) = \sum_{j=1}^n \sum_{i=1}^m \mu(M_j \cap N_k) = \sum_{k=1}^n \mu(N_k).$$

This shows that $\widetilde{\mu}$ is well defined and that (iii) holds.

Now let us show that $\widetilde{\mu}$ is a content. By definition of $\widetilde{\mu}$ we have $\widetilde{\mu}(\varnothing) = 0$ and $\widetilde{\mu}(M) \geq 0$ for all $M \in \mathcal{R}(H)$. For pairwise disjoint $A_1, \dots, A_n \in \mathcal{R}(H)$ choose pairwise disjoint $B_{jk} \in H$ such that $A_j = \dot{\bigcup}_{k=1}^{m_j} B_{jk}$. Then

$$\sum_{j=1}^n \widetilde{\mu}(A_j) = \sum_{j=1}^n \sum_{k=1}^{m_j} \mu(B_{jk} = \mu\Big(\dot{\bigcup} j = 1^n \dot{\bigcup}_{k=1}^{m_j} B_{jk} \Big) = \widetilde{\mu}\Big(\dot{\bigcup}_{j=1}^n A_j \Big).$$

To prove (ii) let us now assume that μ is σ-additive. Let $(A_j)_{j \in \mathbb{N}}$ be pairwise disjoint elements in $\mathcal{R}(H)$ with $A = \dot{\bigcup}_{j=1}^\infty A_j \in \mathcal{R}(H)$. Choose pairwise disjoint $B_{jk} \in H$ and $C_m \in H$ such that

$$A_j = \dot{\bigcup}_{k=1}^{n_j} B_{jk}, \qquad A = \dot{\bigcup}_{m=1}^n C_m.$$

We have to show that $\widetilde{\mu}(A) = \sum_{n=1}^\infty \widetilde{\mu}(A_j)$. Note that

$$\begin{aligned} C_m &= C_m \cap A = C_m \cap \dot{\bigcup}_{j=1}^\infty A_j = \dot{\bigcup}_{j=1}^\infty (C_m \cap A_j) = \dot{\bigcup}_{j=1}^\infty \dot{\bigcup}_{k=1}^{n_j} (C_m \cap B_{jk}), \\ A_j &= A_j \cap A = \dot{\bigcup}_{k=1}^{n_j} B_{jk} \cap \dot{\bigcup}_{m=1}^n C_m = \dot{\bigcup}_{k=1}^{n_j} \dot{\bigcup}_{m=1}^n (B_{jk} \cap C_m), \end{aligned}$$

and that $C_m \cap B_{jk} \in H$. Therefore

$$\begin{aligned} \widetilde{\mu}(A) &= \sum_{m=1}^n \mu(C_m) = \sum_{m=1}^n \mu\Big(\dot{\bigcup}_j^\infty \dot{\bigcup}_k^{n_j} C_m \cap B_{jk} \Big) \\ &= \sum_{m=1}^n \sum_j^\infty \sum_k^{n_j} \mu\big(C_m \cap B_{jk} \big) = \sum_j^\infty \sum_{m=1}^n \sum_k^{n_j} \mu\big(C_m \cap B_{jk} \big) \\ &= \sum_j^\infty \widetilde{\mu}\Big(\dot{\bigcup}_{m=1}^n \dot{\bigcup}_k^{n_j} C_m \cap B_{jk} \Big) = \sum_j^\infty \widetilde{\mu}\Big(A_j \Big) \end{aligned}$$

where the σ-additivity of μ is used in the third step.

For (iv) let μ' be a content on $\mathcal{R}(H)$ with $\mu'|_H = \mu$. Then, for any $A = \dot\bigcup_{m=1}^n C_m \in \mathcal{R}$ with $C_m \in H$, we have

$$\mu'(A) = \sum_{m=1}^n \mu'(C_m) = \sum_{m=1}^n \mu(C_m) = \widetilde{\mu}(A). \qquad \square$$

Theorem 1.34 (Properties of a content). *Let $\mathcal{R} \subseteq \mathbb{P}X$ be a ring, let $A, B, A_j \in \mathcal{R}$, and let $\mu : \mathcal{R} \to \mathbb{R} \cup \{\infty\}$ be a content.*

(i) *If $B \subseteq A$ and $\mu(B) < \infty$, then $\mu(A \setminus B) = \mu(A) - \mu(B)$;*
(ii) *$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$;*
(iii) *subadditivity: $\mu\left(\bigcup_{j=1}^n A_j\right) \le \sum_{j=1}^n \mu(A_j)$;*
(iv) *if the A_j are pairwise disjoint and $\bigcup_{j=1}^\infty A_j \subseteq B$ then*

$$\sum_{j=1}^\infty \mu(A_j) \le \mu(B);$$

(v) *if the A_j are pairwise disjoint and $\dot\bigcup_{j=1}^\infty A_j \in \mathcal{R}$ then*

$$\sum_{j=1}^\infty \mu(A_j) \le \mu\big(\dot\bigcup_{j=1}^\infty A_j\big);$$

(vi) *σ-subadditivity if μ is a premeasure: if μ is a premeasure and $B \subseteq \bigcup_{j=1}^\infty A_j$, then*

$$\mu(B) \le \sum_{j=1}^\infty \mu(A_j).$$

Proof. (i) Note that

$$\mu(A) = \mu((A \setminus B) \,\dot\cup\, B) = \mu(A \setminus B) + \mu(B).$$

Since $\mu(B) < \infty$, we can subtract it on both sides of the equality and obtain the claim.
(ii) follows from

$$\mu(A) + \mu(B) = \mu(A) + \mu((B \setminus A) \,\dot\cup\, (A \cap B)) = \mu(A) + \mu(B \setminus A) + \mu(A \cap B) = \mu(A \cup B) + \mu(A \cap B).$$

(iii) Note that for all $n \in \mathbb{N}$ we have that $\cup_{j=1}^n A_j = \dot\cup_{j=1}^n \big(A_j \setminus \cup_{k=1}^{j-1} A_k\big)$. Therefore

$$\mu\big(\bigcup_{j=1}^n A_j\big) = \mu\Big(\dot\bigcup_{j=1}^n \big(A_j \setminus \cup_{k=1}^{j-1} A_k\big)\Big) = \sum_{j=1}^n \mu\big(A_j \setminus \cup_{k=1}^{j-1} A_k\big) \le \sum_{j=1}^n \mu(A_j).$$

(iv) Note that $B \supseteq \dot\bigcup_{j=1}^n A_j$ for all $n \in \mathbb{N}$. Therefore,

$$\mu(B) \ge \mu\Big(\dot\bigcup_{j=1}^n A_j\Big) = \sum_{j=1}^n \mu(A_j).$$

Taking the limit $n \to \infty$ on both sides shows (iv).
(v) follows directly from (iv) by taking $B = \dot{\bigcup}_{j=1}^{\infty} A_j$.
(vi) Note that

$$B = B \cap \bigcup_{j=1}^{\infty} A_j = B \cap \dot{\bigcup}_{j=1}^{\infty} \Big(A_j \setminus \bigcup_{k=1}^{j-1} A_k \Big) = \dot{\bigcup}_{j=1}^{\infty} \underbrace{B \cap \Big(A_j \setminus \bigcup_{k=1}^{j-1} A_k \Big)}_{\in \mathcal{R}}.$$

Therefore, since μ is σ-additive, we obtain

$$\mu(B) = \mu\Big(\dot{\bigcup}_{j=1}^{\infty} B \cap \Big(A_j \setminus \bigcup_{k=1}^{j-1} A_k \Big) \Big) = \sum_{j=1}^{\infty} \mu \Big(\underbrace{B \cap \Big(A_j \setminus \bigcup_{k=1}^{j-1} A_k \Big)}_{\subseteq A_j} \Big) \leq \sum_{j=1}^{\infty} \mu(A_j) \qquad \square$$

Remark 1.35. (i) Note that in (iv) and (vi) we did not assume that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{R}$.

(ii) Strict inequality in (v) is possible. For example, let $X = \{x_n : n \in \mathbb{N}\}$ be a countable and infinite set, $\mathcal{A} := \{B \subseteq X : B \;\text{ or }\; X \setminus B \;\text{ is finite }\}$ and $\mu : \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ $\mu(B) := 0$ if B is finite and $\mu(B) = \infty$ if $X \setminus B$ is finite. Then $\mathcal{A}$ is an álgebra, μ is a non-σ-finite content and

$$\sum_{j=1}^{\infty} \mu(\{x_j\} = 0 \neq 1 = \mu\Big(\dot{\bigcup} \{x_j\} \Big) = \mu(X) = 1.$$

For another example, see Example 1.41.

Theorem 1.36 (Characterization of σ-additivity). *Let $\mathcal{R}$ be a ring over X and let $\mu : \mathcal{R} \to \mathbb{R} \cup \{\infty\}$ be a content. Consider the following statments:*

(A) *μ is a premeasure.*

(B) *For every sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}$ such that $A_1 \subseteq A_2 \subseteq \dots$ and $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$,*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

(C) *For every sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}$ such that $A_1 \supseteq A_2 \supseteq \dots$ and $A := \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$ and $\mu(A_1) < \infty$,*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

(D) *For every sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}$ such that $A_1 \supseteq A_2 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} A_n = \varnothing$ and $\mu(A_1) < \infty$,*

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

Then

$$\text{(A)} \iff \text{(B)}, \qquad \text{(C)} \iff \text{(C)}, \qquad \text{(B)} \implies \text{(C)},$$

If μ is finite, then also (C) $\iff$ (D),

Proof. (A) $\implies$ (B): Sea $B_1 := A_1$ and $B_j := A_j \setminus A_{j-1}$ for $j \geq 2$. Then $A = \bigcup_{j=1}^{\infty} A_j = \dot{\bigcup}_{j=1}^{\infty} B_j$ and $A_n = \bigcup_{j=1}^{n} B_j$, hence

$$\mu(A) = \mu\Big(\dot{\bigcup}_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j) = \lim_{n\to\infty} \mu\Big(\dot{\bigcup}_{j=1}^{n} B_j\Big) = \lim_{n\to\infty} \mu(A_n).$$

(B) $\implies$ (A): We have to schow that μ is σ-additive. Let $(A_j)_{j\in\mathbb{N}} \subseteq \mathcal{R}$ of pairwise disjoint sets with $A := \dot{\bigcup}_{j=1}^{\infty} A_j \in \mathcal{R}$. Let $C_k := \dot{\bigcup}_{j=1}^{k} A_j$. Then all C_k belong to $\mathcal{R}$, their union is A and $C_1 \subseteq C_2 \subseteq \dots$, hence, by our hypothesis (B), we obtain

$$\mu(A) = \mu\Big(\bigcup_{k=1}^{\infty} C_k\Big) = \lim_{k\to\infty} \mu(C_k) = \lim_{k\to\infty} \mu\Big(\dot{\bigcup}_{k=1}^{n} A_k\Big) = \lim_{k\to\infty} \sum_{k=1}^{n} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k).$$

(B) $\implies$ (C): Note for all $j \in \mathbb{N}$ we have that $\mu(A_j) < \infty$, $A_1 \setminus A_j \in \mathcal{R}$ and $A_1 \setminus A_j \subseteq A \setminus A_{j+1}$. Moreover $\bigcup_{j=1}^{\infty}(A_1 \setminus A_j) = A_1 \setminus \bigcap_{j=1}^{\infty} A_j = A_1 \setminus A$. Therefore,

$$\mu(A_1 \setminus A) = \lim_{n\to\infty} \mu(A_1 \setminus A_k) = \lim_{n\to\infty} (\mu(A_1) - \mu(A_k)) = \mu(A_1) - \lim_{n\to\infty} \mu(A_k) \tag{1.6}$$

and, because $\mu(A_1) < \infty$,

$$\mu(A_1 \setminus A) = \mu(A_1) - \mu(A) \tag{1.7}$$

Hence, (1.6) and (1.7) show that $\mu(A) = \lim_{k\to\infty} \sum_{k=1}^{n} \mu(A_k)$.

(C) $\implies$ (D): is clear.

(D) $\implies$ (C): For all $n \in \mathbb{N}$ let $B_n := A_n \setminus A$. Then for all $n \in \mathbb{N}$ we have that $B_n \in \mathcal{R}$, $B_n \supseteq B_{n+1}$ and $\bigcap_{m=1}^{\infty} B_n = \varnothing$. Therefore,

$$0 = \lim_{n\to\infty} \mu(B_j) = \lim_{n\to\infty} \mu(A_n \setminus A) = \lim_{n\to\infty} (\mu(A_n) - \mu(A)) = \lim_{n\to\infty} \mu(A_n) - \mu(A),$$

hence $\mu(A) = \lim_{n\to\infty} \mu(A_j)$.

Now let us assume that μ is finite. We will show (D) $\implies$ (B): Let $(A_n)_{n\in\mathbb{N}}$ and A as in (C). For all $n \in \mathbb{N}$ let $C_n := A \setminus A_n$. Then for all $n \in \mathbb{N}$ we have that $C_n \in \mathcal{R}$, $C_n \supseteq C_{n+1}$ and $\bigcap_{m=1}^{\infty} C_n = \varnothing$. Therefore,

$$0 = \lim_{n\to\infty} \mu(C_j) = \lim_{n\to\infty} \mu(A \setminus A_j) = \lim_{n\to\infty} (\mu(A) - \mu(A_j)) = \mu(A) - \lim_{n\to\infty} \mu(A_j),$$

hence $\mu(A) = \lim_{n\to\infty} \mu(A_j)$. $\square$

The example in Remark 1.35(ii) shows that in general (D) does not imply (B).

1.3 Finite contents and premeasures on $\mathbb{R}$

In this section we will describe all finite contents and premeasures on the semiring

$$\mathcal{J} = \{[a,b) : a, b \in \mathbb{R},\ a \leq b\}$$

from Example 1.15. By Proposition 1.33 this will then give all finite measures on the ring generated by $\mathcal{J}$.

Theorem 1.37. (i) *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function, and define*

$$\mu_F([a,b)) = F(b) - F(a).$$

Then μ_F is a finite content on $\mathcal{J}$. If $F, G : \mathbb{R} \to \mathbb{R}$ are increasing, then

$$\mu_F = \mu_G \iff F - G \text{ is constant.}$$

(ii) *Conversely, let $\mu : \mathcal{J} \to \mathbb{R}$ be a finite content and define*

$$F(x) = \begin{cases} \mu([0,x)), & x \geq 0, \\ -\mu([x,0)), & x < 0. \end{cases}$$

Then F is increasing, $F(0) = 0$, and $\mu = \mu_F$.

The content μ_F is called the *Stieltjes content* corresponding to F.

Proof. (i): Clearly $\mu_F(\varnothing) = 0$ and $\mu_F([a,b)) \geq 0$ for all $[a,b)\mathcal{J}$. To see finite additivity, let $a \leq b$ and $a_j \leq b_1$ such that $[a,b) = \dot{\bigcup}_{j=1}^{n}[a_j, b_j)$. Without restriction, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. It follows that $a = a_1 \leq b_1 = a_2 \leq b_2 = a_2 \leq \cdots \leq b_n = b$, in particular $F(b_j) = F(a_{j+1})$ for $j = 1, \ldots n-1$. Hence

$$\begin{aligned} \sum_{j=1}^{n} \mu_F([a_j, b_j)) &= \sum_{j=1}^{n} \big(F(b_j) - F(a_j)\big) = \sum_{j=1}^{n} F(b_j) - \sum_{j=1}^{n} F(a_j)\big) \\ &= F(b_n) + \sum_{j=1}^{n-1} F(a_{j+1}) - \sum_{j=1}^{n} F(a_j)\big) = F(b_n) - F(a_1) = F(b) - F(a) = \mu_F([a,b)). \end{aligned}$$

Note that $\mu_F = \mu_G$ if and only if $\mu_f([a,b)) = \mu_G([a,b))$ for all $a < b$, that is, if and only if

$$F(b) - F(a) = G(b) - G(a),$$

that is $F(b) - G(b) = F(a) - G(a)$ for all $a, b \in \mathbb{R}$. This is equivalent to $F - G$ being constant.
(ii): By definition, the function F is increasing. To verify $\mu = \mu_F$, let $a \leq b$.
Case 1: $a \leq b < 0 \implies F(b) - F(a) = -\mu([b,0)) - (-\mu([a,0)) = \mu([a,b))$.
Case 2: $a < 0 \leq b \implies F(b) - F(a) = \mu([b,0)) - (-\mu([a,0)) = \mu([a,b))$.
Case 3: $0 \leq a \leq b \implies F(b) - F(a) = \mu([b,0)) - \mu([0,a)) = \mu([a,b))$. □

Clearly, the function F in (ii) could be replaced by

$$F_c(x) = \begin{cases} \mu([c,x)), & x \geq 0, \\ -\mu([x,c)), & x < 0, \end{cases}$$

or any $c \in \mathbb{R}$. Then $F_c(c) = 0$ and $F - F_c = \mu([0,c))$ if $c > 0$ and $F - F_c = -\mu([c,0))$ if $c < 0$.

Example 1.38. . Let $\mu = \delta_0$ on $\mathcal{J}$, that is

$$\mu(A) = \begin{cases} 1, & 0 \in A, \\ 0, & 0 \notin A. \end{cases}$$

Then $\delta_0 = \mu_F$ for

$$F(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Theorem 1.39 (Finite premeasures on $\mathcal{J}$). *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function and let $\mu := \mu_F$ be as above. Then*

$$\mu \text{ is a premeasure} \quad \iff \quad F \text{ is left-continuous.}$$

Proof. Assume first that μ_F is a premeasure. Fix $a \in \mathbb{R}$, and let $(a_n)_{n\in\mathbb{N}} \subseteq (-\infty, a)$ such that $\lim_{n\to\infty} a_n = a$. Without restriction we may assume that the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing. Since μ is a premeasure, the sequence of sets $([a_n, a))_{n\in\mathbb{N}}$ is decreasing with $\bigcap_{n=1}^{n} [a_n, a) = \varnothing$ and $\mu([a_1, a) < \infty$, we obtain that $F(a) - F(a_n) = \mu([a_n, a))$ is convergent for $n \to \infty$ and

$$\lim_{n\to\infty} \big(F(a) - F(a_n)\big) = \lim_{n\to\infty} \mu([a_n, a)) = \mu\big(\bigcap_{n=1}^{\infty} [a_n, a)\big) = \mu(\varnothing) = 0.$$

Thus $F(a_n) \to F(a)$, so F is left-continuous.
Now suppose that F is left-continuous. Let $a < b$, and let $[a_j, b_j) \in \mathcal{J}$ be pairwise disjoint with

$$[a,b) = \bigcup_{j=1}^{\infty} [a_j, b_j).$$

We have to show $\mu([a,b)) = \sum_{j=1}^{\infty} \mu([a_j, b_j))$. The inequality

$$\mu([a,b)) \geq \sum_{j=1}^{\infty} \mu([a_j, b_j))$$

follows from Theorem 1.34(iv). For the converse inequality, fix $\varepsilon > 0$. By the leftcontinuity of F, we can choose $\beta < b$ such that

$$F(b) < F(\beta) + \varepsilon.$$

For every j, choose $\alpha_j < a_j$ such that

$$F(a_j) > F(\alpha_j) - \frac{\varepsilon}{2^j}.$$

Then $[a,\beta] \subseteq \bigcup_{j=1}^{\infty}(\alpha_j, b_j)$. Since $[a,\beta]$ is compact and the (α_j, b_j) are open, there exists $N \in \mathbb{N}$ such that

$$[a,\beta) \subseteq [a,\beta] \subseteq \bigcup_{j=1}^{N}(\alpha_j, b_j) \subseteq \bigcup_{j=1}^{N}[\alpha_j, b_j).$$

Finite subadditivity gives

$$\begin{aligned}\mu([a,b)) &= \mu([a,\beta)) + \mu([\beta,b)) \leq \sum_{k=1}^{N}\mu([\alpha_k,b_k)) + \mu([\beta,b)) \leq \sum_{k=1}^{N}\Big(\mu([\alpha_k,a_k)) + \mu([a_k,b_k))\Big) + \mu([\beta,b)) \\ &\leq \sum_{k=1}^{N}\mu([\alpha_k,a_k)) + \sum_{k=1}^{N}\mu([a_k,b_k)) + \mu([\beta,b)) \leq \sum_{k=1}^{N}\mu([\alpha_k,a_k)) + \sum_{k=1}^{N}\frac{\varepsilon}{2^k} + \varepsilon \\ &\leq \sum_{k=1}^{\infty}\mu([\alpha_k,a_k)) + 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, equality follows. □

The premeasure μ_F is called the *Lebesgue–Stieltjes premeasure* corresponding to F. For $F(x) = x$, it is the Lebesgue premeasure

$$\lambda([a,b)) := \mu_F([a,b)) = b - a.$$

Remark 1.40. An increasing function need not be left-continuous, but for every $x \in \mathbb{R}$ the left limit

$$F(x-) = \lim_{t\uparrow x} F(t)$$

exists. The function $x \mapsto F(x-)$ is increasing and left-continuous.

Example 1.41. Let

$$F(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0 \end{cases}$$

is not left-continuous, and μ_F is not σ-additive. To see this, observe that $[-1,0) = \bigcup_{n=1}^{\infty}\left[-\frac{1}{n}, -\frac{1}{n+1}\right)$, but

$$\mu_F([-1,0)) = F(0) - F(-1) = 2, \qquad \sum_{n=1}^{\infty}\mu_F\left(\left[-\frac{1}{n}, -\frac{1}{n+1}\right)\right) = 0.$$

Next we want to write a left-continuous function as sum of a continuous function and a so-called jump function.

Definition 1.42. A function $G : \mathbb{R} \to \mathbb{R}$ is called a *jump function* if there exists a countable set $A \subseteq \mathbb{R}$ and a function $p : A \to (0,\infty)$ such that $\sum_{y\in A\cap[-n,n]} p(y) < \infty$ for every $n \in \mathbb{N}$ and there exists $\alpha \in \mathbb{R}$ such that

$$G(x) = \begin{cases} \alpha + \sum\limits_{y\in A\cap[0,x)} p(y), & x \geq 0, \\ \alpha - \sum\limits_{y\in A\cap[x,0)} p(y), & x < 0. \end{cases}$$

Proposition 1.43 (Decomposition of finite premeasures). *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and left-continuous. Then there exist an increasing continuous function H and an increasing left-continuous jump function G such that*

$$F = G + H.$$

The functions G and H are unique up to an additive constant, and

$$\mu_F = \mu_G + \mu_H.$$

Proof. Homework. □

Finite premeasures on $\mathbb{R}^d$

Let

$$\mathcal{J}^d = \{[a, b) : a \leq b\}.$$

as in Corollary 1.18. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, write

$$x^- = (\min\{x_1, 0\}, \dots, \min\{x_d, 0\}), \qquad x^+ = (\max\{x_1, 0\}, \dots, \max\{x_d, 0\}).$$

Theorem 1.44. *Let $\mu : \mathcal{J}^d \to \mathbb{R}$ is a finite premeasure and define*

$$F(x) = \Big(\prod_{j=1}^{d} \operatorname{sign} x_j\Big)\mu([x^-, x^+))$$

is increasing and coordinatewise left-continuous.

Conversely, if $F : \mathbb{R}^d \to \mathbb{R}$ is a coordinatewise left-continuous function, then it defines a finite premeasure on $\mathcal{J}^d$ by

Theorem 1.45. *As special case of Theorem 1.44 we obtain that*

$$\lambda^d : \mathcal{J}^n \to \mathbb{R}, \quad \lambda([a, b)) := \prod_{j=1}^{d}(b_j - a_j)$$

is a premeasure.

We will recover this measure also as the product measure of the Lebesgue measure in $\mathbb{R}$ in Section **??**.

For the definition of increasing functions of several variables and proofs of these results, see [Bau01, I.§4] or [Els05, II §3].

1.4 Extension of Measures

So far we have established:

- there is a bijection between finite contents on $\mathcal{J}$ and increasing functions $\mathbb{R} \to \mathbb{R}$;

- there is a bijection between finite premeasures on $\mathcal{J}$ and increasing left-continuous functions $\mathbb{R} \to \mathbb{R}$;
- every content on $\mathcal{J}$ has a unique extension to the ring $\mathcal{R}(\mathcal{J})$ generated by $\mathcal{J}$;
- the extension is a premeasure if the original content is σ-additive.

Now we will deal with the question if and how premeasures on $\mathcal{J}$ can be extended to the Borel σ-algebra on $\mathbb{R}$ and if such extensions are unique. The idea for extending premeasures is the following:

- Take a ring $\mathcal{R}$ over a set X and a premeasure μ on $\mathcal{R}$.
- Extend the premeasure μ to a so-called outer measure μ^* on $\mathbb{P}X$. See Definition 1.46 and Theorem 1.51.
- Show that there exists a σ-algebra $\mathfrak{M}$ on X with $\mathcal{R} \subseteq \mathcal{A}$ such that $(X, \mathfrak{M}, \mu^*|_{\mathfrak{M}})$ is a measure space. See Theorem 1.50.
- Then $\sigma(\mathcal{R}) \subseteq \mathcal{A}$ and $\overline{\mu} := \mu^*|\mathcal{R}$ is a measure on $\mathcal{R}$.
- We will see that in general the extension $\overline{\mu}$ is not unique. In order to guarantee uniqueness we need that the original measure μ is σ-finite. See Theorem 1.58.

Definition 1.46. Let X be a set. A map

$$\mu^* : \mathbb{P}X \longrightarrow [0, \infty]$$

is called an *outer measure* on X if

(a) $\mu^*(\varnothing) = 0$;
(b) $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$;
(c) for every sequence $(A_j)_{j\in\mathbb{N}} \subseteq \mathbb{P}X$, we have subadditivity:

$$\mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) \le \sum_{j=1}^{\infty} \mu^*(A_j). \tag{1.8}$$

A set $M \subseteq X$ is called μ^**-measurable* if, for every $A \subseteq X$,

$$\mu^*(A) = \mu^*(A \cap M) + \mu^*(A \setminus M). \tag{1.9}$$

Remark 1.47. Subadditivity always gives $\mu^*(A) \le \mu^*(A \cap M) + \mu^*(A \setminus M)$. Therefore, the condition (1.9) for measurability of M can be replaced by

$$\mu^*(A) \ge \mu^*(A \cap M) + \mu^*(A \setminus M). \tag{1.9'}$$

Examples 1.48. (i) Every measure $\mu : \mathbb{P}X \to [0, \infty]$ is an outer measure.

(ii) Let
$$\mu^*(A) = \begin{cases} 0, & A = \varnothing, \\ 1, & A \neq \varnothing. \end{cases}$$
Then μ^* is an outer measure, but in general it is not a measure on $\mathbb{P}X$.

Remark 1.49. If M_1 and M_2 are μ^*-measurable and disjoint, then for every $A \subseteq X$,
$$\mu^*(A \cap (M_1 \mathbin{\dot\cup} M_2)) = \mu^*(A \cap M_1) + \mu^*(A \cap M_2).$$

Proof. This follows from
$$\begin{aligned} \mu^*(A \cap (M_1 \mathbin{\dot\cup} M_2)) &= \mu^*(A \cap (M_1 \mathbin{\dot\cup} M_2) \cap M_1) + \mu^*(A \cap (M_1 \mathbin{\dot\cup} M_2) \setminus M_1) \\ &= \mu^*(A \cap M_1) + \mu^*(A \cap M_2) \end{aligned}$$
where in the first step we used that M_1 is measurable. □

Theorem 1.50 (Carathéodory). *Let $\mu^* : \mathbb{P}X \longrightarrow [0, \infty]$ be an outer measure, and let $\mathfrak{M}$ be the family of all μ^*-measurable sets. Then*

(i) *$\mathfrak{M}$ is a σ-algebra over X;*
(ii) *$\mu = \mu^*|_{\mathfrak{M}}$ is a measure on $\mathfrak{M}$;*
(iii) *if $M \subseteq X$ and $\mu^*(M) = 0$, then $M \in \mathfrak{M}$.*

Proof. (i) Clearly, the empty set is measurable, since for every $A \subseteq X$,
$$\mu^*(A \cap \varnothing) + \mu^*(A \setminus \varnothing) = \mu^*(\varnothing) + \mu^*(A) = \mu^*(A).$$
If $M \in \mathfrak{M}$, then for every $A \subseteq X$,
$$\mu^*(A) = \mu^*(A \cap M) + \mu^*(A \setminus M) = \mu^*(A \setminus (X \setminus M)) + \mu^*(A \cap (X \setminus M)),$$
so $X \setminus M \in \mathfrak{M}$.
Let $M, N \in \mathfrak{M}$. We will show that $M \cup M \in \mathfrak{M}$. To this end, let $A \subseteq \mathfrak{M}$ and note that
$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap M) + \mu^*(A \setminus M) \\ &= \mu^*(A \cap M) + \mu^*((A \setminus M) \cap N) + \mu^*((A \setminus M) \setminus N)) \\ &= \mu^*(A \cap M) + \mu^*((A \setminus M) \cap N) + \mu^*(A \setminus (M \cup N)) \\ &\leq \mu^*\big((A \cap M) \cup ((A \setminus M) \cap N)\big) + \mu^*(A \setminus (M \cup N)) \\ &= \mu^*(A \cap (M \cup N) + \mu^*(A \setminus (M \cup N)) \end{aligned}$$
where in the first step we used that M is measurable, and in the second step we used that N is measurable and the inequality follows from the subadditivity of μ^*. This shows that (1.9') holds, and therefore $M \cup N \in \mathfrak{M}$.
Thus $\mathfrak{M}$ is an algebra. In order to see that $\mathfrak{M}$ is a σ-algebra, it remains to show that it is closed under countable unions. Let $(M_j)_{n=1}^\infty \subseteq \mathfrak{M}$. We have to show that $M := \bigcup_{n=1}^\infty M_n \in \mathfrak{M}$. Without

restriction we may assume that the M_n are pairwise disjoint. If not, we can replace the M_j by $\widetilde{M}_1 := M_1$ and $\widetilde{M}_n := M_n \setminus \bigcup_{j=1}^{n-1} M_j \in \mathfrak{M}$, $n \geq 2$ if we note that $\bigcup_{j=1}^{\infty} M_j = \dot{\bigcup}_{j=1}^{\infty} \widetilde{M}_j$.
Set

$$N_n = \bigcup_{j=1}^{n} M_j, \qquad M = \bigcup_{j=1}^{\infty} M_j.$$

For all $A \subseteq X$ and all $n \in \mathbb{N}$ we obtain

$$\begin{aligned} \mu^*(A) = \mu^*(A \cap N_n) + \mu^*(A \setminus N_n) &= \sum_{j=1}^{n} \mu^*(A \cap M_n) + \mu^*(A \setminus N_n) \\ &\geq \sum_{j=1}^{n} \mu^*(A \cap M_n) + \mu^*(A \setminus M) \end{aligned}$$

where the first equality is due to Remark 1.11 and the inequality comes from the subadditivity or μ^*. Since this is true for all $n \in \mathbb{N}$, we can take the limit $n \to \infty$ and then use σ-subadditivity of μ^* to obtain

$$\begin{aligned} \mu^*(A) &\geq \sum_{j=1}^{\infty} \mu^*(A \cap M_n) + \mu^*(A \setminus M) \\ &\geq \mu^*\Big(\bigcup_{j=1}^{\infty} A \cap M_n \Big) + \mu^*(A \setminus M) \\ &\geq \mu^*(A \cap M) + \mu^*(A \setminus M) \end{aligned} \tag{1.10}$$

Therefore $M \in \mathfrak{M}$ by Remark 1.49, showing that $\mathfrak{M}$ is a σ-algebra.

(ii) Let $\overline{\mu} := \mu^*|_{\mathfrak{M}}$. Since μ^* is an outer measure, we know that $\overline{\mu}(\varnothing) = 0$ and that $\mu^*(M) \geq 0$ for all $M \in \mathfrak{M}$. In order to show that $\overline{\mu}$ is σ-additive take pairwise disjoint $M_j \in \mathfrak{M}$ and, as before, set $M := \dot{\bigcup}_{n=1}^{\infty}$. Since $\mathfrak{M}$ is a σ-algebra, we have that $M \in \mathfrak{M}$ and the inequality (1.10) gives, with $A = M$,

$$\mu^*(M) \geq \sum_{j=1}^{\infty} \mu^*(M \cap M_j) + \mu^*(M \setminus M) = \sum_{j=1}^{n} \mu^*(M_j) \geq \mu^*(M) \tag{1.11}$$

where the last inequality follows from the subadditivity of μ^*. Therefore, all inequalities in (1.11) are indeed equalities, proving the σ-addivity of $\mu^*|_{\mathfrak{M}}$.

(iii) Suppose $M \subseteq X$ and $\mu^*(M) = 0$. Let $A \subseteq X$. Then $\mu^*(A \cap M) = 0$ because $0 \leq \mu^*(A \cap M) \leq \mu^*(M) = 0$ and therefore

$$\mu^*(A) = \mu^*(A) + \mu^*(A \cap M) \geq \mu^*(A \setminus M) + \mu^*(A \cap M)$$

Consequently M is μ^*-measurable by (1.9'). $\square$

The next theorem shows that every premeasure can be extended to an outer measure. The construction also explains why it is called "outer" measure.

Theorem 1.51 (Extension of a premeasure to an outer measure). *Let H be a semiring over X, and let $\mu : H \longrightarrow [0,\infty]$ be a content. We define*

$$\mu^* : \mathbb{P}X \to \mathbb{R} \cup \{\infty\}, \qquad \mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in H, \quad A \subseteq \bigcup_{j=1}^{\infty} A_j \right\} \tag{1.12}$$

Note that $\mu^*(A) = \infty$ if $A \not\subseteq \bigcup_{B\in H} B$ (in that case, the set in the definition of μ^*a is empty, so $\mu^*(A) = \inf \varnothing = \infty$). Set

$$\mathfrak{M} = \{M \subseteq X : M \ \text{ is } \ \mu^*\text{-measurable}\}.$$

(i) *μ^* is an outer measure, and $H \subseteq \mathfrak{M}$.*

(ii) *If μ is a premeasure, then $\mu^*|_H = \mu$.*

The outer measure μ^* from (1.12) is called the *outer measure generated by* μ.

Proof. (i) From the definition of μ^* it is clear that $\mu^*(\varnothing) = 0$ and that $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$. To prove σ-subadditivity, let $(A_j)_{j\in\mathbb{N}} \subseteq \mathbb{P}X$ and set $A := \bigcup_{j=1}^{n} A_j$. We have to show that $\mu^*(A) \le \sum_{j=1}^{\infty} \mu^*(A_j)$.
If $\mu^*(A_j) = \infty$ for at least one $j \in \mathbb{N}$, then both sides of the inequality are equal to ∞ by montonicity of μ^*.
Now assuwme that $\mu^*(A_j) < \infty$ for all $j \in \mathbb{N}$ and fix $\varepsilon > 0$. For every j, we can choose $(N_{jk})_{k\in\mathbb{N}} \subseteq H$ such that

$$A_j \subseteq \bigcup_{j=1}^{\infty} A_{jk} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \mu(A_{jk}) \le \mu^*(A_j) + \frac{\varepsilon}{2^j}.$$

Clearly, $A = \bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j,k=1}^{\infty} N_{jk}$ and the union is a countable union of elements in H. Hence

$$\mu^*(A) = \mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) \le \sum_{j,k=1}^{\infty} \mu(N_{jk}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu(N_{jk}) \le \sum_{j=1}^{\infty} \Big(\mu^*(A_j) + \frac{\varepsilon}{2^j}\Big) = \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\mu^*(A) \le \sum_{j=1}^{\infty} \mu^*(A_j)$.
Now let $M \in H$. In order to show that $M \in \mathfrak{M}$ it suffices to show that

$$\mu^*(A) \ge \mu^*(A \cap M) + \mu^*(A \setminus M) \qquad \text{for all } A \subseteq X. \tag{1.13}$$

If $\mu^*(A) = \infty$, this is clear. Now assuwme that $\mu^*(A) < \infty$ and fix $\varepsilon > 0$. Choose $(A_j)_{j\in\mathbb{N}} \subseteq H$ with

$$A \subseteq \bigcup_{j=1}^{\infty} A_j, \qquad \sum_{j=1}^{\infty} \mu(A_j) \le \mu^*(A) + \varepsilon,$$

and for every $j \in \mathbb{N}$ choose $N_{jk} \in H$ with $N_{jk} \cap N_{j\ell} = \varnothing$ if $k \ne \ell$ and $A_j \setminus M = \dot{\bigcup}_{k=1}^{n_j} N_{jk}$. Therefore

$$\mu(A_j) = \mu((A_j \cap M) \,\dot\cup\, (A \setminus M)) = \mu\Big(A_j \cap M \,\dot\cup\, \Big(\dot{\bigcup}_{k=1}^{n_j} N_{jk}\Big)\Big) = \mu(A_j \cap M) + \sum_{k=1}^{n_j} \mu(N_{jk})$$

and consequentely, using the definition of μ^* in the second step,

$$\begin{aligned}
\mu^*(A) + \epsilon \geq \sum_{j=1}^{\infty} \mu(A_j) &= \sum_{j=1}^{\infty} \mu(A_j \cap M) + \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \mu(N_{jk}) \\
&\geq \mu^*\Big(\bigcup_{j=1}^{\infty} A_j \cap M \Big) + \mu^*\Big(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{n_j} N_{jk}) \Big) \\
&= \mu^*(A \cap M) + \mu^*(A \setminus M).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitary, we obtain the desired inequality (1.13).
(i) Now assume the μ is σ-subadditive and let $M \in H$. Then $\mu^*(M) \leq \mu(M)$ is clear (take $A_1 = M$ and $A_j = \varnothing$, $j \geq 2$, in the definition of $\mu^*(M)$). Now let $(A_j)_{j\in\mathbb{N}} \subseteq H$ with $M \subseteq \bigcup_{n=1}^{\infty} A_n$. Subadditivity of μ shows that $\mu(M) \leq \sum_{n=1}^{\infty} \mu(A_n)$. Hence, by definition of μ^*, we obtain that $\mu(M) \leq \mu^*(M)$. □

Remark 1.52. If μ is only a content and *not* a premeasure, then $\mu^*|_{\mathfrak{M}} \neq \mu$, because we showed that μ^* is σ-additive, hence any restriction of μ^* is also σ-additive.
Consequentely, if μ is not a premeasure, there must exist $B \in H$ such that $\mu^*(B) < \mu(B)$.

This can also seen as follows: If μ is not σ-additive, then, by Theorem 1.34(vi), there exist B and $A_j \in H$ such that $B \subseteq \bigcup_{j=1}^{\infty} A_j$ and $\mu(B) > \sum_{j=1}^{\infty} \mu(A_j)$, hence $\mu^*(B) < \mu(B)$.

Remark 1.53. Let H be a semiring with a content μ. Let $\mathcal{R}$ be the ring generated by H and $\overline{\mu}$ the unique extension of μ to $\mathcal{R}$ (see Proposition 1.33). Then it is not hard to see that $\mu^* = \overline{\mu}^*$. (Proof!)

Let H be a semiring H with a content μ and let μ^* be the outer measure generated by μ on $\mathbb{P}X$. Let $\sigma(H)$ be the σ-algebra generated by H. Then $\sigma(H) \subseteq \mathfrak{M}$ and $\mu^*|_{\mathcal{R}}$ is a measure on $\mathcal{R}$ because it is a restriction of $\mu^*_{\mathfrak{M}}$ which is a measure by Theorem 1.51. Therefore, every premeasure on a semiring can be extended to a measure on $\sigma(H)$. However, this extension is not unique in general as the next example shows.

Example 1.54. Let $X \neq \varnothing$ be a set and let $H = \{\varnothing\}$ with content $\mu : H \to \mathbb{R} \cup \{\infty\}$. Then the σ-algebra generated by H is $\sigma(H) = \{\varnothing, X\}$ and for every $\alpha \in [0, \infty]$ the following is an extension of μ to a measure on $\sigma(H)$:

$$\mu_\alpha : \sigma(H) \to \mathbb{R} \cup \{\infty\}, \qquad \mu_\alpha(\varnothing) = 0,\ \mu_\alpha(X) = \alpha.$$

Note that $\mu^*(X) = \infty$, so the restricion of the exteriore measure to $\sigma(H)$ gives the largest possible extension of μ to $\sigma(H)$.

In order to guarantee uniqueness of the extension of a premeasure to the corresponding σ-algebra need the so-called σ-finitness as defined next.

Definition 1.55. Let $\mathcal{R}$ be a ring over a set X. A premeasure $\mu : \mathcal{R} \longrightarrow [0, \infty]$ is called *σ-finite* if there exist $S_j \in \mathcal{R}$ such that

$$X = \bigcup_{j=1}^{\infty} S_j \quad \text{and} \quad \mu(S_j) < \infty \quad \text{for all } j \in \mathbb{N}. \tag{1.14}$$

The same definition is used for a premeasure or content on a semiring, algebra or σ-algebra, etc.

Remark 1.56. Let S_j as in Definition 1.55.

- Set $\widetilde{S}_j = \bigcup_{k=1}^{j} S_j$. Then $\widetilde{S}_1 \subseteq \widetilde{S}_2 \subseteq \dots$, and the $\widetilde{S}_j$ satisfy (1.14)
- Set $\widehat{S}_1 = S_1$ and $\widehat{S}_j = S_j \setminus \bigcup_{k=1}^{j-1} S_j$ for $j \geq 2$. Then $\widehat{S}_j$ are pairwise dijoint and they satisfy (1.14).

Therefore we may assume without restriction that sets S_j from Definition 1.55 are either increasing or that they are pairwise disjoint.

Example 1.57. The Lebesgue premeasure λ on the semiring $\mathcal{J} = \{[a,b) : a,b \in \mathbb{R},\ a \leq b\}$ is σ-finite because

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n,n) \quad \text{and} \quad \lambda([-n,n)) = 2n < \infty,$$

but the outer measure λ^* restricted to the σ-algebra generated by $\mathcal{J}$ is not finite because $\lambda^*(\mathbb{R}) = \infty$ (use that $\lambda^*(\mathbb{R}) \geq \lambda^*([-n,n)$ for all $n \in \mathbb{N}$ by monotony of λ^*).
Similarly, every finite content μ on $\mathcal{J}$ is σ-finite because if we choose an increasing function $F : \mathbb{R} \to \mathbb{R}$ such that $\mu = \mu_F$, then $\mu_F([-n,n)) = F(n) - F(-n) < \infty$. The outer measure μ_F^* is finite if and only if F is bounded.

Theorem 1.58 (Hahn's extension theorem). *Let $\mathcal{R}$ be a ring over X, let $\sigma(\mathcal{R})$ be the σ-algebra generated by $\mathcal{R}$, and let μ be a σ-finite premeasure on $\mathcal{R}$. Then there exists exactly one measure $\widetilde{\mu}$ on $\sigma(\mathcal{R})$ such that*

$$\widetilde{\mu}|_{\mathcal{R}} = \mu.$$

Proof. Existence: By Theorem 1.51, μ^* is a measure on $\mathfrak{M} \subseteq \sigma(\mathcal{R})$ with $\mu^*|_{\mathcal{R}} = \mu$. Hence $\overline{\mu} := \mu^*|_{\sigma(\mathcal{R})}$ is a measure on $\sigma(\mathcal{R})$ that extends μ.
Uniqueness: Let ν be a measure on $\sigma(\mathcal{R})$ with $\nu|_{\mathcal{R}} = \mu$. Let $M \in \sigma(\mathcal{R})$. We have to show $\overline{\mu}(M) = \nu(M)$.
Case 1. Assume that there exists $A \in \mathcal{R}$ such that $M \subseteq A$ and $\mu(A) < \infty$. The inequality $\overline{\mu}(M) \geq \nu(M)$ follows from

$$\begin{aligned}
\overline{\mu}(M) = \mu^*(M) &= \inf\Big\{ \sum_{k=1}^{\infty} \mu(A_k) : A_k \in \mathcal{R},\ M \subseteq \bigcup_{k=1}^{\infty} A_k \Big\} \\
&= \inf\Big\{ \sum_{k=1}^{\infty} \nu(A_k) : A_k \in \mathcal{R},\ M \subseteq \bigcup_{k=1}^{\infty} A_k \Big\} \\
&\geq \inf\Big\{ \nu\Big(\bigcup_{k=1}^{\infty} A_k \Big) : A_k \in \mathcal{R},\ M \subseteq \bigcup_{k=1}^{\infty} A_k \Big\} \geq \nu(M).
\end{aligned}$$

Since $A \subseteq A \setminus M \in \sigma(\mathcal{R})$, the same argument shows that $\overline{\mu}(A \setminus M) \geq \nu(A \setminus M)$. Note that $A = (A \setminus M) \mathbin{\dot{\cup}} M$. Therefore, since $\overline{\mu}$ and ν are measures on $\sigma(\mathcal{R})$ that coincide on $\mathcal{R}$, we obtain

$$\overline{\mu}(A \setminus M) + \overline{\mu}(M) = \overline{\mu}(A) = \nu(A) = \nu(A \setminus M) + \nu(M).$$

Reorganizing the terms gives

$$\overline{\mu}(A \setminus M) - \nu(A \setminus M) = \nu(M) - \overline{\mu}(M).$$

Since we already know that $\overline{\mu}(A \setminus M) - \nu(A \setminus M) \geq 0$ and $\nu(M) - \overline{\mu}(M) \leq 0$, we must have

$$\overline{\mu}(A \setminus M) - \nu(A \setminus M) = \nu(M) - \overline{\mu}(M) = 0,$$

hence $\overline{\mu}(M) = \nu(M)$.

Case 2. $M \in \sigma(\mathcal{R})$ arbitrary. Since μ is σ-finite, we can choose pairwise disjoint S_j such that $X = \dot{\bigcup}_{j=1}^{\infty} S_j$ and $\mu(S_j) < \infty$ for all $j \in \mathbb{N}$. Let $M_j := M \cap S_j$, hence $M = \dot{\bigcup}_{j=1}^{\infty} M_j$. Since $M, M_j \in \sigma(\mathcal{R})$, and $\overline{\mu}$ and ν are measures on $\sigma(\mathcal{R})$, we have

$$\overline{\mu}(M) = \sum_{j=1}^{\infty} \overline{\mu}(M_j) \qquad \text{and} \qquad \nu(M) = \sum_{j=1}^{\infty} \nu(M_j). \tag{1.15}$$

For each j, the set M_j satisfies the assumptions form Case 1 with $A = S_j$. Therefore, $\overline{\mu}(M_j) = \nu(M_j)$ for all $j \in \mathbb{N}$, hence also $\overline{\mu}(M) = \nu(M)$ by (1.15). $\square$

The following theorem is more general and contains the uniqueness claim from Theorem 1.58.

Theorem 1.59. *Let $\mathcal{A}$ be a σ-algebra over X. Assume that there exists a family $\mathcal{E}$ of sets in $\mathcal{A}$ such that $\sigma(\mathcal{E}) = \mathcal{A}$ (that is, $\mathcal{A}$ is the σ-ring generated by $\mathcal{E}$). Suppose that $\mathcal{E} \subseteq \mathcal{A}$ be stable under finite intersections and that $X = \bigcup_j E_j$ for some $E_j \in \mathcal{E}$. If ν and ρ are measures on $\mathcal{A}$ such that*

$$\nu|_{\mathcal{E}} = \rho|_{\mathcal{E}} \qquad \textit{and} \qquad \nu(E_j) = \rho(E_j) < \infty \qquad \textit{for all } j \in \mathbb{N},$$

then $\nu = \rho$.

Proof. Fix $S \in \mathcal{E}$ with $\nu(S) = \rho(S) < \infty$, and define

$$\mathcal{D}_S = \{M \in \mathcal{A} : \nu(M \cap S) = \rho(M \cap S)\}.$$

The family $\mathcal{D}_S$ is a Dynkin system (see Definition 1.25) because

- clearly, $X \in \mathcal{D}_A$;
- if $M \in \mathcal{D}_S$, then $X \setminus M \in \mathcal{D}_A$ because, noting that $(X \setminus M) \cap S = S \setminus M$ and that $\mu(S) = \rho(S) < \infty$, we obtain

$$\nu((X \setminus M) \cap S) = \nu(S \setminus M) = \nu(S) - \nu(S \cap M) = \rho(S) - \rho(S \cap M) = \rho((X \setminus M) \cap S);$$

- if $A_j \in \mathcal{D}_S$ are pairwise disjoint and $A := \dot{\bigcup}_{j=1}^{\infty} A_j$, then

$$\nu(A \cap S) = \lim_{n \to \infty} \nu\Big(\bigcup_{j=1}^{n} A_j \cap S\Big) = \lim_{n \to \infty} \sum_{j=1}^{n} \nu(A_j \cap S) = \lim_{n \to \infty} \sum_{j=1}^{n} \rho(A_j \cap S) = \cdots = \rho(A \cap S).$$

Moreover, $\mathcal{E} \subseteq \mathcal{D}_S$ because for $B \in \mathcal{E}$ we have $B \cap S \in \mathcal{E}$, hence $\nu(B \cap S) = \rho(B \cap S)$. Therefore, $D(\mathcal{E}) \subseteq \mathcal{D}_S$ where $D(\mathcal{E})$ is the Dynkin system generated by $\mathcal{E}$. By Theorem 1.28, we have $D(\mathcal{E}) = \sigma(\mathcal{E})$, which by assumption is equal to $\mathcal{A}$. Hence $\mathcal{A} = \sigma(\mathcal{E}) = D(\mathcal{E}) \subseteq \mathcal{D}_S \subseteq \mathcal{A}$, so

$$\mathcal{D}_S = \sigma(\mathcal{E}),$$

in particular, $\nu(S \cap M) = \rho(S \cap M)$ for all $M \in \mathcal{A}$.
Since for all $j \in \mathbb{N}$, the sets E_j satisfy $\nu(E_j) = \rho(E_j) < \infty$, the above gives $\mathcal{D}_{E_j} = \mathcal{A}$, hence

$$\nu(E_j \cap M) = \rho(E_j \cap M) \qquad \text{for all } j \in \mathbb{N} \text{ and } M \in \mathcal{A}. \tag{1.16}$$

In particular, for $S_j := E_j \setminus \bigcup_{k=1}^{j-1} E_k = E_j \cap \left(X \setminus \bigcup_{k=1}^{j-1} E_k\right)$ we obtain

$$\nu(S_j) = \rho(S_j) < \infty, \qquad j \in \mathbb{N},$$

so also $\mathcal{D}_{S_j} = \mathcal{A}$ and (1.16) holds also for S_j instead of E_j. Since by construction $X = \dot{\bigcup}_{j=1}^{\infty} S_j$, we finally have, for every $M \in \mathcal{A}$

$$\nu(M) = \nu\Big(\dot{\bigcup}_{j=1}^{\infty} M \cap S_j\Big) = \sum_{j=1}^{\infty} \nu(M \cap S_j) = \sum_{j=1}^{\infty} \rho(M \cap S_j) = \rho\Big(\dot{\bigcup}_{j=1}^{\infty} M \cap S_j\Big) = \rho(M). \qquad \square$$

We saw that for ring $\mathcal{R}$ with a premeasure μ we have $\mathcal{R} \subseteq \sigma(\mathcal{R}) \subseteq \mathfrak{M}$. Now we will show that in general $\sigma(\mathcal{R}) \subsetneq \mathfrak{M}$.

Definition 1.60. Let $(X, \mathcal{A}, \mu)$ be a measure space. A set $M \in \mathcal{A}$ is called a *zero set* if $\mu(M) = 0$. A set $N \subseteq X$ is called a *negligible set* if there exists a zero set $M \in \mathcal{A}$ such that $N \subseteq M$.
The measure space $(X, \mathcal{A}, \mu)$ is called a *complete measure space* if the sigma algebra $\mathcal{A}$ contains all negligible sets; in other words, if every subset of a zero set belongs itself to $\mathcal{A}$.

Example 1.61. Let $\mathcal{R}$ be a ring with a premeasure μ and let μ^* and $\mathfrak{M}$ as Carathéodory's theorem (Theorem 1.50). Then $(X, \mathfrak{M}, \mu^*_{\mathfrak{M}})$ is a complete measure space.

Proof. Let $M \in \mathfrak{M}$ with $\mu^*(M) = 0$, and let $N \subseteq M$. Then $0 \leq \mu^*(N) \leq \mu^*(M) = 0$ by monotony μ^*, hence $\mu^*(N) = 0$, and therefore $N \in \mathfrak{M}$ by Theorem 1.50(iii). $\square$

Remark 1.62. In the following we will frequently use that for $A, C \in \mathcal{A}$ with $\mu(C) = 0$ we have that

$$\mu(A) = \mu(A \cup C) = \mu(A \setminus C). \tag{1.17}$$

This follows from $\mu(A) \leq \mu(A \cup C) \leq \mu(A) + \mu(C) = \mu(A)$ and $\mu(A \setminus C) = \mu(A) + \mu(A \cap C) = \mu(A)$.

Theorem 1.63. *Let $(A, \mathcal{A}, \mu)$ be a measure space and let $\mathfrak{N}$ be the set of all negligible sets onf X, that is,*

$$\mathfrak{N} := \{N \subseteq X : \text{exists } A \in \mathcal{A} \text{ such that } N \subseteq A,\ \mu(A) = 0\}.$$

Define

$$\widetilde{\mathcal{A}} := \{A \cup N : A \in \mathcal{A},\ N \in \mathfrak{N}\}, \qquad \text{and} \qquad \widetilde{\mu} : \widetilde{\mathcal{A}} \to \mathbb{R} \cup \{\infty\}, \quad \widetilde{\mu}(A \cup N) = \mu(A). \tag{1.18}$$

(i) $\mathfrak{N}$ *is a σ-ring.*

(ii) *$\widetilde{\mu}$ is well-defined and $(X, \widetilde{\mathcal{A}}, \widetilde{\mu})$ is complete measure space and $\widetilde{\mu}$ is the unique extension of μ to a measure in $\widetilde{\mathcal{A}}$.*

(iii) *$(X, \widetilde{\mathcal{A}}, \widetilde{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$. That means: If $(X, \mathcal{B}, \nu)$ is a complete measure space with $\mathcal{A} \subseteq \mathcal{B}$ and $\nu|_{\mathcal{A}} = \mu$, then $\widetilde{\mathcal{A}} \subseteq \mathcal{B}$ and $\nu|_{\widetilde{\mathcal{A}}} = \widetilde{\mu}$.*

Proof. (i) is clear from the definition of $\mathfrak{N}$.

(ii) First we show that $\widetilde{\mu}$ is well-defined. Let $A, A' \in \mathcal{A}$ and $N, N' \in \mathfrak{N}$ such that $A \cup N = A' \cup N'$. We have to show that $\mu(A) = \mu(A')$. Choose $C, C' \in \mathcal{A}$ such that $N \subseteq C$, $N' \subseteq C'$ and $\mu(C) = \mu(C') = 0$. Then, using (1.17), and the fact that $A \cup N \cup N' = A' \cup N \cup N'$, we obtain

$$\mu(A) = \mu(A \cup C' \cup C) = \mu(A' \cup C' \cup C) = \mu(A').$$

Now let us show that $\widetilde{A}$ is a σ-algebra. First note that $X \in \mathcal{A} \subseteq \widetilde{\mathcal{A}}$. Now let $M \in \widetilde{\mathcal{A}}$. Then there exist $A, C \in \mathcal{A}$, $N \in \mathfrak{N}$ such that $M = A \cup N$, $N \subseteq C$ and $\mu(C) = 0$. We can write

$$X \setminus M = X \setminus (A \cup N) = (X \setminus (A \cup C)) \cup (C \setminus N).$$

The first set in in $\mathcal{A}$ while the second set is $\mathfrak{N}$ (because $C \setminus N \subseteq C$), therefore $X \setminus M \in \widetilde{A}$.
Let $(M_j)_{j\in\mathbb{N}} \subseteq \widetilde{A}$. Then there exist $A_j, C_j \in \mathcal{A}$, $N_j \in \mathfrak{N}$ such that $M_j = A_j \cup N_j$, $N_j \subseteq C_j$ and $\mu(C_j) = 0$. Then $M := \bigcup_{j=1}^{\infty} M_j = \bigcup_{j=1}^{\infty} A_j \cup \bigcup_{j=1}^{\infty} N_j$. The first union belongs to $\mathcal{A}$ and the second one belongs to $\mathfrak{N}$, therefore $M \in \widetilde{\mathcal{A}}$.
Next we show that $\widetilde{\mu}$ is a measure on $\widetilde{\mathcal{A}}$. It is clear that $\widetilde{\mu}(\varnothing) = 0$ and that $\widetilde{\mu}(M) \geq 0$ for all $M \in \widetilde{\mathcal{A}}$. Let $M_j \in \widetilde{\mathcal{A}}$ be pairwise disjoint and choose $A_j, C_j \in \mathcal{A}$, $N_j \in \mathfrak{N}$ such that $M_j = A_j \cup N_j$, $N_j \subseteq C_j$ and $\mu(C_j) = 0$. Then also the A_j are pairwise disjoint and

$$\dot{\bigcup_{j=1}^{\infty}} M_j = \dot{\bigcup_{j=1}^{\infty}} A_j \ \cup\ \dot{\bigcup_{j=1}^{\infty}} N_j.$$

Since the second union belongs to $\mathfrak{N}$, have that

$$\widetilde{\mu}\Big(\dot{\bigcup_{j=1}^{\infty}} M_j\Big) = \mu\Big(\dot{\bigcup_{j=1}^{\infty}} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \widetilde{\mu}(A_j \cup N_j).$$

Now we show that $\widetilde{\mu}$ is the only measure on $\widetilde{\mathcal{A}}$ that extends μ. Assume that ν is an extension of μ to $\widetilde{\mathcal{A}}$. Then clearly $\nu(N) = 0$ for all $N \in \mathfrak{N}$. Therefore, for all $A \in \mathcal{A}$, $N \in \mathfrak{N}$ we have, again using (1.17),

$$\nu(A \cup N) = \nu(A) = \mu(A) = \widetilde{\mu}(A \cup N).$$

Finally, we show that $(X, \widetilde{\mathcal{A}}, \widetilde{\mu})$ is complete. Let $M \in \widetilde{\mathcal{A}}$ with $\widetilde{\mu}(M) = 0$. Let $D \subseteq M$. We have to show that $N \in \widetilde{\mathcal{A}}$. Since $M \in \widetilde{\mathcal{A}}$, we can choose $A, C \in \mathcal{A}$ and $N \in \mathfrak{N}$ such that $M = A \cup N$, $N \subseteq C$ and $\mu(C) = 0$. Then $D \subseteq A \cup C$ and $\mu(A \cup C) = \mu(A) = \widetilde{\mu}(M) = 0$, hence $D \in \mathfrak{N}$.

(iii) Let $(X, \mathcal{B}, \nu)$ be a complete measure space with $\mathcal{A} \subseteq \mathcal{B}$ and $\nu|_{\mathcal{A}} = \mu$. Then clearly $\mathfrak{N} \subseteq \mathcal{B}$, hence $\widetilde{\mathcal{A}} \subseteq \mathcal{B}$. Then $\nu|_{\widetilde{\mathcal{A}}}$ is a measure on $\widetilde{\mathcal{A}}$ that extends μ. So, by (ii), it must coincide with $\widetilde{\mu}$. $\square$

Theorem 1.64. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Then $(X, \mathfrak{M}, \mu^*|_{\mathfrak{M}})$ is its completion.*

Proof. By Theorem 1.63 we only have to show that $\mathfrak{M} = \widetilde{\mathcal{A}}$ as defined in (1.18). Since $(X, \mathfrak{M}, \mu^*|_{\mathfrak{M}})$ is a complete extension of $(X, \mathcal{A}, \mu)$ by Example 1.61, we already have $\mathfrak{M} \subseteq \widetilde{\mathcal{A}}$. To show the other inclusion, let $B \in \mathfrak{M}$.

Case 1. $\mu^*(B) < \infty$. Choose $A_{nk} \in \mathcal{A}$ such that $B \subseteq \bigcup_{k=1}^\infty A_{nk}$ and $\mu^*(B) + \frac{1}{n} \geq \sum_{k=1}^\infty \mu(A_{nk})$. Then

$$B \subseteq \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty A_{nk} =: A \in \mathcal{A}$$

and for all $n \in \mathbb{N}$

$$\mu(A) \leq \mu\Big(\bigcup_{k=1}^\infty A_{nk}\Big) \leq \sum_{k=1}^\infty \mu(A_{nk}) \leq \mu^*(B) + \frac{1}{n}.$$

Since this holds for every $n \in \mathbb{N}$, we have that $\mu(A) = \mu^*(B)$.
Now $A \setminus B \in \mathfrak{M}$ and $\mu^*(A \setminus B) \leq \mu^*(A) < \infty$. Therefore, as above, we find $C \in \mathcal{A}$ such that $A \setminus B \subseteq C$ and $\mu(C) = \mu^*(A \setminus B)$. Since $A, B \in \mathfrak{M}$ and $\mu^*|_{\mathfrak{M}}$ is a measure, we have

$$\mu(C) = \mu^*(A \setminus B) = \mu^*(A) - \mu^*(B) = \mu(A) - \mu^*(B) = 0.$$

We can write

$$B = (B \setminus C) \cup (B \cap C) = (A \setminus C) \cup (B \cap C).$$

The first set belongs to $\mathcal{A}$ because $A, C \in \mathcal{A}$ and the second set belongs to $\mathfrak{N}$ because $B \cap C \subseteq C$ and $\mu(C) = 0$. Therefore $B \in \widetilde{\mathcal{A}}$.

Case 2. $B \in \mathfrak{M}$ arbitrary. Since $(X, \mathcal{A}, \mu)$ is σ-finite, we can choose $S_j \in \mathcal{A}$ such that $X = \bigcup_{j \in \mathbb{N}} S_j$ and $\mu(S_j) < \infty$ for all $j \in \mathbb{N}$. Set $B_j := B \cap S_j$. Then $\mu(B_j) < \infty$, hence $B_j \in \widetilde{A}$ and by Case 1. Since $\widetilde{\mathcal{A}}$ is a σ-algebra and $B = \bigcup_{j \in \mathbb{N}} B_j$, it follows that $B \in \widetilde{\mathcal{A}}$. □

Remark 1.65. Other descriptions of $\mathfrak{M}$ are:

$$\begin{aligned} \mathfrak{M} &= \text{set of all } \mu^*\text{-measurable sets in } X \\ &= \text{completion of } \mathcal{A} \\ &= \{A \cup N : A \in \mathcal{A},\, N \in \mathfrak{N}\} \\ &= \{A \triangle N : A \in \mathcal{A},\, N \in \mathfrak{N}\} \\ &= \{M \subseteq X : \exists\, A \in \mathcal{A},\, N, N' \in \mathfrak{N} \text{ such that } \ M \cup N = A \cup N'\}. \end{aligned}$$

1.5 The Lebesgue measure on $\mathbb{R}$

Recall that on $\mathbb{R}$ we have the semiring

$$\mathcal{J} = \{[a, b) : a \leq b,\, a, b \in \mathbb{R}\} \tag{1.19}$$

as in (1.2) and the finite content

$$\mu : \mathcal{J} \to \mathbb{R} \cup \{\infty\}. \tag{1.20}$$

We know that we can extend μ to an exterior measure μ^* on $\mathbb{P}\mathbb{R}$.

Definition 1.66. The σ-algebra $\mathfrak{M}$ of μ^*-measurable sets is called the *Lebesgue σ-algebra* and $\lambda := \mu^*|_{\mathfrak{M}}$ is called the *Lebesgue measure on* $\mathbb{R}$.

We also know from Theorems 1.58 and 1.64 that λ is the unique extension of μ to $\mathfrak{M}$ and that $(\mathbb{R}, \mathfrak{M}, \lambda)$ is a complete meausure space.

Definition 1.67. Let (X, τ) be a topological space with τ being its topology (the collection of all open sets). The σ-algebra generated by τ is called the *Borel σ-algebra* on X. Usually it is denoted by $\mathfrak{B}(\tau)$ or $\mathfrak{B}(X)$.

Clearly, another set of generators of $\mathfrak{B}(X)$ is the collection of all closed sets of X.

Remark 1.68 (Borel σ-algebra in $\mathbb{R}$). Let $\mathfrak{B}$ be the Borel σ-algebra on $\mathbb{R}$ (considered with the usual Euclidean topology). Then

$$\begin{aligned} \mathfrak{B} &:= \{\sigma\text{-algebra generated by the open sets of } \mathbb{R}\} \\ &= \{\sigma\text{-algebra generated by } \mathcal{J}\} \\ &= \{\sigma\text{-algebra generated by the intervals } (a, b] \text{ with } a \leq b,\, a, b \in \mathbb{R}\} \end{aligned}$$

Proof. To see that $\mathfrak{B} \subseteq \sigma(\mathcal{J}$, we only need to proof that all open sets belong to $\sigma(\mathcal{J}$. This follows from $(a, b) = \bigcup_{n\in\mathbb{N}}[a + \frac{1}{n},\, b)$, $(-\infty, b) = \bigcup_{n\in\mathbb{N}}[-n,\, b)$, and $(a, \infty) = \bigcup_{n\in\mathbb{N}}[a,\, n,) \in \sigma(\mathcal{J})$.
To see that $\sigma(\subseteq J\mathfrak{B}$, we only need to proof that all halfopen intervals belong to $\mathfrak{B}$. This follows from $[a, b) = \bigcap_{n\in\mathbb{N}}(a - \frac{1}{n},\, b)$. □

The remark shows that

$$\mathcal{J} \subseteq \sigma(\mathcal{J}) = \mathfrak{B} \subseteq \mathfrak{B}$$

and that $\mathfrak{M}$ is the completion of $\mathfrak{B}$.

Definition 1.69. The measure $\beta := \lambda|_{\mathfrak{B}}$ is called the *Borel measure* on $\mathbb{R}$.

Theorem 1.70. *Let λ be the Lebesgue measure on $\mathbb{R}$ and let A be a Lebesgue measurable set. Then, for every $\epsilon > 0$ there exist*

(i) *an open set U such that $A \subseteq U$ and $\lambda(U \setminus A) < \varepsilon$,*

(ii) *a closed set C such that $C \subseteq A$ and $\lambda(A \setminus C) < \varepsilon$.*

Proof. (i): First assume that $\lambda(A) < \infty$. By definition of λ, there exist $A_n := [a_n, b_n)$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and $\lambda(A) + \frac{\varepsilon}{2} \geq \sum_{n\in\mathbb{N}} \lambda(A_n)$. Now choose open intervals $B_n \supseteq A_n$ such that $\lambda(B_n) - \lambda(A_n) < \frac{\varepsilon}{2^{n+1}}$, for instance $B_n = (a_n - \frac{\varepsilon}{2^{n+1}},\, b_n)$. Then $U := \bigcup_{n\in\mathbb{N}} B_n$ is open, $A \subseteq U$ and

$$\lambda(U) = \lambda\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) \leq \sum_{n\in\mathbb{N}} \lambda(B_n) \leq \sum_{n\in\mathbb{N}} \Big(\lambda(A_n) + \frac{\varepsilon}{2^{n+1}}\Big) \leq \sum_{n\in\mathbb{N}} \lambda(A_n) + \frac{\varepsilon}{2} \leq \lambda(A) + \varepsilon.$$

Therefore, $\lambda(U \setminus A) = \lambda(U) - \lambda(A) \leq \varepsilon$.
Now let A be an arbitrary Lebesgue measurable set. Since λ is σ-finite, we can choose $S_j \in \mathfrak{M}$, pairwise disjoint, such that $\mathbb{R} = \dot{\bigcup}_{j\in\mathbb{N}} S_j$ and $\lambda(S_j) < \infty$ for all $j \in \mathbb{N}$. Let $A_j := A \cap S_j$. Then

$A_j \in \mathfrak{M}$ and $\lambda(A_j) < \infty$, hence, by what we already proved, we can choose open sets U_j such that $A_j \subset U_j$ and $\lambda(U_j \setminus A_j) < \frac{\varepsilon}{2^j}$. Then $U := \bigcup_{j=1}^{\infty} U_j$ is open, $A \subseteq U$ and

$$\lambda(U \setminus A) = \lambda\Big(\bigcup_{j\in\mathbb{N}} U_j \setminus A_j\Big) \leq \sum_{j\in\mathbb{N}} \lambda(U_j \setminus A_j) < \sum_{j\in\mathbb{N}} \frac{\varepsilon}{2^j} = \varepsilon.$$

(ii): By (i) applied to $\mathbb{R} \setminus A$, we can choose $U \subseteq \mathbb{R}$ open such that $A \subseteq U$ and $\lambda(U \setminus (\mathbb{R} \setminus A)) < \varepsilon$. Set $C := \mathbb{R} \setminus U$. Then C is closed, $C \subseteq A$ and

$$\lambda(A \setminus C) = \lambda(A \cap U) = \lambda(U \setminus (\mathbb{R} \setminus A)) < \varepsilon. \qquad \square$$

Proposition 1.71. *The Lebesgue measure is translation invariant. That means: For every $c \in \mathbb{R}$, we have that $A \in \mathfrak{M}$ if and only if $c + A \in \mathfrak{M}$ and $\lambda(A) = \lambda(c + A)$.*

Proof. Let $\mathcal{J}$ be the semiring of halfopen intervals as in (1.2) and μ as in (1.20) and let $c \in \mathbb{R}$. Clearly, for every $[a, b) \in \mathcal{J}$ we have that $\mu([a, b)) = \mu(c + [a, b))$. Moreover, $M \subseteq \bigcup_{n\in\mathbb{N}}[a_n, b_n)$ if and only if $c + M \subseteq \bigcup_{n\in\mathbb{N}}(c + [a_n, b_n))$. Therefore, $\mu^*(M) = \mu^*(c + M)$ for every $M \subseteq \mathbb{R}$. In particular, it follows that $M \in \mathfrak{M}$ if and only if $c + M \in \mathfrak{M}$ and $\lambda(M) = \lambda(c + M)$ since $\lambda = \mu^*|_{\mathfrak{M}}$. $\square$

Remark 1.72. For the Lebesgue σ-algebra $\mathfrak{M}$ and the Borel σ-algebra $\mathfrak{B}$ it can be shown that $\mathfrak{B} \subsetneq \mathfrak{M}$ by a cardinality argument. Since the Cantor set C belongs to $\mathfrak{M}$ and has measure zero, all its subsets must also belong to $\mathfrak{M}$. The Cantor set is uncountable, therefore $|\mathbb{P}C| = 2^{2^{\aleph_0}} \leq |\mathfrak{M}| \leq |\mathbb{P}[0, 1]| = 2^{2^{\aleph_0}}$. On the other hand, $|\mathfrak{B}| = 2^{\aleph_0} < |\mathfrak{M}|$. Therefore $\mathfrak{M} \neq \mathfrak{B}$. See [HS75, (10.21)].

Taller 5, Exercise 1 gives another proof that $\mathfrak{B} \subsetneq \mathfrak{M}$ without using a counting argument.

If we assume the axiom of choice, then one can construct a non-Lebesgue measurable set in $\mathbb{R}$, showing that $\mathfrak{M} \subsetneq \mathbb{P}\mathbb{R}$. (Homework.)

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