Theorems without proof:

<u>Theorem</u> (Rudin, Thm. 7.13). $x \in \mathbb{R}^s$, $(A_n)_n$ nicely shrinking to x.

μ complex Borel measure, $\mu \perp \lambda$

$$\Rightarrow \lim_{n\to\infty} \frac{\mu(A_n)}{\lambda(A_n)} = 0 \quad \lambda\text{-ae.}$$

<u>Theorem</u> (Rudin, Thm 7.14)

$x \in \mathbb{R}^s$, $(A_n)_n$ nicely shrinking to x.

μ complex Borel measure on $\mathbb{R}^s$ with Lebesgue decomposition $\mu = f \odot \lambda + \mu_s$ $(\mu_s \perp \lambda)$.

$$\Rightarrow \lim_{n\to\infty} \frac{\mu(A_n)}{\lambda(A_n)} = f(x) \quad \lambda\text{-ae.}$$

In particular; $\mu \perp m \Rightarrow (D\mu)(x) = 0$ λ-a.e.

<u>Theorem</u> (Rudin, Thm 7.15)

μ pos. Borel measure on $\mathbb{R}^s$, $\mu \perp m$.

$$\Rightarrow (D\mu)(x) = 0 \quad \mu\text{-ae.}$$

6.4. The fundamental theorem of calculus.

<u>Recall from Analysis 1:</u>

$f: [a,b] \to \mathbb{R}$ cont, $F: [a,b] \to \mathbb{R}$, $F(x) := \int_a^x f(t)\,dt$

$\Rightarrow$ F differentiable and $F'(x) = f(x)$.

<u>Questions:</u>

- $f \in \mathcal{L}_1 \Rightarrow F$ differentiable ? $\rightsquigarrow$ F differentiable λ-ae (Cor. 6.18)
  (Note: the proof uses that F' can be approximated by <u>cont</u> fct's in $\|\cdot\|_1$)
- F continuous, and diff. ae. $\Rightarrow$ $F' \in \mathcal{L}_1$?
  $F(x) - F(a) = \int_a^x F'(t)\,dt$?

No! Extra condition "F abs. continuous" is needed!

<u>Examples:</u>

(1) $F: [0,1] \to \mathbb{R}$, $F(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0. \end{cases}$

$\to$ F continuous and everywhere diff'able $\left(F'(x) = \begin{cases} 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos(\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0 \end{cases} \right)$

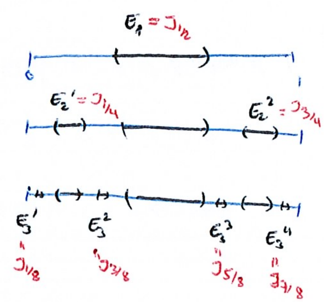
but $F' \notin \mathcal{L}_1([0,1])$ because $\int_{[0,1]} |F'(t)|\,dt = \infty$.

(2) <u>Cantor function.</u>

<u>Recall:</u> $C :=$ Cantor set is cpt, uncountable.

$$C' := [0,1] \setminus C = \bigcup_{n\in\mathbb{N}} \bigcup_{k=1}^{2^{n-1}} E_n^k = \bigcup_{n\in\mathbb{N}} \bigcup_{k=1}^{2^{n-1}} J_{\frac{2k-1}{2^n}}$$

Where $E_n^k = J_{\frac{2k-1}{2^n}}$ = k-th open interval from the left of length 3^{-n}



$\lambda(C') = 1$, $\lambda(C) = 0$

Define for $x \in E_n^k$: $\psi_0(x) := \frac{2k-1}{2^n}$ $\leftarrow$ ψ_0 is increasing!

Note also: $\psi_0|_{E_n^k}$ = arithmetic mean of ψ on neighbouring "black" intervals.

Define $\psi: [0,1] \to \mathbb{R}$ by:

$\psi(0) := 0$, $\psi(x) := \sup\{\psi_0(t) \mid t \leq x\}$

$\to$ $\psi(1) = 1$, $\psi(x) = \psi_0(x)$ for $x \in C'$ and ψ is increasing.

- Ψ <u>continuous</u>:

  * $x \in E_n^k$ for some n, k. $\Rightarrow$ Ψ cont. in x, because ~~locally~~ const. on the open neighbourh. E_n^k of x.

  * $x \in C$; $x \neq 1$ } let $\varepsilon > 0$. Choose n large enough s.t. $2^{-n} < \varepsilon$ and $x_0 < \mathring{J}_{\frac{2k-1}{n}}$ for some k.

  $\rightarrow$ Choose k_0 s.t. $J_{k_0/2^n} < x < J_{\frac{k_0+1}{2^n}}$ and $\alpha \in J_{k_0/2^n}$, $\beta \in J_{\frac{k_0+1}{n}}$.

  $\Rightarrow$ $\Psi(\alpha) = \frac{k_0}{2^n}$, $\Psi(b) = \frac{k_0+1}{2^n}$

  Ψ increasing $\Rightarrow$ $\forall y \in (\alpha, \beta)$: $|\Psi(y) - \Psi(x)| \leq \Psi(\beta) - \Psi(\alpha) = \frac{1}{2^n} < \varepsilon$.

  $\Rightarrow$ Ψ cont. in x.

  * $x = 1$: ✓

- <u>Ψ diff λ-a.e.</u> because Ψ is piecewise constant on the <u>open</u> set C':

  $\rightarrow$ $\Psi'(t) = 0$, $t \in C'$.

  <u>But</u>: $\boxed{\underbrace{\Psi(1) - \Psi(0)}_{=1-0=1} \neq \underbrace{\int_0^1 \Psi' \, d\lambda}_{=\int 0 \, d\lambda = 0}}$

- If μ_Ψ = measure induced by Ψ $\Rightarrow$ $\mu_\Psi \perp \lambda$ because $\mu_\Psi(C') = 0 = \lambda(C)$

  $(\mu_\Psi([a,b)) := \Psi(b) - \Psi(a))$

- <u>Ψ not abs. continuous</u>:

  let $\delta > 0$ and choose n s.t. $(2/3)^n < \delta$.

  Observe: total length of "blue" intervals in the n-th step
  $= \lambda([0,1] \setminus \bigcup_{k=1}^{2^n - 1} J_{k/2^n}) = (2/3)^n$.

  Denote the "blue" intervals at the n-th step by $[\alpha_1, \beta_1], \dots, [\alpha_p, \beta_p]$

  Ψ const. on (α_j, β_j) and f cont $\Rightarrow$ $\Psi(\beta_j) = f(\alpha_{j+1})$

  $$\Rightarrow \begin{cases} \sum_{k=1}^{p} \beta_k - \alpha_k = (2/3)^n < \delta \\ \sum_{k=1}^{p} |\Psi(\beta_k) - \Psi(\alpha_k)| = \sum_{k=1}^{p} \Psi(\beta_k) - \Psi(\alpha_k) = \Psi(\beta_p) - \Psi(\alpha_1) = \Psi(1) - \Psi(0) = 1. \end{cases}$$

  $\Rightarrow$ f not abs. continuous.

  <u>Remark</u> This follows also from theorem 6.24 which says that

  Ψ abs. cont $\Leftrightarrow$ $\Psi \ll \lambda$. $\Leftrightarrow$ $\Psi = \int \Psi' \, d\lambda$

<u>Recall</u>

<u>Definition 6.19</u> $f : [a, b] \rightarrow \mathbb{C}$ absolutely continuous

$:\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for $a \leq \alpha_1 < \beta_1 < \alpha_2 < \dots < \alpha_n < \beta_n \leq b$:
with $\sum_{j=1}^n \beta_j - \alpha_j < \delta$: $\sum_{j=1}^n |f(\beta_j) - f(\alpha_j)| < \varepsilon$. (*)

Similar for $f : (a, b) \rightarrow \mathbb{C}$ with $-\infty \leq a < b \leq \infty$

<u>Def</u>

$P = \{x_0, x_1, \dots, x_n\}$ partition of $[a, b]$. $(x_0 = a < x_1 < \dots < x_n = b)$

$\Rightarrow$ $V(P, f) := \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$ =: variation of f with respect to P

$V_a^b(P, f) := \sup \{ V(P, f) \mid P \text{ partition of } [a,b] \}$ =: total variation of f

f is of bounded variation $:\Leftrightarrow$ $V_a^b f < \infty$.

If $f : (a, b) \rightarrow \mathbb{C}$ with $-\infty \leq a < b \leq \infty$

$\Rightarrow$ $V_a^b f := \lim_{\alpha \searrow a} \lim_{\beta \nearrow b} V_\alpha^\beta f$.

<u>Observation</u>.

i) f abs. cont $\Leftrightarrow$ (*) is true for an "infinite partition", that is:
$(a_j)_j, (b_j)_j \subseteq (a, b)$ with $a \leq a_j < b_j < a_{j+1} < b_{j+1} < b$ $(j \in \mathbb{N})$

ii) f abs. cont $\Rightarrow$ f uniformly continuous

iii) f cont $\not\Rightarrow$ f abs. cont. (take f = Cantor function)

<u>Observations</u>.

i) $f, g : [a, b] \rightarrow \mathbb{C}$, $\alpha \in \mathbb{C}$ $\rightarrow$ $V_a^b(f + g) \leq V_a^b(f) + V_a^b(g)$
$V_a^b(\alpha f) = |\alpha| \, V_a^b(f)$.

ii) $V_a^b f = 0$ $\Leftrightarrow$ f constant

iii) f monotonic $\Rightarrow$ $V_a^b f = |f(b) - f(a)| < \infty$

iv) $\forall c \in [a, b]$: $V_a^b f = V_a^c f + V_c^b f$, $V_a^a f = 0$

v) f of bdd variation $\Rightarrow$ f bdd.

<u>Proof</u> $\forall x \in [a, b]$: $|f(x)| \leq |f(a)| + |f(x) - f(a)|$
$\leq |f(a)| + V_a^b f < \infty$

$\square$

Theorem 6.20. $f : [a,b] \to \mathbb{R}$ continuous, non-decreasing.

The following is equivalent:

i) f is abs. continuous

ii) $f\ \forall A \in \mathcal{L}$: $\lambda(A) = 0 \Rightarrow \lambda(f(A)) = 0$

iii) f differentiable λ-ae, $f' \in \mathcal{L}_1([a,b])$ and

$$f(x) - f(a) = \int_a^x f'(t)\, d\lambda.$$

Proof.

i) ⇒ ii) Let $A \in \mathcal{L}^1$, $A \subseteq J$ with $\lambda(A) = 0$. Without restriction: $a, b \notin A$.

~~f cont ⇒ f bdd ⇒ $f \in \mathcal{L}_1([a,b])$~~

Let $\delta > 0$ and choose $\varepsilon > 0$ as in the def of "abs. cont.".

By regularity of λ ex. $(a_j)_j, (b_j)_j$ st. $A \subseteq \dot{\bigcup}_{j=1}^{\infty} (a_j, b_j) =: V \subseteq (a,b)$ st. $\lambda(V) < \delta$

$$\Rightarrow \sum_{j=1}^{\infty} b_j - a_j < \delta \xRightarrow{f \text{ abs. cont}} \underbrace{\sum_{j=1}^{\infty} |f(b_j) - f(a_j)|}_{\geq \lambda(f(A))} < \varepsilon.$$

because $f(A) \subseteq \bigcup_j (f(a_j), f(b_j))$ since f is monotonic.

$\Rightarrow f(A) \in \mathcal{L}_1$ and $\lambda(f(A)) = 0$.

ii) ⇒ iii) Let $g : [a,b] \to \mathbb{R}$, $g(x) := x + f(x)$.

g satisfies ii) because: $g((\alpha,\beta)) \subseteq (\alpha,\beta) + (f(\alpha), f(\beta)) \subseteq (\alpha + f(\alpha), \beta + f(\beta))$

$\rightsquigarrow \lambda(g(\alpha,\beta)) \leq (\beta - \alpha) + |f(\beta) - f(\alpha)|$.

Let $A \in \mathcal{L}^1$, $A \subseteq [a,b]$.

By inner regularity: $\exists K_n \subseteq A$, K_n cpt. st $A = \bigcup_{n \in \mathbb{N}} K_n \cup A_0$ with $\lambda(A_0) = 0$.

$\Rightarrow g(\lambda(A_0)) = 0$, and $g(K_n)$ cpt for all $n \in \mathbb{N}$ (g cont!).

in particular: $g(K_n) \in \mathcal{L}^1$.

Define $\mu : \mathcal{L}^1 \to [0,\infty]$, $\mu(A) = \lambda(g(A))$

$\to \mu$ is a measure (σ-additivity follows from injectivity of g)

$\mu \ll \lambda$ because g satisfies ii).

$\Rightarrow \exists h \in \mathcal{L}_1(\lambda)$ st. $\mu = h \odot \lambda$ (Radon-Nikodym)

$\Rightarrow g(x) - g(a) = \lambda([g(a), g(x)]) = \mu([a,x]) = \int_{[a,x]} h\, d\lambda$

$\Rightarrow g$ diff. λ-ae and $g'(x) = h(x)$ λ-ae.

$\Rightarrow f$ diff λ-ae and $f'(x) = \frac{d}{dx}(g(x) - x) = h(x) - 1$ λ-ae.

iii) ⇒ i) ~~f' induces a measure: $\mu = f' \odot \lambda$~~

Übungsblatt 5, Aufgabe 3. □

Theorem 6.21. $f : [a,b] \to \mathbb{C}$ abs. continuous.

$\Rightarrow f$ diff λ-ae, $f' \in \mathcal{L}_1([a,b])$ and

$$f(x) - f(a) = \int_a^x f'(t)\, dt$$

Proof. Without restriction: f real valued.

By Lemma 6.22 below ex. f_1, f_2 abs. cont, non-decreasing, st. $f = f_1 - f_2$.

Application of Thm 6.20 to f_1, f_2 & linearity of integration and differentiation give the assertion. □

Lemma 6.22. $f : [a,b] \to \mathbb{R}$, of bdd variation

Let $f_\pm := \frac{1}{2}(V_a^x f \pm f(x))$.

$\Rightarrow f_\pm$ are non-decreasing and $V_a^b(f_\pm) \leq V_a^b(f)$.

Proof $\forall x \leq y$: $f(y) - f(x) \leq |f(y) - f(x)| \leq V_x^y f = V_a^y f - V_a^x f$.

$$\Rightarrow \underbrace{V_a^y f + f(y)}_{= f_+(y)} \geq \underbrace{V_a^x f + f(x)}_{f_+(x)} \Rightarrow f_+ \text{ non-decreasing.}$$

$$V_a^b(f_+) \leq \frac{1}{2}\big(\underbrace{V_a^b(V_a^* f)}_{= V_a^b f - V_a^a f = V_a^b f}\big) + \frac{1}{2} V_a^b f = V_a^b f.$$

because $x \mapsto V_a^x f$ is non-increasing

Similar for f_-. □

Alternative proof

f of bdd variation ⇒ f induces finite measure μ_f

$\Rightarrow \mu_f = (\mu_f)^+ + (\mu_f)^-$ by Hahn's decomposition thm

$\to (\mu_f)^\pm$ define non-decreasing fct's $\varphi_\pm$

$\Rightarrow f = \varphi_+ - \varphi_-$ □

Also. f monotonic ⇒ measure $\mu_f = \mu_c + \mu_\perp$ with $\mu_c \ll \lambda$, $\mu_\perp \perp \lambda$ (Lebesgue decomposition)

$\Rightarrow f = f_c + f_s$. f_c cont and f_c: piecewise cond. with at most countably many jumps

Lemma 6.23. $f: [a,b] \to \mathbb{C}$ abs cont;

$$f_\pm := \frac{1}{2}\left(V_a^x f \pm f(x)\right) \quad \text{abs. cont.}$$

Proof. Obviously, it suffices to show that $V_a^x f$ is of bdd variation.
Let $\varepsilon > 0$ and choose $\delta > 0$ as in the def of abs. cont. for f.
Let $a \le a_1 < b_1 < \dots < b_n \le b$. with $\sum_{j=1}^n b_j - a_j < \delta$.

$$\Rightarrow \sum_{j=1}^n |V_a^{b_j} f - V_a^{a_j} f| = \sum_{j=1}^n V_{a_j}^{b_j} f$$

$$= \sum_{j=1}^n \sup\left\{ \sum_{k=1}^m |f(t_k) - f(t_{k-1})| \;\middle|\; a_j = t_0 < t_1 < \dots < t_m = b \right\}$$

$$= \sup\left\{ \sum_{j=1}^n \sum_{k=1}^m |\dots| \;\middle|\; \dots \right\} \le \varepsilon.$$

□

Corollary 6.24

f of bdd variation $\Rightarrow$ f has at most finitely discontinuities and in every $x \in (a,b)$ the left- and right limit exist.

Proof The conclusion is true for monotonic fct's (Analysis 1)
By Lemma 6.22. f is sum of two mon. fcts □

Lemma 6.25. $f \in BV[a,b]$, $c \in [a,b]$

$$\Rightarrow \quad f \text{ left cont. in } b \iff \lim_{c \nearrow b} V_a^c f = V_a^b f$$

$$f \text{ right cont. in } a \Leftarrow \lim_{c \searrow a} V_c^b f = V_a^b f.$$

Proof.

"$\Rightarrow$" $|f(b) - f(c)| \le |V_c^b f| \longrightarrow 0 \quad (c \nearrow b)$

"$\Leftarrow$" f left cont in b $\Rightarrow$ $f_\pm$ can be chosen left-cont. in b.

$$\Rightarrow |V_a^b f - V_a^c f| = |V_c^b f| \le |V_c^b f_+| + |V_c^b f_-|$$

$$= f_+(b) - f_+(c) + |f_-(b) - f_-(c)|$$

$$\longrightarrow 0 \qquad (c \nearrow b)$$

Analogously for a. □

$\rightarrow$

① ϕ of bdd variation $\Rightarrow$ ϕ has a derivative μ-a.e.

② $\phi = \int \phi' \, d\lambda \iff \phi$ abs. cont.

③ $\phi \in \mathcal{L}_1 \Rightarrow \phi = \frac{d}{dx} \int \phi \, dx \quad \lambda$-a.e.

Proposition ϕ of bdd variation on $(-\infty, \infty)$. Then:

ϕ abs. cont $\iff \mu_\phi$ is abs. cont. wrt λ $\quad (\mu_\phi([a,b)) := \phi(b) - \phi(a))$.

Integration by parts

$F, G : [a,b] \to \mathbb{C}$ abs. cont.

$\Rightarrow$ $f(x) := F'(x)$, $g(x) := G'(x)$ ex. λ-a.e (Thm 6.21), and

$$\int_a^b F(x) g(x)\, dx = F(x) G(x) \Big|_a^b - \int_a^b f(x) G(x)\, dx.$$

Proof. It can be shown: $F \cdot G$ abs cont.

$\Rightarrow$ $F \cdot G$ diff almost everywhere, $(FG)'(x) = F(x) g(x) + f(x) G(x) \quad \lambda$-a.e.

$$\Rightarrow (F \cdot G)(x) \Big|_a^b = \int_a^b \frac{d}{dx}(F G)\, dx$$

$$= \int_a^b f(x) G(x) + F(x) g(x)\, dx.$$

□