7. Lebesgue spaces:

$(X, \mathcal{A}, \mu)$ measure space, $f: X \to \mathbb{K}$ mb.

Recall. $f \in \mathcal{L}_1(X,\mu) \Leftrightarrow \int_X |f| \, d\mu =: \|f\|_1 < \infty$

Now for $1 \le p < \infty$: $\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{1/p}$

$p = \infty$ $\|f\|_\infty := \operatorname{ess\,sup} f := \inf\{\alpha \in \mathbb{R} \mid |f| \le \alpha \ \ \lambda^5\text{- casio sempre}\}$

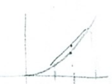
$= \inf\{\alpha \in \mathbb{R} \mid \lambda(|f|^{-1}(\alpha,\infty)) = 0\}$.

$\mathcal{L}_p(X,\mu) := \{f : X \to \mathbb{K} \text{ mb} \mid \|f\|_\infty < \infty\}$.

Observation: $\mathcal{L}_p(X,\mu)$ are vector spaces.

Clearly: $f \in \mathcal{L}_p(X,\mu)$ $\lambda \in \mathbb{K} \Rightarrow \lambda f \in \mathcal{L}_p(X,\mu)$.

- $p = \infty \Rightarrow \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty \Rightarrow f+g \in \mathcal{L}_p(X,\mu)$



- $1 \le p < \infty$: for $p \ge 1$ the fct $t \to t^p$ is convex $\Rightarrow \left(\frac{|f(x)+g(x)|}{2}\right)^p \le \frac{1}{2}(|f(x)|^p + |g(x)|^p)$

$\Rightarrow \|f-g\|_p^p \le 2^{p-1}(\|f\|_p^p + \|g\|_p^p) \le 2^{p-1}(\|f\|_p + \|g\|_p)^p$.

Goal. $\mathcal{L}_p$ are seminormed spaces with $\|\cdot\|_p$ seminorm on $\mathcal{L}_p$.

$(\mathcal{L}_p/\mathcal{N}, \|\cdot\|_p)$ Banach space where $\mathcal{N} = \{f \in \mathcal{L}_p \mid \|f\|_p = 0\}$.

7.1 Inequalities.

Lemma 7.1 (Young's inequality) $p, q \in (1,\infty)$ an $\frac{1}{p} + \frac{1}{q} = 1$, $a, b \ge 0$

$\Rightarrow ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

Proof 1 Clear if $ab = 0$.

$f(\lambda x + (1-\lambda)y) \ge \lambda f(x) + (1-\lambda) f(y)$

Now assume $ab \ne 0$. Since $\ln$ is concave and $\frac{1}{p} + \frac{1}{q} = 1$:

$\ln\left(\frac{1}{p}a^p + \frac{b^q}{q}\right) \ge \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) = \ln(a) + \ln(b) = \ln(ab)$

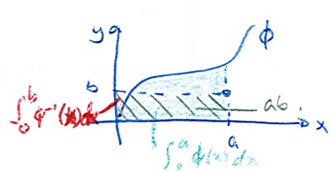
Since exp is mon. increasing on $\mathbb{R}$ the assertion is proved. $\square$

Proof 2. Clear if $ab = 0$. Let $\phi : [0,\infty) \to [0,\infty)$ continuous, increasing with $\phi(0) = 0$

For $a < b$: The picture shows:

$ab \le \int_0^a \phi(x)\,dx + \int_0^b \phi^{-1}(x)\,dx$

with "=" if and only if $b = \phi(a)$.



Special case: $\phi(x) = x^{p-1} \Rightarrow \phi^{-1}(x) = x^{\frac{1}{p-1}}$ (note: $\frac{1}{p} + \frac{1}{q} = 1$

$\Rightarrow q = \frac{p}{p-1} = 1 + \frac{1}{p-1}$.)

$\Rightarrow ab \le \int_0^a x^{p-1}\,dx + \int_0^b x^{\frac{1}{p-1}}\,dx$

$= \frac{1}{p}a^p + \frac{1}{1+\frac{1}{p-1}} b^{1+\frac{1}{p-1}} = \frac{1}{p}a^p + \frac{1}{q}b^q$. $\square$

Theorem 7.2 (Hölder's inequality). $(X, \mathcal{A}, \mu)$ measure space, $f, g : X \to \mathbb{K}$ mb, $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (Convention: $p = 1 \Rightarrow q = \infty$)

$\Rightarrow \|f \cdot g\|_1 \le \|f\|_p \|g\|_q$ (with convention $0 \cdot \infty = 0$ on the right side).

Proof

Case 1 $\|f\|_p$ or $\|g\|_q = 0 \Rightarrow f = 0$ μ-a.e. or $g = 0$ μ-a. $\Rightarrow f \cdot g = 0$ μ-a.e. $\Rightarrow \|f \cdot g\|_1 = 0$

Case 2. $\|f\|_p \ne 0 \ne \|g\|_q$, $\|f\|_p = \infty$ or $\|g\|_q = \infty \Rightarrow$ The inequality is trivially satisfied.

Case 3. $\|f\|_p, \|g\|_q \in (0,\infty)$

Without restriction: $\|f\|_p = \|g\|_q = 1$.

By lemma 7.1:

$\|f g\|_1 = \int_X |f(x)||g(x)| \, d\mu \le \int \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q \, d\mu = \frac{1}{p}\|f\|_p + \frac{1}{q}\|g\|_q$

$= \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \cdot \|g\|_q$. $\square$

Remarks

- Generalized Hölder inequality: $\frac{1}{p_1} + \dots + \frac{1}{p_n} = 1 \Rightarrow \|f_1 \cdots f_n\| \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}$.
  (Proof by induction).
- $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} \Rightarrow \|f g\|_s \le \|f\|_p \cdot \|g\|_q$.

  Proof. Apply Hölder's inequality to $|f|^s |g|^s$ with exponents p/s and $(p/s)'$.
- Interpolation inequalities: $1 \le s \le r \le t$; $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$

  $\Rightarrow \|f\|_r \le \|u\|_s^\theta \cdot \|u\|_t^{1-\theta}$.

  Proof. Apply Hölder's inequality to $u^{\theta r} \cdot u^{(1-\theta)r}$ with exponents $\frac{s}{\theta r}, \left(\frac{s}{\theta r}\right)' = \frac{t}{(1-\theta)r}$

Theorem 7.3. (Minkowski inequality).

$(X, \mathcal{A}, \mu)$ measure space, $f, g : X \to \mathbb{K}$ mb, $1 \le p \le \infty$.

$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof Clear if $p = \infty$, or $\|f+g\|_p = 0$, or $\|f\|_p = \infty$ or $\|g\|_q = \infty$. $\quad q = \frac{p}{p-1} = 1 + \frac{1}{p-1}$

Now assume $\|f+g\|_p \neq 0$ and let q s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

$$\Rightarrow \|f+g\|_p^p = \int_X |f+g|^{p-1}|f+g| \, d\mu \le \int_X |f+g|^{p-1}|f| \, d\mu + \int_X |f+g|^{p-1}|g| \, d\mu$$

$$\le \Big(\int_X |f+g|^{\overset{=p}{(p-1)q}} d\mu\Big)^{\frac{1}{q}} \Big(\int |f|^p d\mu\Big)^{\frac{1}{p}} + \Big(\int |f+g|^{(p-1)q} d\mu\Big)^{\frac{1}{q}} \Big(\int |g|^p d\mu\Big)^{\frac{1}{p}}$$

$$= \|f+g\|_p^{p/q} \left(\|f\|_p + \|g\|_p\right)$$

If $\|f\|_p \neq \infty \neq \|g\|_p \Rightarrow \|f+g\|_p < \infty$ since $\mathcal{L}_p(X)$ is a VS.

$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$ using that $p - p/q = p(1 - \frac{1}{q}) = 1$ □

Corollary. $\|\cdot\|_p$ is a seminorm on $\mathcal{L}_p(X, \mu)$

(Homogeneity ✓, Δ-inequality ✓, $f = 0 \Rightarrow \|f\|_p = 0$ ✓)

But: $\|f\|_p = 0 \not\Rightarrow f = 0$.

Define equivalence class $\sim$ on $\mathcal{L}_p(X)$: $f \sim g :\Leftrightarrow f - g = 0$ μ-ae.

$\leadsto f \sim g \Leftrightarrow \|f-g\|_p = 0 \Leftrightarrow f - g \in \mathcal{N} := \{\varphi : X \to \mathbb{K} \text{ mb} \mid \|\varphi\|_p = 0\}$. (*)

$\Rightarrow \boxed{(L_p(X), \|\cdot\|_p) \text{ is a normed space.}}$

where $L_p(X) := \mathcal{L}_p(X)/\mathcal{N}$, $\|[f]\|_p := \|f\|_p$ is well-defined by (*).

Usually: f is identified with its equivalence class.

Next goal: $(L_p(X), \|\cdot\|_p)$ is a Banach space, i.e., complete.

Lemma 7.4. $(X, \mathcal{A}, \mu)$ measure space, $1 \le p \le \infty$, $(f_n)_n$ CS in $\mathcal{L}_p(X)$

$\rightarrow$ $\exists$ subsequence ~~which~~ $(f_{n_k})_k$ which converges to some $f \in \mathcal{L}_p(X)$ μ-ae.

Proof Case 1: $\boxed{1 \le p < \infty}$

Choose subsequence $(f_{n_k})_k$ with $\|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{2^k}$ (*)

and set $g_m := \sum_{k=1}^{m} |f_{n_{k+1}} - f_{n_k}|$,

$g := \lim_{m \to \infty} g_m = \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$. (the limit ex. because of (*))

$\Rightarrow \forall m \in \mathbb{N}$: $\|g_m\|_p \overset{\text{Minkowski}}{\le} \sum_{k=1}^{m} \|f_{n_{k+1}} - f_{n_k}\|_p < \sum_{k=1}^{m} \frac{1}{2^k} < 1$.

$\Rightarrow \|g\|_p^p = \int_X |g|^p d\mu \underset{\text{mon. conv.}}{=} \lim_{m \to \infty} \int_X |g_m|^p d\mu = \lim_{m \to \infty} \|g_m\|_p^p \le 1$.

$\Rightarrow g(x) \neq \infty$ μ-ae.

$\Rightarrow f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$ converges for μ-a.e. x.

Let
$$f(x) := \begin{cases} \lim_{m \to \infty} f_{n_1}(x) + \sum_{k=1}^{m} f_{n_{k+1}}(x) - f_{n_k}(x) & \text{si el límite existe} \\ 0, & \text{si el límite no existe} \end{cases}$$
$$= \begin{cases} \lim_{m \to \infty} f_{n_m}(x) & \text{si el límite existe} \\ 0, & \text{si el límite no existe} \end{cases} \qquad (*)$$

$\Rightarrow f(x) = \lim_{m \to \infty} f_{n_m}(x)$ for μ-a.e. x.

It remains to show: $f \in \mathcal{L}_p(X)$.

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ s.t. $\forall n, m \ge N$: $\|f_n - f_m\| < \varepsilon$.

$\Rightarrow \|f - f_m\|_p^p = \int_X |f - f_m|^p d\mu \underset{\text{Lemma von Fatou}}{\le} \liminf_{k \to \infty} \int |f_{n_k} - f_m|^p d\mu \le \varepsilon^p$

$\Rightarrow f - f_m \in \mathcal{L}_p(X) \Rightarrow f = (f - f_m) + f_m \in \mathcal{L}_p(X)$.

Case 2 $\boxed{p = \infty}$

Let $A_k := \{x \in X \mid f_k(x) \ge \|f_k\|_\infty\}$

$B_{n,m} := \{x \in X \mid |f_n(x) - f_m(x)| \ge \|f_n - f_m\|_\infty\}$

$C := \bigcup_k A_k \cup \bigcup_{n,m} B_{n,m}$

$\rightarrow \mu(C) = 0$ and $(f_n)_n$ converges uniformly to a bdd fct f on $X \setminus C$.

(Note: $(\|f_n\|_\infty)_n$ is a bdd sequence in $\mathbb{R}$)

Extend f to all of X, e.g. by setting $f(x) = 0$ $(x \in C)$. □

Theorem 7.5 (Riesz-Fischer)

$(L_p(X), \|\cdot\|_p)$ is a Banach space.

Proof. By Lemma 7.4, every $\|\cdot\|_p$-Cauchy sequence contains a convergent subsequence, so it is convergent (with the same limit). □

7.2. Approximation of functions in $\mathcal{L}_p$

Definition 7.6. X Banach space, $M \subseteq X$.

M is dense in X $:\Longleftrightarrow$ $\overline{M} = X$

$\Longleftrightarrow$ $\forall x_0 \in X \;\; \exists (y_n)_n \subseteq M$ s.t. $y_n \longrightarrow x_0 \;(n \to \infty)$.

Questions.

- Can functions in $\mathcal{L}_p$ be approximated by "simpler" functions with "nice properties"?
- If so, in what sense (pointwise convergence, convergence in $\|\cdot\|_p$-norm...)?

We already know: $f \in \mathcal{L}_p$ and $(f_n)_n \subseteq \mathcal{L}_p$ with $f_n \longrightarrow f$.

Then: There exists a subsequence which converges almost everywhere to f.

Proof. $(f_n)_n$ conv $\Rightarrow$ Cauchy sequence $\Rightarrow$ $\exists g \in \mathcal{L}_p$ and subseq. st. $f_{n_k} \to g$ μ-a.e.

$\Rightarrow \|f-g\|_p \leq \|f-f_n\|_p + \|f_n - g\|_p \longrightarrow 0 \Rightarrow f = g$ μ-a.e.

SIMPLE FUNCTIONS.

Theorem 7.7 $(X, \mathfrak{A}, \mu)$ measure space, $1 \leq p \leq \infty$, $S := E(X, \mathfrak{A}) \cap \mathcal{L}_p(X)$ = set of all simple function which belong to $\mathcal{L}_p(X)$.

$$\Longrightarrow \overline{S}^{\|\cdot\|_p} = \mathcal{L}_p(X)$$

(that is: $\forall f \in \mathcal{L}_p(X) \;\; \exists (s_n)_n \subseteq S$ st. $\|f - s_n\|_p \longrightarrow 0 \;(n \to \infty)$)

Proof. Let $f \in \mathcal{L}_p(X)$. Since $f = (\mathrm{Re} f)^+ - (\mathrm{Re} f)^- + i(\mathrm{Im} f)^+ - i(\mathrm{Im} f)^-$ is linear combination of 4 positive fct's in $\mathcal{L}_p(X)$, we may assume without loss of generality that $f \geq 0$.

Let $(s_n)_n$ as in Theorem 2.15. $\leadsto s_n \uparrow f$.

If $p = \infty$ then $s_n \xrightarrow{\|\cdot\|_\infty} f$ (it converges uniformly outside a set of measure 0)

If $1 \leq p < \infty$: Observe that $|f(x) - s_n(x)|^p \leq (|f(x)| + |s_n(x)|)^p$

$\leq 2|f(x)|^p$ $\quad (*)$

and all $s_n \in \mathcal{L}_p(X)$ because $0 \leq s_n \leq f$. Hence,

$$\lim_{n\to\infty} \|f - s_n\|_p^p = \lim_{n\to\infty} \int_X |f - s_n|^p \, d\mu \underset{\uparrow}{=} \int_X \lim_{n\to\infty} |f - s_n|^p \, d\mu = \int_X 0 \, d\mu = 0.$$

dominated convergence thm by $(*)$ because $|f|^p \in \mathcal{L}_1(X)$.

CONTINUOUS FUNCTIONS.

Theorem (Lusin's theorem).

$(X, \mathfrak{A}, \mu)$ measure space where X locally compact Hausdorff space, μ regular measure, Borel measure.

$A \in \mathfrak{A}$ with $\mu(A) < \infty$ and

$f: X \longrightarrow \mathbb{C}$ mb s.t. $f(x) = 0 \quad (x \notin A)$

$\Rightarrow \forall \varepsilon > 0 \quad \exists g \in C_c(X)$ st. $\mu\{x \in X \mid f(x) \neq g(x)\} < \varepsilon$ $\quad (**)$

g can be chosen such that $\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$. $\quad (*)$

That is: "A measurable function is almost continuous".

Proof. Assume $\boxed{0 \leq f \leq 1 \text{ and } A \text{ cpt.}}$

Choose sequence of simple fct's s_n as in the proof of thm 2.15.

Let $t_1 := s_1$, $t_n := s_n - s_{n-1}$ $\quad (n \geq 2)$.

Note: $\|t_n\|_\infty \leq 2^{-n}$. $\leadsto$ $2^n t_n = \chi_{T_n}$ where $T_n \subseteq A$,

$T_n = \{x \in X \mid t_n(x) \neq 0\}$.
$= \{x \in X \mid t_n(x) = 2^{-n}\}$

and $f(x) = \lim_{n\to\infty} s_n(x) = \sum_{n=1}^{\infty} t_n(x)$.

Choose V open, s.t. $A \subseteq V$ and $\overline{V}$ cpt.

Choose K_n cpt, V_n open, such that $K_n \subseteq T_n \subseteq V_n \subseteq V$, $\mu(V_n \setminus K_n) < \frac{\varepsilon}{2^n}$

By Urysohn's lemma ex. h_n continuous fct's with $h_n|_{K_n} = 1$, $\mathrm{supp}\, h_n \subseteq V_n$ and $0 \leq h_n \leq 1$.

Let $g(x) := \sum_{n=1}^{\infty} 2^{-n} h_n(x)$, $\quad (x \in X)$.

The series converges uniformly (because $\sum_{n=1}^{\infty} \|2^{-n} h_n\|_\infty = \sum_{n=1}^{\infty} 2^{-n} < \infty$)

$\Rightarrow g$ continuous

Observe:

- $\mathrm{supp}(g) \subseteq \overline{V}$ cpt $\Rightarrow g \in C_c(X)$
- $x \in X \setminus (V_n \setminus K_n) = K_n \cup (X \setminus V_n) \Rightarrow 2^{-n} h_n(x) = t_n(x) = \begin{cases} 0, & x \in X \setminus V_n \\ 2^{-n}, & x \in K_n \subseteq T_n \end{cases}$

$\Rightarrow f(x) = g(x)$ on $X \setminus \underbrace{\left(\bigcup_{n=1}^{\infty} V_n \setminus K_n\right)}_{\mu(\cdot) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus K_n) < \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon}$.

$\square$

Note: • The theorem also holds for f bdd, measurable (rescale!)

• If A not compact: Choose $K \subseteq A$ cpt with $\mu(A \setminus K)$ small, and extend the fct g obtained by the first steps for K and f to A by $g(x) = 0$ on $A \setminus K$. $\leadsto$ continuous because supp $g \subseteq K$.

• If f complex, mb.

Let $B_n := \{x \in X \mid |f(x)| > n\}$

$\Rightarrow \bigcap_{n=1}^{\infty} B_n = \emptyset \Rightarrow \mu(B_n) \to 0$

$\Rightarrow \mu(\{f - \underbrace{(1-\chi_{B_n}) f}_{\text{bdd!}} \neq 0\}) \to 0 \quad (n \to \infty)$

$\leadsto$ Apply what we already proved to $\chi(1-\chi_{B_n}) f$ for n large enough.

To prove $(*)$:

Define $R := \sup\{|f(x)| \mid x \in X\}$ and

$\varphi : \mathbb{C} \to \mathbb{C}, \quad \varphi(z) := \begin{cases} z & \text{if } |z| \le R \\ \frac{Rz}{|z|} & \text{if } |z| > R \end{cases}$

$\Rightarrow \varphi$ continuous and $|\varphi(z)| \le R \quad (z \in \mathbb{C})$

If g satisfies $(**)$, then so does $\tilde{g} := \varphi \circ g$ and $|\tilde{g}(x)| \le R \quad (x \in X)$ $\square$

Theorem 7.8. (Approximation by continuous fcts)

$(X, \mathcal{A}, \mu)$ measure space, X locally compact Hausdorff, μ regular Borel measure.

Then: $\quad C_c(X) \subseteq \mathcal{L}_p(X)$ dense for $1 \le p < \infty$.

Proof. Let $S = \mathcal{E}(X, \mathcal{A}) \cap \mathcal{L}_p(X)$, $\varepsilon > 0$

Let $f \in \mathcal{L}_p(X) \Rightarrow \exists s \in S$ s.t. $\|f - s\|_p < \varepsilon/2$.

s satisfies the hypothesis of Lusin's theorem.

$\Rightarrow \exists g \in C_c(X)$ s.t. $g(x) = s(x)$ on $X \setminus B$ where $\mu(B) < \left(\frac{\varepsilon}{2\|g\|_\infty}\right)^p$

and $|g| \le \|s\|_\infty$.

$\Rightarrow \|f - g\|_p \le \|f - s\|_p + \underbrace{\|s - g\|_p}_{\substack{= \left(\int_X |s-g|^p d\mu\right)^{1/p} = \left(\int_B |s-g|^p d\mu\right)^{1/p} \\ \le \left(\int_B \|g\|_\infty^p d\mu\right)^{1/p} = \|g\|_\infty \mu(B)^{1/p}}} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$ $\square$

Note. $\overline{C_c(X)}^{\|\cdot\|_\infty} \neq \mathcal{L}_\infty(X)$ in general.

Example $X = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$

$\leadsto f \in \mathcal{L}_\infty(\mathbb{R})$, but $\forall \varphi \in C_c(\mathbb{R}) : \|f - \varphi\|_\infty = 1$. $\square$

Definition 7.9.

X locally compact Hausdorff space. Then: f vanishes at infinity

$:\Longleftrightarrow \forall \varepsilon > 0 \ \exists K$ cpt s.t. $|f(x)| < \varepsilon$ for $x \in X \setminus K$.

$C_0(X) := \{f : X \to \mathbb{K} \mid f$ continuous, f vanishes at infinity$\}$.

Theorem 7.10. X locally compact Hausdorff space

$\Rightarrow \overline{C_c(X)}^{\|\cdot\|_\infty} = C_0(X)$.

Proof.

Let $f \in C_0(X)$ and $\varepsilon > 0$.

$\Rightarrow \exists K$ cpt s.t. $|f(x)| < \varepsilon$ on $X \setminus K$.

By Urysohn's lemma $\exists g \in C_c(X)$ s.t. $0 \le g \le 1$, $g|_K = 1$.

Let $h = f \cdot g$. $\Rightarrow h \in C_c(X)$ and $\|f - h\|_\infty < \varepsilon$.

Now we show that $C_0(X)$ is complete.

Let $(f_n)_n \subseteq C_0(X)$ Cauchy sequence.

$\Rightarrow$ the pointwise limit exists and is continuous.

Let f be the limit fct, and $\varepsilon > 0$.

Choose $n \in \mathbb{N}$ s.t. $\|f_n - f\| < \varepsilon/2$ and K cpt s.t. $|f_n(x)| < \varepsilon/2$ on $X \setminus K$.

$\Rightarrow \forall x \in X \setminus K$: $\quad |f(x)| \le |f_n(x)| + |f(x) - f_n(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\Rightarrow \quad f \in C_0(X)$. $\square$

7.3. Dual spaces

Definition. Let X be a (real or complex) Banach space and $\varphi: X \to \mathbb{K}$ linear.
φ is called bounded if

$$\|\varphi\| := \sup\left\{ \frac{|\varphi(x)|}{\|x\|} \;\middle|\; x \in X \text{ with } \|x\| = 1 \right\} < \infty.$$

In this case, φ is called a bounded linear functional on X.
The set of all bounded linear functionals is called the dual space of X, denoted by X'.

Let $1 \le p < \infty$ and q s.t. $\frac{1}{p} + \frac{1}{q} = 1$, $(X, \mathcal{A}, \mu)$ measure space.
Then obviously every $g \in L_q(X)$ defines a bdd linear fct'nl on $L_p(X)$ by:

$$\varphi_g : L_p(X) \to \mathbb{C}, \quad f \mapsto \int fg \, d\mu \qquad (*)$$

- Linearity is clear
- Boundedness: Let $f \in L_p(X)$ with $\|f\|_p = 1$.

$$\Rightarrow |\varphi_g(f)| = \left|\int fg\, d\mu\right| \underset{\text{Hölder}}{\le} \|f\|_p \|g\|_q = \|g\|_q$$

$$\Rightarrow \varphi \text{ is bdd. and } \|\varphi_g\| \le \|g\|_q.$$

It turns out that $L_q(X)$ "is" the dual space of $L_p(X)$:

Theorem $(X, \mathcal{A}, \mu)$ σ-finite measure space, $1 \le p < \infty$.
Then: $\Phi: L_q(X) \to (L_p(X))'$, $\quad g \mapsto \varphi_g$
is a surjective isometry. (φ_g defined as in $(*)$)

Proof. We already saw that Φ is well-defined; obviously Φ is linear.
Now we prove surjectivity of Φ.
Let $\varphi \in \Phi$.

Case 1 $\mu(X) < \infty$.

$\forall A \in \mathcal{A}$ define $\nu(A) := \varphi(\chi_A)$.

Then: φ is a finite measure on X because $\nu(\emptyset) = \varphi(0) = 0$,
and for $A = \dot{\bigcup}_{j=1}^{\infty} A_j$ with A_j mb.

Dominated convergence thm (note: $\mu(X) < \infty$!): $\overbrace{\left\|\sum_{j=1}^{n} \chi_{A_j} - \chi_A\right\|_p \to 0}^{\text{is not true for } p=\infty}$

$\overset{\varphi \text{ cont.}}{\Longrightarrow} \varphi\left(\sum_{j=1}^{n} \chi_{A_j}\right) - \varphi(\chi_{A_j}) \to 0 \quad (n \to \infty)$

$\overset{\varphi \text{ lin.}}{\Longrightarrow} \sum_{j=1}^{n} \nu(A_j) - \nu(A) \to 0 \quad (n \to \infty).$

$\Rightarrow \sum_{j=1}^{\infty} \nu(A_j) - \nu(A) = 0.$

If $\mu(A) = 0 \Rightarrow \nu(A) = \varphi(\chi_A) = 0$ because $|\varphi(\chi_A)| \le \|\varphi\| \overbrace{\|\chi_A\|_p}^{=0} = 0$
$\Rightarrow \nu \ll \mu$.

$\overset{\text{Radon-Nikodym}}{\Longrightarrow} \exists g \in \mathcal{L}^1(\mu)$ s.t. $\varphi(\chi_A) = \int_A g\, d\mu = \int \chi_A g\, d\mu$. $\quad (A \in \mathcal{A}) \quad (*)$

$\Rightarrow \varphi(f) = \int_A fg\, d\mu \quad$ ($f \in \mathcal{L}_\infty(X)$, because L_∞-fcts can be approx. by simple fcts)

Now we show: $g \in \mathcal{L}_q(X)$.

(A) $p = 1$: $\forall A \in \mathcal{A}$: $\left|\int_A g\, d\mu\right| = |\varphi(\chi_A)| \le \|\varphi\| \|\chi_A\|_1 = \|\varphi\| \mu(A)$.

$\Rightarrow |g(x)| \le \|\varphi\|$ μ-a.e.

Assume not. $\Rightarrow \exists \alpha \in \mathbb{C}$, $|\alpha| > \|\varphi\|$ and $\varepsilon > 0$ t.q. $\|\varphi\| < \alpha - \varepsilon$
& $\mu\{g^{-1}(B_\varepsilon(\alpha))\} > 0$. Let $A := g^{-1}(B_\varepsilon(\alpha))$

$$\Rightarrow \begin{cases} \left|\frac{1}{\mu(A)}\int_A g\, d\mu - \alpha\right| = \frac{1}{\mu(A)}\left|\int_A g - \alpha\, d\mu\right| \le \frac{1}{\mu(A)} \int_A \overbrace{|g - \alpha|}^{\le \varepsilon} d\mu \le \varepsilon & g(x) \in B_\varepsilon(\alpha) \Leftrightarrow A \\ \left|\frac{1}{\mu(A)}\int_A g\, d\mu - \alpha\right| \ge \frac{-1}{\mu(A)}\left|\int g\, d\mu\right| + \alpha \ge \alpha - \|\varphi\| > \varepsilon. \;\text{↯} \end{cases}$$

(B) $1 < p < \infty$:

$\Rightarrow \exists \alpha: X \to \mathbb{C}$ mb, s.t. $g = \alpha |g|$.

$\forall n$ let $E_n := \{x \in X \mid |g(x)| \le n\}$.
and $f := \chi_{E_n} g^{q-1} \cdot \alpha \quad \Rightarrow f \in L^\infty(X)$ and $|f|^p = |g|^q$

Then, by $(*)$:

$$\int_X \chi_{E_n} |g|^q d\mu = \int_{E_n} |g|^q d\mu = \int_X fg\, d\mu = \Phi(f) \le \|\varphi\| \|f\|_p = \|\varphi\| \left(\int_{E_n} |g|^q d\mu\right)^{1/p}$$

$$\left(\int_{E_n} |g|^q d\mu\right)^{1/q} \le \|\varphi\| \quad \cancel{\left(\int |g|^q d\mu\right)^{1/p}} \overset{\text{mon. conv.}}{\Longrightarrow} \left(\int_X |g|^q d\mu\right)^{1/q} \le \|\varphi\| \quad (**)$$

Now (*) holds for all $f \in L_p(X)$ because both sides define bdd. operators on $L_\infty(X)$ and $L_\infty(X)$ is dense in $L_p(X)$ ($\|\cdot\|_p$-dense).
It also follows that ϕ is an isometry because
$\|\varphi_g\| \le \|g\|_q$ (by Hölder), $\|\varphi_g\| \ge \|g\|_q$ by (**)

Case 2 $\boxed{\mu(X) = \infty}$

Let $S_1 \subseteq S_2 \subseteq \dots$ with $X = \bigcup_{j=1}^{\infty} S_j$ and $\mu(S_j) < \infty$ ($j \in \mathbb{N}$).
$\Rightarrow$ Apply case 1 to S_j. $\rightsquigarrow g_j$.
By uniqueness of the g_j's: $g_{j+1}|_{S_j} = g_j$.
$\rightsquigarrow g$ on X, and $\|g\|_q \le \|\varphi\|$ because $\|g_n\|_q \le \|\varphi\|$ for every $n \in \mathbb{N}$. $\square$

- $1 \le p < \infty$, $n \in \mathbb{N}$, $f \in \mathcal{L}_p(\mathbb{R}^n)$.

  $\Rightarrow \exists$ step functions $(t_n)_n$ s.t. $\|f - t_n\|_p \to 0 \quad (n \to \infty)$

  where $t_n := \sum_{j=1}^{n} \alpha_j \chi_{A_j}$ step function if A_j n-cells, that is a ~~gener~~ finite cube with or without boundary

Proof.

① $f = \chi_U$ for some U open.

$f \in \mathcal{L}_p \Rightarrow \mu(U) < \infty$.

Known: $U = \dot{\bigcup}_{j=1}^{\infty} W_j$ for W_j n-cells

Let $t_n := \sum_{j=1}^{n} \chi_{W_j}$, $t := \lim_{n\to\infty} t_n$.

$\Rightarrow \|f - t_n\|_p^p = \int_X (\chi_U - \chi_{\dot\bigcup_{j=1}^n W_j})^p \, d\lambda = \int_{U \setminus \bigcup_{j=1}^n W_j} d\lambda = \lambda(U \setminus \dot\bigcup_{j=1}^{n} W_j)$

$= \lambda(\bigcup_{j=n+1}^{\infty} W_j) \to 0$ (because $\lambda(U) < \infty$).

② $f = \chi_E$ for some $E \in \mathcal{L}^s$ = Lebesgue σ-algebra

Choose $U \subseteq \mathbb{R}^s$ open with $E \subseteq U$, $\lambda(U \setminus E) < (\varepsilon/2)^p$

Note $\mu(U) \le \mu(E) + (\varepsilon/2)^p < \infty$.

By ① $\exists t$ step fct s.t. $\|t - \chi_U\|_p < \varepsilon/2$.

$\Rightarrow \|f - t\|_p \le \|f - \chi_U\|_p + \|\chi_U - t\|_p < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

④ f simple fct: ✓

⑤ f mb ✓ because we already know that $E(\mathbb{R}^s, \lambda^s) \cap \mathcal{L}_p(\mathbb{R}^s) \subseteq \mathcal{L}_p(\mathbb{R}^s)$ dense.

- $1 \le p < \infty$, $n \in \mathbb{N}$, Q n-cell, $\varepsilon > 0$

  $\Rightarrow \exists q_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ st. $\|\chi_Q - q_\varepsilon\|_p < \varepsilon$.

Proof. Let $\{\varphi_\varepsilon \mid \varepsilon > 0\}$ be an approx. unity, $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Let $q_\delta := \varphi_\delta * \chi_Q$.

Without restriction $Q = \overline{Q}$ because $\|\chi_Q - \chi_{\overline{Q}}\|_p = 0$

$\Rightarrow Q = [a_1, b_1] \times \dots \times [a_n \times b_n]$.

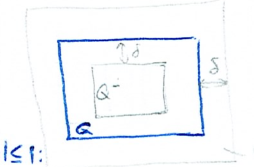
If $a_j = b_j$ for some j $\Rightarrow$ choose $q_\varepsilon = 0 \Rightarrow \|\chi_Q - q_\varepsilon\|_p = 0$.

If $a_j \ne b_j$ $(j = 1, \dots n)$ let $\delta_0 = \frac{1}{2}\min\{b_j - a_j \mid j = 1, \dots, n\} > 0$.

For $0 < \delta < \delta_0$ let

$Q_\delta^+ := (a_1 - \delta, b_1 + \delta) \times \dots \times (a_n - \delta, b_n + \delta)$

$Q_\delta^- := (a_1 + \delta, b_1 - \delta) \times \dots \times (a_n + \delta, b_n - \delta)$



$\Rightarrow$ for $x \in Q_\delta^- \cup (\mathbb{R}^n \setminus Q_\delta^+)$: and $t \in \mathbb{R}^n$ with $|t| \le 1$:

$\chi_Q(x) - \chi_Q(x + \delta t) = 0$.

$\Rightarrow$ For $0 < \delta < \delta_0$:

$|\chi_Q(x) - \varphi_\delta * \chi_Q(x)| = \left| \int_{\mathbb{R}^s} \varphi_\delta(x - y)(\chi_Q(x) - \chi_Q(y))\, dy \right|$

$\le \int_{\mathbb{R}^s} |\underbrace{\varphi_\delta(x-y)}_{= \varphi(t)}| \, |\chi_Q(x) - \chi_Q(y)| \, dy$

$\overset{t = \frac{x-y}{\delta}}{=} \int_{\mathbb{R}^s} \delta^{-n} |\varphi_\delta(t\delta)| \, \underbrace{|\chi_Q(x) - \chi_Q(x - t\delta)|}_{\le 1} \, dy$

$= \begin{cases} 0, & x \in Q_\delta^- \cup (\mathbb{R} \setminus Q_\delta^+) \\ 1, & x \in Q_\delta^+ \setminus Q_\delta^- \end{cases}$

$\Rightarrow \|\chi_Q - \varphi_\delta * \chi_Q\|_p^p = \int_{\mathbb{R}^s} |\chi_Q(x) - \varphi_\delta * Q(x)|^p \, d\lambda^s$

$\le \int_{Q_\delta^+ \setminus Q_\delta^-} d\lambda^s = \mu(Q_\delta^+ \setminus Q_\delta^-) \longrightarrow 0$.

$\square$

Corollary $C_c^\infty(\mathbb{R}^s)$ dense in $\mathcal{L}_p(\mathbb{R}^s)$.