- Def bdd lin. fctal
- Def HS, Satz von Riesz-Fréchet

Radon Nikodym & Lebesgue decomposition (Teschl)

Theorem. μ, ν σ-finite measures on $\mathcal{A}$

$\Rightarrow \exists f \geq 0$ μ-a.e., mb, and $\exists N \in \mathcal{A}$ with $\mu(N) = 0$

such that $\nu(A) = \nu(A \cap N) + \int_A f \, d\mu$. $\quad (*)$

Proof.

Case 1 μ, ν finite

$\Rightarrow \alpha := \mu + \nu$ is a finite measure.

Then: $\varphi : \mathcal{L}_2(X, \alpha) \to \mathbb{C}, \quad h \mapsto \varphi(h) = \int_X h \, d\nu$

is a bdd. linear functional on $\mathcal{L}_2(X, \alpha)$ because:

- linearity: ✓
- bdd: $|\varphi(h)|^2 = \left|\int_X h \, d\nu\right|^2 \underset{\uparrow C.S.}{\leq} \left(\int_X 1^2 d\nu\right)\left(\int_X |h|^2 d\nu\right)$
  $= \nu(X) \cdot \|h\|^2_{\mathcal{L}_2(\alpha)}$.

$\Rightarrow \exists g \in \mathcal{L}_2(X, \alpha)$ s.t. $\forall h \in \mathcal{L}_2(X, \alpha)$ $\varphi(h) = \int_X h \, d\nu = \int_X h \cdot g \, d\alpha$

$\Rightarrow \forall A \in \mathcal{A}$: $\boxed{\nu(A) = \int_X \chi_A \, d\nu = \varphi(\chi_A) = \int_X \chi_A \cdot g \, d\alpha}$ $\int_A g \, d\nu + \int_A g \, d\mu$

$\Rightarrow g \geq 0$ α-a.e (take $A = \{x \mid g(x) < \infty\}$)

let $N := \{x \in X \mid g(x) \geq 1\}$

$\Rightarrow \nu(N) = \int_A g \, d\alpha \geq \alpha(N) = \mu(N) + \nu(N)$

$\Rightarrow \mu(N) = 0$

let $f = \frac{g}{1-g} \cdot \chi_{X \setminus N}$

$\Rightarrow \forall A \in \mathcal{A}$:

$\mu = \alpha - \nu$
$= (1-g)\,d\alpha$

$\int_A f \, d\mu = \int_X \frac{g}{1-g} \chi_A \chi_{X \setminus N} \, d\mu = \int_X \chi_{A \cap (X \setminus N)} \, g \, \underbrace{\frac{\chi_{X\setminus N}}{1-g} d\mu}_{= \chi_{X \setminus N} d\alpha}$

$= \int_X \chi_{A \cap (X \setminus N)} \underbrace{g \, d\alpha}_{= d\nu} = \int_X \chi_{A \cap (X \setminus N)} \, d\nu = \nu(A \cap (X \setminus N))$

$\Rightarrow \nu(A) = \nu(A \cap N) + \nu(A \cap (X \setminus N)) = \nu(A \cap N) + \int_A f \, d\mu$.

Uniqueness of f: Assume $\exists g$ mb. s.t. $(*)$ holds for all $A \in \mathcal{A}$ with g

$\Rightarrow \forall A \in \mathcal{A}$: $\int_A f - g = 0 \Rightarrow (f-g)_+, (f-g)_- = 0$ μ-a.e.

(take $A_+ = (f-g)^{-1}((0,\infty))$, $A_- = X \setminus A_+$)

Case 2. μ, ν σ-finite.

Choose $(X_n)_n, (Y_n)_n \subseteq X$ s.t. $X = \bigcup_{n=1}^\infty X_n = \bigcup_{n=1}^\infty Y_n$ and $\mu(X_n) < \infty$, $\nu(Y_n) < \infty$.
Without restr. $X_1 \subseteq X_2 \subseteq \dots$, $Y_1 \subseteq Y_2 \subseteq \dots$

$\Rightarrow X = \bigcup_{n=1}^\infty (X_n \cap Y_n)$ and $\alpha(X_n \cap Y_n) < \infty$.

$\rightsquigarrow$ Apply Case 1 and obtain sets N_n with $\mu(N_n) = 0$ and fct's f_n
with $\nu(A) = \nu(A \cap N_n) + \int_A f_n \, d\mu$ $\quad (A \in \mathcal{A}, A \subseteq X_n \cap Y_n)$.

Note: $f_{n+1}|_{X_n} = f_n$ μ-a.e.

$\Rightarrow \exists f$ mb s.t. $f|_{X_n} = f_n$ μ-a.e.

And let $N = \bigcup_{n=1}^\infty N_n$

$\Rightarrow \mu(N) = 0$ and

$\nu(A \cap (X \setminus N)) = \lim_{n\to\infty} \nu(A \cap (X_n \setminus N))$

$= \lim_{n \to \infty} \int_{A \cap X_n} f \, d\mu = \int_A f \, d\mu$ $\quad \square$.

Theorem (Lebesgue decomposition)

μ, ν σ-finite measures on a σ-algebra $\mathcal{A}$.

$\Rightarrow \exists$ measures ϱ, σ s.t. $\nu = \varrho + \sigma$, $\varrho \ll \mu$, $\sigma \perp \varrho$.
also: $\sigma \perp \mu$

Proof. let $\sigma(A) := \nu(A \cap N)$ $\quad (A \in \mathcal{A})$,
$\varrho := f \odot \nu$ $\quad$ (f as in the thm before)

$\Rightarrow \nu = \sigma + \varrho$, $\varrho \ll \mu$ and $\sigma \perp \varrho$ because $\sigma(X \setminus N) = 0$,
$\varrho(N) = \int_N f \, d\mu = 0$ because $\mu(N) = 0$.

Uniqueness. $\nu = \varrho + \sigma = \varrho' + \sigma'$. and $N' \in \mathcal{A}$ s.t. $\mu(N') = 0$, $\sigma'(X \setminus N') = 0$

$\Rightarrow \forall A \in \mathcal{A}$:

$\varrho(A) = \varrho(A \setminus (N \cup N')) + \underbrace{\varrho(N \cup N')}_{= 0 \text{ because } \varrho \ll \mu}$

$= \nu(A \setminus (N \cup N')) - \underbrace{\sigma(A \setminus (N \cup N'))}_{= 0}$

$= \nu(A \setminus (N \cup N'))$

Analogously: $\varrho'(A) = \nu(A \setminus (N \cup N'))$ $\quad \Rightarrow \varrho(A) = \varrho'(A)$.

$$\sigma(A) = \sigma(A\cap N) + \overbrace{\sigma(A\setminus N)}^{=0}$$
$$= \sigma(A\cap N) + \overbrace{\varsigma(A\cap N)}^{=0} = \nu(A\cap N) = \nu(A\cap N\cap N') + \overbrace{\nu((A\cap N)\setminus N')}^{=0\ (*)}$$
$$= \nu(A\cap N\cap N')$$

$(*)$: $\nu((A\cap N)\setminus N') = \varsigma'((A\cap N)\setminus N') + \sigma'((A\cap N)\setminus N')$
$\overset{\varsigma=\varsigma'}{=} \varsigma((A\cap N)\setminus N') + 0 = 0.$

Analogously: $\sigma'(A) = \nu(A\cap N\cap N')$.

□

Theorem. (Radon-Nikodym)

μ, ν σ-finite measures on a σ-algebra $\mathfrak{A}$.

Then: $\nu \ll \mu \iff \exists f \geq 0$, mb. s.t. $\nu = f\odot\mu$.

f is determined uniquely by μ and ν ~~as~~ μ-a.e.

Notation: $f =: \frac{d\nu}{d\mu}$ = Radon-Nikodym derivative of ν wrt μ.

Proof. "$\Leftarrow$" ✓

"$\Rightarrow$" Let ς, σ as in the Lebesgue decomp. and N as in the first thm.

$\Rightarrow \nu(N) = 0$ (because $\mu(N)=0$, ~~as~~ $\nu\ll\mu$)

$\Rightarrow \forall A: \sigma(A) = \nu(A\cap N) = 0$.

$\Rightarrow \nu = \varsigma \ll \mu$.

□

Examples.

- β Lebesgue measure on $\mathbb{R}$;
  ν = Dirac measure on $\mathbb{R}$, $\nu(A) := \begin{cases} 1, & 0\in A \\ 0, & 0\notin A\end{cases}$ $(A\in\mathfrak{A})$

  $\Rightarrow \nu \perp \beta$ because: $\nu(\mathbb{R}\setminus\{0\}) = 0 = \beta(\{0\})$.

- β Lebesgue measure on ~~$\mathbb{R}$~~ $[0,1]$.
  ψ = Cantor function ... see p.122

Remarks

- Radon-Nikodym and Lebesgue decomposition also true if μ σ-finite measure, ν complex measure.

- Radon-Nikodym is <u>not</u> true for arbitrary measures μ.

  Examples

  ① $X = \emptyset$, $\mathfrak{A} = \{X, \emptyset\}$. $\mu(\emptyset) = 0 = \nu(\emptyset)$, $\mu(X) = \infty$, $\nu(X) = 1$
  $\Rightarrow \mu, \nu$ measures on $\mathfrak{A}$, $\nu\ll\mu$, but ν has no μ-density.
  (also: $\mu \ll \nu$)

  ② $X = [0,1]$, μ counting measure, β = Borel measure on $[0,1]$.
  $\Rightarrow \beta\ll\mu$, but β has no ν-density.

  Proof. Assume $\exists f\in\mathcal{L}_1(\mu)$ s.t. $\forall A\in\mathfrak{A}$ $\beta(A) = \int_A f\,d\mu$.
  Assume $\exists x_0 \in [0,1]$ with $f(x_0)\neq 0$.
  $\Rightarrow \beta(\{x_0\}) = \int_{\{x_0\}} f\,d\mu = f(x_0) \neq 0$ ↯
  $\Rightarrow f\equiv 0 \Rightarrow \beta = 0$ ↯.

  □

Corollary 6.11. μ σ-finite, $\nu\ll\mu$, $g\in\mathcal{L}_1(\nu)$.
$\Rightarrow \int g\,d\nu = \int g f\,d\mu$ where $\nu = f\odot\mu$ $(*)$

Proof. $(*)$ is true for characteristic functions, hence, by linearity, for simple functions.
The monotone convergence theorem shows then, that $(*)$ is true for all $g\in\mathcal{L}_1(\nu)$, with $g\geq 0$. Again, by linearity, $(*)$ holds for all $g\in\mathcal{L}_1(\nu)$.

□

Examples for complex measures:

$\phi: [a,b] \to \mathbb{C}$ continuous from the left.
For $a\leq\alpha<\beta\leq b$ define $\mu_\phi([\alpha,\beta)) := \phi(\beta)-\phi(\alpha)$
and $\mu_\phi\left(\dot{\bigcup}_{j=1}^{\infty}[\alpha_j,\beta_j)\right) = \sum_{j=1}^{\infty}\mu_\phi([\alpha_j,\beta_j))$
In addition assume that ϕ is of bounded variation
$\Rightarrow \mu_\phi$ is σ-additive on $\mathcal{J}([a,b))$
$\Rightarrow \mu_\phi$ can be extended uniquely to a complex measure on $\mathcal{B}([a,b))$.
(Jordan decomposition & Hahn's extension theorem; uniqueness by Hahn's decomposition & extension theorems)

$\Rightarrow \boxed{|\mu_\phi|(A) = \int_A |\phi|\,d\beta}$ (β = Borel-~~Lebesgue~~ measure on $[a,b)$).
(see Teschl, Lemma 9.17)

6.3. Derivatives of measures

Definition 6.12. μ complex Borel measure on $\mathbb{R}^s$; $x \in \mathbb{R}^s$

$$(\overline{D}\mu)(x) := \limsup_{\varepsilon \searrow 0} \frac{\mu(B_\varepsilon(x))}{|B_\varepsilon(x)|} =: \text{upper derivative of } \mu \text{ in } x$$

$$:= \lim_{\varepsilon \to 0} \left(\sup \left\{ \frac{\mu(B_\delta(x))}{|B_\delta(x)|} \;\middle|\; 0 < \delta < \varepsilon \right\} \right)$$

$$(\underline{D}\mu)(x) := \liminf_{\varepsilon \searrow 0} \frac{\mu(B_\varepsilon(x))}{|B_\varepsilon(x)|} =: \text{lower} \quad "$$

$$(D\mu)(x) := \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{|B_\varepsilon(x)|} =: \text{derivative of } \mu \text{ in } x, \text{ if the limit ex.}$$

where we use the notation $B_\varepsilon(x_0) := \{x \in \mathbb{R}^s \mid \|x - x_0\| < \varepsilon\}$,

$|A| := \lambda^s(A)$, $A \in \mathcal{L}^s$ = Lebesgue σ-algebra on $\mathbb{R}^s$.

Remark. $\left.\begin{array}{l} \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\pm\infty\},\; x \longmapsto (\overline{D}\mu)(x) \\ \qquad\qquad\qquad\qquad\;\; x \longmapsto (\underline{D}\mu)(x) \end{array}\right\}$ are measurable.

Idea. Show: $x \longmapsto \mu(B_\varepsilon(x))$, $x \longmapsto |B_\varepsilon(x)|$ are cont.

$\Rightarrow x \longmapsto \sup\left\{ \frac{\mu(B_\delta(x))}{|B_\delta(x)|} \;\middle|\; 0 < \delta < \varepsilon \right\}$ is lower semi cont; hence measurable.

$\Rightarrow x \longmapsto \lim_{n \to \infty} \sup\{ \dots \mid 0 < \delta < \frac{1}{n}\}$ is measurable (limit of mb. functions)

Lemma 6.13. (Wiener covering lemma).

$B_1, \dots, B_n$ open balls in $\mathbb{R}^s$ with $B_j = B_{r_j}(x_j)$.

$\Rightarrow \exists\, j_1, \dots, j_m \in \{1, \dots, n\}$ st. B_{j_k} pairwise disjoint and

$$\left| \bigcup_{m=1}^{n} B_j \right| \leq 3^s \sum_{k=1}^{m} |B_{j_k}|.$$

Proof. Without restriction $r_1 \geq r_2 \geq \dots \geq r_n$.

Take $j_1 = 1$ and remove all balls from the collection that intersect B_1.

$\Rightarrow$ all removed balls are contained in $B_{3r_1}(x_1)$, hence:

$$\left| \bigcup_{\substack{\text{removed}\\\text{balls}}} B \right| \leq |B_{3r_1}(x_1)| = 3^s |B_1|.$$

Repeat this with the remaining balls. The process ends after finite steps, because there are only finitely many balls.

$$\Rightarrow \left| \bigcup_{j=1}^{n} B_j \right| = \left| \bigcup_{k=1}^{m} \bigcup_{\substack{\text{balls removed}\\\text{in the } k\text{-th step}}} B \right| \leq \sum_{k=1}^{m} 3^s |B_{j_k}|.$$

$\square$

Lemma 6.14. μ complex Borel measure on $\mathbb{R}^s$, $\alpha > 0$, $A \in \mathcal{B}^s$, $\mu(A) < \infty$.

$\Rightarrow$ i) $|\{x \in A \mid (\overline{D}\mu)(x) > \alpha\}| \leq 3^s \frac{\mu(A)}{\alpha}$

ii) $|\{x \in A \mid (\overline{D}\mu)(x) > 0\}| = 0$ if $\mu(A) = 0$.

Proof. Define $A_\alpha := \{x \in A \mid (\overline{D}\mu)(x) > \alpha\}$.

i) Since λ^s is regular, it suffices to show:

$$|K| \leq 3^s \frac{\mu(A)}{\alpha} \quad \text{for every } K \subseteq A_\alpha \text{ cpt.}$$

Note that μ is finite (because a complex measure) $\Rightarrow \mu$ is regular (*)

$\Rightarrow$ For $\varepsilon > 0$ $\exists\, U_\varepsilon$ open, st. $U_\varepsilon \supseteq A$ and $\mu(U) \leq \mu(A) + \varepsilon$ (note: $\mu(A) < \infty$)

Now let $K \subseteq A_\alpha$, cpt.

$\Rightarrow \forall x \in K\; \exists\, r_x > 0$ st. $\frac{\mu(B_{r_x}(x))}{|B_{r_x}(x)|} > \alpha$ (*) and $B_{r_x}(x) \subseteq U$.

K cpt $\Rightarrow$ finitely many of the open balls $B_{r_x}(x)$ cover K, and from these we can choose a finite disjoint collection according to Lemma 6.13.

$$\Rightarrow \cancel{\mu(K)}\; |K| \leq 3^s \sum_{k=1}^{n} |B_{r_x}(x_k)| \overset{(*)}{\leq} \frac{3^s}{\alpha} \sum_{k=1}^{n} \mu(B_{r_x}(x_k))$$

$$= \frac{3^s}{\alpha} \mu\Big(\underbrace{\textstyle\bigcup B_{r_k}(x_k)}_{\subseteq U}\Big) \leq \frac{3^s}{\alpha} \mu(U) < \frac{3^s}{\alpha}(\mu(A) + \varepsilon)$$

let $\varepsilon \to 0 \Rightarrow |K| \leq \frac{3^s}{\alpha} \mu(A)$.

ii) If $\mu(A) = 0$, then:

$$|A_0| = \left| \bigcup_{j=1}^{\infty} A_{\frac{1}{j}} \right| \leq \sum_{j=1}^{\infty} |A_{\frac{1}{j}}| \leq \sum_{j=1}^{\infty} \frac{3^s}{1/j} \cdot \underbrace{\mu(A)}_{=0} = 0$$

$\square$

(*) In $\mathbb{R}^s$: Every open set is countable union of open balls and every open ball is ~~count~~ countable union compact sets $\Rightarrow$ Every open set is countable union of compact sets.

$\Rightarrow$ Every open set is inner regular.

$\rightsquigarrow$ It can be shown that $R := \{A \subseteq \mathcal{B}^s \mid A \text{ is } \mu \text{ regular}\}$ is a σ-ring.

Since all open sets belong to R $\rightsquigarrow$ $\mathcal{B}^s \subseteq R \Rightarrow \mathcal{B}^s = R$.

$\forall \varepsilon > 0\; \exists\, U$ open, K cpt. st $K \subseteq A \subseteq U$, $\mu(U \setminus K) < \varepsilon$.

Theorem 6.15 $f \in \mathcal{L}^1(\mathbb{R}^s)$ (Should also work for $f \in \mathcal{L}^1_{loc}(\mathbb{R}^s)$)

$\Rightarrow \lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f - f(x)|\, d\lambda^s = 0$ for λ-a.e. $x\in\mathbb{R}^s$. (*)

A point $x \in \mathbb{R}^s$ which satisfies (*) is called a Lebesgue point of f.

Proof Define

$(D_r f)(x) := \frac{1}{|B_r(x)|}\int_{B_r(x)} |f-f(x)|\, d\lambda^s$.

$(Df)(x) := \limsup_{r\to0} (D_r f)(x)$.

Fix $\varepsilon>0$.

Choose $g\in\mathcal{L}_1(\mathbb{R}^s)$, $h \in C(\mathbb{R})$ s.t. $f = g+h$, $\|g\|_1 < \varepsilon$ (possible because $C(\mathbb{R}^s)\cap\mathcal{L}_1(\mathbb{R}^s) \subseteq \mathcal{L}_1(\mathbb{R})$ dense)

h cont. $\Rightarrow (Dh)(x) = 0$

$$\Rightarrow (Df)(x) \le (Dg)(x) + \overbrace{(Dh)(x)}^{=0}$$
$$= \limsup_{r\to0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |g - g(x)|\, d\lambda^s$$
$$\le \underbrace{\limsup_{r\to0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |g|\, d\lambda^s}_{=(D\mu)(x)} + |g(x)|.$$

where $\mu = g\odot\lambda$ (finite measure because $g\in\mathcal{L}_1$)

$\Rightarrow \{x \in\mathbb{R}^s \mid (Df)(x) \ge 2\delta\} \subseteq \{x\in\mathbb{R}^s \mid (D\mu)(x)\ge\delta\}$ ①
$\cup \{x\in\mathbb{R}^s \mid |g(x)|\ge\delta\}$. ②

①: $|①| \le \frac{3^s}{\delta}\cdot\mu(\mathbb{R}^s) = \frac{3^s}{\delta}\cdot\|g\|_1 = \frac{3^s}{\delta}\cdot\varepsilon$.

②: $|\{x\in\mathbb{R}^s \mid |g(x)|\ge\delta\}| = \int_{\{g\ge\delta\}} d\lambda^s \le \int_{\{g\ge\delta\}} \frac{|g(x)|}{\delta}\, d\lambda^s$
$\le \frac{1}{\delta}\int_{\mathbb{R}^s}|g(x)|\,d\lambda^s = \frac{\|g\|_1}{\delta} = \varepsilon/\delta$

$\Rightarrow |\{x\in\mathbb{R}^s \mid (Df)(x)\ge 2\delta\}| < \varepsilon\left(\frac{3^s}{\delta}+1\right)$

Since ε arbitrary $\Rightarrow |\{..\}| = 0$.

$\Rightarrow |\{x\in\mathbb{R}^s \mid Df(x) \ne 0\}| = |\bigcup_n \{x\in\mathbb{R}^s \mid Df(x)\ge \frac1n\}| = 0$. □

Definition 6.16 $x\in\mathbb{R}^n$, $(A_n)_n$ sequence of sets in $\mathbb{R}^n$. $(A_n)_n$ shrinks nicely to x

$:\Leftrightarrow \exists (r_n)_n \subseteq (0,\infty)$ s.t. $r_n\to0$ and $\exists \varepsilon>0$ st.

$A_n \subseteq B_{r_n}(x)$ and $|A_n| \ge \varepsilon|B_{r_n}|$ $(n\in\mathbb{N})$.

Examples. $A_n := (0,\frac1n)\times(0,\frac2n)$ shrink nicely to $(0,0)\in\mathbb{R}^2$

$A_n := (0,\frac1n)\times(0,\frac{2}{n^2})$ does not.

Lemma 6.17 $f\in\mathcal{L}_1(\mathbb{R}^s)$, x Lebesgue point of f and $(A_n)_n$ shrinks nicely to x

$\Rightarrow f(x) = \lim_{n\to\infty}\frac{1}{|A_n|}\int_{A_n} f\, d\lambda$.

Proof Let $\varepsilon>0$ and B_n as in the def. of nicely shrinking.

$$\Rightarrow \left|f(x) - \frac{1}{|A_n|}\int_{A_n} f\, d\lambda^s\right| = \frac{1}{|A_n|}\left|\int_{A_n} f(x) - f\, d\lambda^s\right|$$
$$\le \frac{1}{|A_n|}\int_{A_n}|f(x)-f|\,d\lambda^s \le \frac{1}{\varepsilon|B_n|}\int_{B_n}|f(x)-f|\,d\lambda^s$$
$$\xrightarrow{n\to\infty} 0 \quad \text{(Thm. 6.16)}$$
□

Corollary 6.18. $f\in\mathcal{L}_1(\mathbb{R})$ and $F(x) = \int_{-\infty}^x f\, d\lambda$.

$\Rightarrow F$ diff at every Lebesgue point of f (that is, λ-a.e.), and $F'(x) = f(x)$.

Proof. Let x be a Lebesgue point of f and $A_n := [x, x+\frac1n]$, $B_n := [x-\frac1n, x]$

$\rightarrow A_n$ and B_n shrink nicely to x, and by Lemma 6.17:

$$f(x) = \lim_{n\to\infty} \frac{1}{1/n}\int_{A_n} f\, d\lambda = \lim_{n\to\infty} \frac{1}{1/n}\left(F(x+\tfrac1n) - F(x)\right)$$
$$= \dots = \lim_{n\to\infty} \frac{1}{1/n}\left(-F(x-\tfrac1n)+F(x)\right)$$

$\Rightarrow$ left and right derivative exist and are equal in x

$\Rightarrow F$ diff in x and $F'(x) = f(x)$ □

Remark

Corollary 6.18 applies, if, e.g. μ is a complex Borel measure with $\mu \ll \lambda$.

$\rightsquigarrow \mu = f\odot\lambda$ for some $f\in\mathcal{L}_1(\mathbb{R})$

and $F =$ distribution fct of μ, that is $F(x) = \begin{cases} \mu([0,x)), & x>0 \\ 0, & x=0 \\ -\mu([x,0)), & x<0. \end{cases}$

(Note: μ measure $\Rightarrow F$ left cont).

Theorems without proof:

Theorem (Rudin, Thm. 7.13). $x \in \mathbb{R}^s$, $(A_n)_n$ nicely shrinking to x.
μ complex Borel measure, $\mu \perp \lambda$

$$\Rightarrow \lim_{n\to\infty} \frac{\mu(A_n)}{\lambda(A_n)} = 0 \qquad \lambda\text{-a.e.}$$

Theorem (Rudin, Thm. 7.14)
$x \in \mathbb{R}^s$, $(A_n)_n$ nicely shrinking to x.
μ complex Borel measure on $\mathbb{R}^s$ with Lebesgue decomposition $\mu = f \odot \lambda + \mu_s$ ($\mu_s \perp \lambda$).

$$\Rightarrow \lim_{n\to\infty} \frac{\mu(A_n)}{\lambda(A_n)} = f(x) \qquad \lambda\text{-a.e.}$$

In particular, $\mu \perp m \Rightarrow (D\mu)(x) = 0$ λ-a.e.

Theorem (Rudin, Thm. 7.15)
μ pos. Borel measure on $\mathbb{R}^s$, $\mu \perp m$.

$$\Rightarrow (D\mu)(x) = 0 \qquad \mu\text{-a.e.}$$

6.4. The fundamental theorem of calculus.

Recall from Analysis 1:

$f: [a,b] \to \mathbb{R}$ cont., $F: [a,b] \to \mathbb{R}$, $F(x) := \int_a^x f(t)\,dt$

$\Rightarrow F$ differentiable and $F'(x) = f(x)$.

Questions:

- $f \in \mathcal{L}_1 \Rightarrow F$ differentiable? $\rightsquigarrow$ F differentiable λ-a.e. (Cor. 6.18)
  (Note: the proof uses that F' can be approximated by cont. fct's in $\|\cdot\|_1$)
- F continuous, and diff. a.e. $\Rightarrow$ $F' \in \mathcal{L}_1$?
  $F(x) - F(a) = \int_a^x F'(t)\,dt$?

No! Extra condition "F abs. continuous" is needed!

Examples:

① $F: [0,1] \to \mathbb{R}$, $F(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0. \end{cases}$

$\to F$ continuous and everywhere diff'able $\left(F'(x) = \begin{cases} 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos(\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0 \end{cases} \right)$

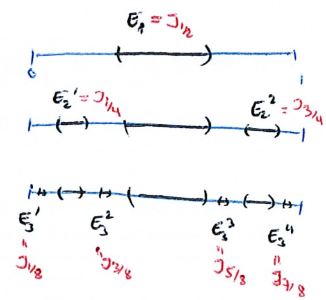
but $F' \notin \mathcal{L}_1([0,1])$ because $\int_{[0,1]} |F'(t)|\,dt = \infty$.

② Cantor function.

Recall: $C :=$ Cantor set is cpt, uncountable.

$$C' := [0,1] \setminus C = \bigcup_{n\in\mathbb{N}} \bigcup_{k=1}^{2^{n-1}} E_n^k = \bigcup_{n\in\mathbb{N}} \bigcup_{k=1}^{2^{n-1}} J_{\frac{2k-1}{2^n}}$$

where $E_n^k = J_{\frac{2k-1}{2^n}}$ = k-th open interval from the left of length 3^{-n}



$\lambda(C') = 1$, $\lambda(C) = 0$

Define for $x \in E_n^k$: $\psi_0(x) := \frac{2k-1}{2^n}$ ← ψ_0 is increasing!
Note also: $\psi_0|_{E_n^k}$ = arithmetic mean of ψ on neighbouring "black" intervals.

Define $\psi: [0,1] \to \mathbb{R}$ by:

$\psi(0) := 0$, $\psi(x) := \sup\{\psi_0(t) \mid t \leq x\}$

$\to \psi(1) = 1$, $\psi(x) = \psi_0(x)$ for $x \in C'$ and ψ is increasing.