Application. (Construction of the Borel measure on $\mathbb{R}^n$)

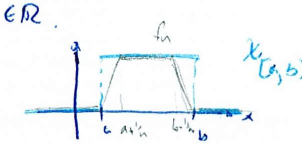
The Riemann integral gives a well defined pos. linear form on $C_c(X)$:

$$J: C_c(\mathbb{R}) \to \mathbb{R} \qquad f \mapsto \int_{-\infty}^{\infty} f(x)\,dx$$

By the Riesz representation theorem exists a unique Radon measure μ on $\mathcal{B}(\mathbb{R})$ s.t.
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{\mathbb{R}} f\,d\mu, \qquad f \in C_c(X)$$

To show that $\mu = \beta$ = Borel measure it suffices to show that $\mu([a,b)) = \beta([a,b)) = b-a$ for all $a<b \in \mathbb{R}$.

To this end, fix $a<b \in \mathbb{R}$



$\Rightarrow \exists (f_n)_n \subseteq C_c(X)$ s.t.

$$0 \le f_n \le \chi_{[a,b)}, \quad f_n \nearrow \chi_{[a,b)} \quad \& \quad \int_{-\infty}^{\infty} \chi_{[a,b)}(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

$$\Rightarrow \mu([a,b)) = \int_{\mathbb{R}} \chi_{[a,b)}\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,d\mu$$
$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} \chi_{[a,b)}(x)\,dx = b-a.$$

$\square$

6. Representation of measures and the Radon-Nikodym theorem.

6.1. Decomposition theorems.

Definition 6.1. $\mathcal{A}$ σ-algebra, $\nu: \mathcal{A} \to \mathbb{R}$ is called a *signed measure* if

i) $\nu(\emptyset) = 0$

ii) $\nu\left(\dot{\bigcup}_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \nu(A_j)$ for $A_j \in \mathcal{A}$, pairwise disjoint.

$\nu: \mathcal{A} \to \mathbb{C}$ is called a *complex measure* if i) and ii) hold.

Observations.

i) The series in (ii) is abs. convergent because it does not depend on the order of summation (l.h.s. does not depend on ordering of the A_j's)

ii) Sometimes: a signed measure is allowed to have values in $[-\infty,\infty)$ or $(-\infty,\infty]$.

For us: a signed/complex measure ~~take~~ is finite valued.

Lemma 6.2 μ signed or complex measure on a σ-algebra $\mathcal{A}$.

$\Rightarrow \exists M \in \mathbb{R}$ s.t. $\forall A \in \mathcal{A}$ $|\mu(A)| < M$.

Proof. Assume μ is not finite $\Rightarrow$ Either real or imaginary part of μ is not bdd.
Without restriction: real part of μ is not bdd from above.

$\Rightarrow \exists (A_n)_n \in \mathcal{A}$ and $0 < a_1 < b_1 < a_2 < b_2 < \dots$
with $a_n > b_{n-1} + \dots + b_2 + b_1 + 1$
and $a_n < \operatorname{Re}\mu(A_n) < b_n$.

Define $B_n := A_n \setminus (A_1 \cup \dots \cup A_{n-1})$

$$\Rightarrow \operatorname{Re}(\mu(B_n)) \ge \operatorname{Re}(\mu(A_n)) - \sum_{j=1}^{n-1} \operatorname{Re}(\mu(A_j)) \ge a_n - \sum_{j=1}^{n-1} b_j \ge 1.$$

$$\Rightarrow \operatorname{Re}\left(\mu\left(\dot{\bigcup}_{n=1}^{\infty} B_n\right)\right) \ge \sum_{n=1}^{\infty} 1 = \infty$$

B_n's pairwise disjoint.

On the other hand

$$\operatorname{Re}\Big(\mu\big(\underbrace{\textstyle\bigcup_{n=1}^{\infty} B_n}_{\in \mathcal{A}}\big)\Big) < \infty$$

Observation. As for measures, one can prove for $(A_j)_j \subseteq \mathcal{A}$:

i) $A_1 \subseteq A_2 \subseteq \dots \implies \mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n)$

ii) $A_1 \supseteq A_2 \supseteq \dots \implies \mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n)$.

Example. $(X, \mathcal{A}, \mu)$ measure space. $f \in \mathcal{L}_1(X, \mathbb{R})$ or $f \in \mathcal{L}_1(X, \mathbb{C})$.

$\implies \nu : \mathcal{A} \to \mathbb{R}$ (or $\mathbb{C}$), $\nu(A) := \int_A f \, d\mu$

is a signed (or complex) measure on X.

ν is called the signed (or complex) measure on X with density f.

Notation: $\nu = f\,d\mu$ or $\nu = f(\cdot)\,d\mu$

Definition 6.3. $\mu : \mathcal{A} \to \mathbb{R}$ signed measure, $A \in \mathcal{A}$ is called

i) positive (ν-positive) $:\iff \forall B \in \mathcal{A}$ with $B \subseteq A$: $\nu(B) \geq 0$

ii) negative (ν-negative) $:\iff \forall B \in \mathcal{A}$ with $B \subseteq A$: $\nu(B) \leq 0$

iii) ν-zero set $:\iff \forall B \in \mathcal{A}$ with $B \subseteq A$: $\nu(B) = 0$.

Lemma 6.4. $\nu : \mathcal{A} \to \mathbb{R}$ signed measure; $A \in \mathcal{A}$.

$\implies \exists B \in \mathcal{A}$ s.t. $B \subseteq A$, B is positive and $\nu(B) \geq \nu(A)$.

Proof. It suffices to show:

$$(*) \begin{cases} \forall n \in \mathbb{N} \ \exists B_n \subseteq A,\ B_n \in \mathcal{A} \text{ s.t. } \nu(B_n) \geq \nu(A) \text{ and } \forall C \in \mathcal{A} \text{ with } C \subseteq B_n: \\ \nu(C) \geq -\frac{1}{n} \end{cases}$$

Then obviously we can choose the B_n such that $B_1 \supseteq B_2 \supseteq \dots$

Then $B := \bigcap_{j=1}^{\infty} B_j$ is positive and

$\nu(B) = \nu\left(\bigcap_{j=1}^{\infty} B_j\right) = \lim_{n\to\infty} \nu(B_n) \geq \nu(A)$.

Proof of $(*)$: Let $n \in \mathbb{N}$ and suppose no B_n as in $(*)$ exists.

$\implies$ Every $B' \in \mathcal{A}$ with $B' \subseteq A$ and $\nu(B') \geq \nu(A)$ contains some $D \in \mathcal{A}$ s.t. $\nu(D) \leq -\frac{1}{n}$.

Inductively we can construct a sequence $(D_n)_n \subseteq \mathcal{A}$ with

$D_1 \subseteq A$, $D_n \subseteq A \setminus (D_1 \cup \dots \cup D_{n-1})$ $(n \geq 2)$ and $\nu(D_n) \leq -\frac{1}{n}$

(Start with $B' = A \leadsto D_1$

$\implies A \setminus D_1 \subseteq A$ and $\nu(A \setminus D_1) = \nu(A) - \nu(D_1) \geq \nu(A) \dots$)

The D_n's are pairwise disjoint.

$\implies \nu\left(\bigcup_{j=1}^{\infty} D_j\right) = \sum_{j=1}^{\infty} \nu(D_j) = -\infty$ ↯ (Lemma 6.2). □

Theorem 6.5 (Hahn's decomposition theorem)

$\nu : \mathcal{A} \to \mathbb{R}$ signed measure.

$\implies \exists P, N \in \mathcal{A}$, P positive, N negative such that $X = P \cup N$, $P \cap N = \emptyset$.

The decomposition is not unique, but: Let $X = P' \cup N'$ be another decomposition. Then: $P \triangle P' = N \triangle N'$ is a ν-zero set.

Proof. Let $\alpha := \sup\{\nu(A) \mid A \in \mathcal{A}\} < \infty$ (Lemma 6.2)

By Lemma 6.4. $\exists$ $(P_n)_n \subseteq \mathcal{A}$, P_n positive, s.t. $\nu(P_n) \to \alpha$.

$P := \bigcup_{n=1}^{\infty} P_n$ $(\in \mathcal{A}!)$

$\implies P$ positive and $\nu(P) \geq \nu(P_n)$ $(n \in \mathbb{N})$ $\implies \nu(P) = \alpha$.

$N := X \setminus P$. $\implies N$ negative. Because: Assume $\exists C \in \mathcal{A}$ with $C \subseteq N$ and $\nu(C) > 0 \implies \nu(P \cup C) = \nu(P) + \nu(C) > \alpha$ ↯.

Let $X = P' \cup N'$ another decomp. with P' pos, N' neg.

For $B \in \mathcal{A}$ with ~~$B \in P \triangle P'$~~ $B \subseteq P \setminus P' = P \cap (X \setminus P') = P \cap N'$

$\implies \begin{cases} \nu(B) \geq 0 & \text{because } B \subseteq P \\ \nu(B) \leq 0 & \text{because } B \subseteq N' \end{cases} \implies \nu(B) = 0$

$\implies P \setminus P'$ ν-zero set.

Analogously: $P' \setminus P$ ν-zero set.

$\implies P \triangle P' = (P' \setminus P) \cup (P \setminus P')$ is a ν-zero set. □

Definition 6.6. $\nu : \mathcal{A} \to \mathbb{R}$ signed measure; $X = P \cup N$ as in the Hahn decomp. theorem. Then:

$\nu^+ : \mathcal{A} \to \mathbb{R}$, $\nu^+(A) := \nu(A \cap P) \geq 0$ positive variation

$\nu^- : \mathcal{A} \to \mathbb{R}$, $\nu^-(A) := -\nu(A \cap N) \geq 0$ negative variation

$|\nu| : \mathcal{A} \to \mathbb{R}$, $|\nu|(A) := \nu^+(A) + \nu^-(A)$ variation of μ.

Observe:
- $\nu^\pm$, $|\nu|$ do not depend on the choice of P and N.
- $\nu^\pm$ are measures on $\mathcal{A}$
- $\nu = \nu^+ - \nu^-$
- $|\nu| = \nu^+ + \nu^-$.

Theorem 6.7. $\nu: \mathcal{A} \to \mathbb{R}$ signed measure, $A \in \mathcal{A}$.

$\Rightarrow$ i) $\nu^+(A) = \sup\{\nu(B) \mid B \in \mathcal{A}, B \subseteq A\}$

ii) $\nu^-(A) = -\inf\{\nu(B) \mid B \in \mathcal{A}, B \subseteq A\}$

iii) $|\nu|(A) = \sup\{\sum_{j=1}^n |\nu(A_j)| \mid A_1, \dots, A_n \in \mathcal{A}$, disjoint, $A = \bigcup_{j=1}^n A_j\} =: s_1$

$= \sup\{\sum_{j=1}^\infty |\nu(A_j)| \mid (A_j)_j \subseteq \mathcal{A}$ disjoint, $A = \bigcup_{j=1}^\infty A_j\}. =: s_2$

Proof. Let P, N as in the Hahn decomposition theorem $(X = P \mathbin{\dot\cup} N)$

i) Let $B \in \mathcal{A}$ with $B \subseteq A$.

$\Rightarrow \nu(B) = \nu(B \cap P) + \underbrace{\nu(B \cap N)}_{\leq 0} \leq \nu(B \cap P) \leq \nu(A \cap P) = \nu^+(A)$.

$\Rightarrow \sup\{\nu(B) \mid B \in \mathcal{A}, B \subseteq A\} \leq \nu^+(A)$.

On the other hand: $\nu^+(A) = \nu(P \cap A) \leq \sup\{\dots\}$.
take $B = A \cap P$.

ii) Analogously, or apply i) to $-\nu$.

iii) $|\nu|(A) = \nu^+(A) + \nu^-(A) = |\nu(A \cap P)| + |\nu(A \cap N)| \leq s_1 \leq s_2$.

Let $A_1, \dots, A_n \in \mathcal{A}$ disjoint with $A \subseteq \bigcup_{j=1}^n A_j$

$\Rightarrow \sum_{j=1}^n \underbrace{|\nu(A_j)|}_{= |\nu^+(A_j) - \nu^-(A_j)|} \leq \sum_{j=1}^n \nu^+(A_j) + \nu^-(A_j) = \nu^+(A) + \nu^-(A) = |\nu|(A)$

$\Rightarrow s_2 = s_1 \leq |\nu|(A)$.

$\square$

Definition 6.8. μ, ν signed measures on a σ-algebra $\mathcal{A}$.

- μ, ν are called mutually singular (μ is ν-singular, ν is μ-singular) if there exist $A, B \in \mathcal{A}$ s.t. A is a μ-zero set, B is a ν-zero set and $A \cap B = \emptyset$, $X = A \mathbin{\dot\cup} B$
  Notation: $\mu \perp \nu$.
- μ is ν-continuous if for all $A \in \mathcal{A}$: A ν-zero set $\Rightarrow$ A is μ-zero set.
  Notation: $\mu \ll \nu$.

Analogous definitions for measures μ, ν:

- $\mu \perp \nu :\Leftrightarrow \exists A, B \in \mathcal{A}$ s.t. $\mu(A) = \nu(B) = 0$, $X = A \mathbin{\dot\cup} B$
- $\mu \ll \nu :\Leftrightarrow \forall A \in \mathcal{A}$ $\nu(A) = 0 \Rightarrow \mu(A) = 0$

Theorem 6.9. (Jordan decomposition theorem).

i) ν signed measure on $\mathcal{A}$.

$\Rightarrow$ there exist measures $\nu^\pm$ on $\mathcal{A}$, s.t. $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$ (*)

If $\nu^\pm$ are as in Definition 6.6, the decomposition is minimal in the following sense:

If $g^\pm : \mathcal{A} \to [0, \infty]$ are measures with $\nu = g^+ - g^-$, then

$g^+ \geq \nu^+$, $g^- \geq \nu^-$.

ii) ν complex measure on $\mathcal{A}$.

$\Rightarrow$ there exist measures $\nu_1, \dots, \nu_4$ s.t. $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$.

Proof. ii) is clear because obviously real and imaginary part of ν are signed measures.

i) Obviously, $\nu^\pm$ from definition 6.6 satisfy (*).

Let $g^\pm$ be measures on $\mathcal{A}$ with $\nu = g^+ - g^-$.

$\Rightarrow \forall A \in \mathcal{A}$: $\nu^+(A) = \nu(A \cap P) = g^+(A \cap P) - \overbrace{g^-(A \cap P)}^{\geq 0}$

$\leq g^+(A \cap P) \leq g^+(A)$

g^+ is a measure, hence isotonic

$\square$

Integration with respect to a complex measure:

Let $\nu = \nu^+ - \nu^- + i\mu^+ - i\mu^-$ and $f \in \mathcal{L}_1(\nu^+) \cap \mathcal{L}_1(\nu^-) \cap \mathcal{L}_1(\mu^+) \cap \mathcal{L}_1(\mu^-)$

$\Rightarrow \int_X f \, d\nu := \int_X f \, d\nu^+ - \int_X f \, d\nu^- + i \int_X f \, d\mu_+ - i \int f \, d\mu_-$.

where $\nu^\pm = (\operatorname{Re} \nu)^\pm$, $\mu^\pm = (\operatorname{Im}(\nu))^\pm$ according to definition 6.6.

6.2. The Radon–Nikodym theorem and Lebesgue decomposition

$(X, \mathcal{A})$ measurable space, $\mu : \mathcal{A} \to [0, \infty]$ measure; $f : X \to [0, \infty]$, $f \in \mathcal{L}_1(X, \mu)$.

Then: $\nu : \mathcal{A} \to [0, \infty]$, $\nu(A) := \int_A f \, d\mu$

is a measure on $\mathcal{A}$, denoted by $\nu = f \odot \mu$. (f = density of ν wrt μ)

Then: $\nu \ll \mu$.

Question. Given measures ν, μ with $\nu \ll \mu$.
Does there exist a $f \in \mathcal{L}_1(\mu)$ s.t. $\nu = f \odot \mu$?

$\to$ Yes! (Radon–Nikodym).

Theorem 6.10: (Radon – Nikodym).
$(X, \mathcal{A}, \mu)$ σ-finite measure space, ν complex measure on X with $\nu \ll \mu$.

$\Rightarrow$ $\exists f \in \mathcal{L}_1(\mu)$ s.t. $\forall A \in \mathcal{A}$ $\nu(A) = \int_A f \, d\mu$. (*)

All functions satisfying (*) are equal μ-a.e.
If ν is a pos. measure, then f can be chosen ≥ 0.

Remark The thm. remains valid if ν is a signed measure with values in $(-\infty, \infty]$ or $[-\infty, \infty)$. The functions in (*) then are quasi-integrable, i.e. at least one of the fct's f_+ or f_- are in $\mathcal{L}_1(\mu)$, not necessarily both. See Elstrodt, VII §2.

Proof. (Widom, Thm. IV.6, p. 65)
Uniqueness. Let $f, g \in \mathcal{L}_1(\mu)$ with (*)

$\Rightarrow$ $\forall A \in \mathcal{A}$ $\int_A f - g \, d\mu = 0$

Let $A_+ := \{x \in X \mid (f-g)(x) \geq 0\}$
$A_- := \{x \in X \mid (f-g)(x) < \infty\}$

$\Rightarrow$ $\int_X (f-g)_+ \, d\mu = \int_{A_+} (f-g) \, d\mu = 0$
$\int_X (f-g)_- \, d\mu = -\int_{A_-} (f-g) \, d\mu = 0$

$\Rightarrow$ $(f-g)_+ = (f-g)_- = 0$ μ-a.e. (Cor. 2.22)

$\Rightarrow$ $f - g = 0$ μ-a.e.

Existence

Case 1 $\mu(X) < \infty$, $\nu \geq 0$.

Observe: If a function f satisfies (*), then: for all mb. g with $g \leq f$:

$\nu(A) \geq \int_A g \, d\mu$ $(A \in \mathcal{A})$ (**)

$\rightsquigarrow$ f should be the "largest" fct satisfying (**).

$\rightsquigarrow$ Define $G = \{ g : X \to [0, \infty]$ mb $\mid \boxed{\int_A g \, d\mu \leq \nu(A)}$ for all $A \in \mathcal{A} \}$.

Goal: Find $f \in G$ s.t. $\int_X f \, d\mu \geq \int_X g \, d\mu$ for all $g \in G$.

Observe: $\forall g \in G$: $\int_X g \, d\mu \leq \nu(X) < \infty$ (ν signed measure $\rightsquigarrow$ finite!)

$\Rightarrow$ $M := \sup\{ \int_X g \, d\mu \mid g \in G \} \leq \nu(X) < \infty$. Show: the sup is a max.

Let $(g_n)_n \subseteq G$ st. $\int_X g \, d\mu \geq M - 1/n$.
and define $f_n := \max\{g_1, \dots, g_n\}$. (pointwise maximum).

Now we show: $\forall n$ $f_n \in G$.

Obviously: $f_1 = g_1 \in G$.
Now for all $A \in \mathcal{A}$: Let $A_+ = \{x \in X \mid g_1(x) \geq g_2(x)\}$, $A_- = \{x \in X \mid g_1(x) < g_2(x)\}$.

$$\int_A f_2 \, d\mu = \int_A \max\{g_1, g_2\} \, d\mu = \int_{A_+} g_1 \, d\mu + \int_{A_-} g_2 \, d\mu$$
$$\leq \nu(A_+) + \nu(A_-) = \nu(A)$$

$\Rightarrow$ $f_2 \in G$.

Note: $f_n = \max\{g_n, f_{n-1}\}$, $n \geq 2$
$\to$ Repeating the argument above shows that $f_n \in G$.

Let $f := \lim_{n\to\infty} f_n$. Obviously $f_n \nearrow f$.

$\Rightarrow$ $\int_A f \, d\mu \underset{\text{mon. conv.}}{=} \lim_{n\to\infty} \int_A f_n \, d\mu \leq \nu(A)$, $A \in \mathcal{A}$.

$\Rightarrow$ $f \in G$.

Now we show: $\forall A \in \mathcal{A}$ $\nu(A) = \int_A f \, d\mu$.
Since $f \in G$, is suffices to show $\nu(A) \leq \int_A f \, d\mu$. $(\overset{*}{**})$

Assume $(\overset{*}{**})$ is not true.

$\Rightarrow$ $\exists A_0 \in \mathcal{A}$, $\varepsilon > 0$ s.t. $\nu(A_0) > \int_{A_0} (f + \varepsilon) \, d\mu$. (Note: $\mu(A_0) < \infty$ because $\mu < \infty$!)

Define the signed measure

$$\varrho : \mathcal{A} \to \mathbb{R}, \quad \varrho(A) := \nu(A) - \int_A (f+\varepsilon)\, d\mu .$$

And let $X = P \,\dot\cup\, N$ a Hahn decomposition with respect to ϱ.

Define $$g : X \to \mathbb{R}, \quad g(x) = \begin{cases} f(x), & x \in N \\ f(x) + \varepsilon, & x \in P \end{cases}$$

Note: $\varrho(P) \geq \varrho(A_0) > 0$ by choice of A.

$$\Rightarrow \int_X g \, d\mu > \int_X f \, d\mu \Rightarrow g \notin G .$$

On the other hand:

$\forall A \in \mathcal{A}$: $$\int_A g\, d\mu = \int_{A\cap N} g\, d\mu + \int_{A \cap P} g \, d\mu = \overbrace{\int_{A\cap N} f\, d\mu}^{\leq \nu(A\cap N) \text{ because } f \in G} + \int_{A\cap P} f + \varepsilon \, d\mu$$
$$\leq \nu(A\cap N) + \nu(A \cap P) - \underbrace{\varrho(A\cap P)}_{\geq 0}$$
$$\leq \nu(A\cap N) + \nu(A\cap P)$$
$$= \nu(A).$$

$\Rightarrow g \in G$. ↯

Case 2. $\mu(X) < \infty$, ν complex measure.

$\Rightarrow$ $\mathrm{Re}(\nu)$, $\mathrm{Im}(\nu)$ signed measures and $\mathrm{Re}\,\nu \ll \mu$, $\mathrm{Im}\,\nu \ll \mu$

By Hahn's decomposition theorem: $(\mathrm{Re}\,\nu)^{\pm} \ll \mu$, $(\mathrm{Im}\,\nu)^{\pm} \ll \mu$.

$\Rightarrow$ Apply case 1 to $(\mathrm{Re}\,\nu)^{\pm}$, $(\mathrm{Im}\,\nu)^{\pm}$.

Case 3. $\mu(X) = \infty$, ν complex measure.

Let $(S_j)_j \subseteq X$, pairwise disjoint, by $\mu(S_j) < \infty$, $X = \bigcup_{j=1}^{\infty} S_j$

$\Rightarrow$ $\forall j$ $\exists \tilde{f}_j \in \mathcal{L}_1(S_j, \mu|_{S_j})$ s.t. $\nu|_{S_j} = \tilde{f}_j \odot \mu|_{S_j}$.

Extend $\tilde{f}_j$ to $f_j \in \mathcal{L}_1(X, \mu)$ by setting $f_j(x) = 0$ for $x \in X \setminus S_j$.

Let $f := \sum_{j=1}^{\infty} f_j$

$f \in \mathcal{L}_1(X, \mu)$ because:

$$\int_X |f|\, d\mu \underset{\text{mon. conv.}}{=} \lim_{n\to\infty} \int_{\bigcup_{j=1}^n S_j} |f|\, d\mu = \lim_{n\to\infty} \overbrace{\int_{\bigcup_{j=1}^n S_j} \Big|\sum_{j=1}^n f_j\Big|\, d\mu}^{\leq |\nu|_{\bigcup S_j}(\bigcup_{j=1}^n S_j) \,\leq\, |\nu|(X)}$$
$$\leq |\nu|(X) < \infty .$$

and for all $A \in \mathcal{A}$:

$$\int_A f\, d\mu \underset{\substack{\uparrow \\ \text{dom. conv. } (|f| \text{ dominating fct})}}{=} \lim_{n\to\infty} \int_A \sum_{j=1}^n f_j \, d\mu = \lim_{n\to\infty} \sum_{j=1}^{n} \int_{A\cap S_j} f_j\, d\mu = \lim_{n\to\infty} \sum_{j=1}^{\infty} \nu(A \cap S_j)$$
$$= \nu(A).$$

□

Definition. The function f is called the Radon-Nikodym derivative of ν wrt. μ; $f =: \frac{d\nu}{d\mu}$.

Remark. Thm. 6.10. is not true for arbitrary measures μ.

Examples.

i) $X \neq \emptyset$, $\mathcal{A} = \{X, \emptyset\}$, $\mu(\emptyset) = \nu(\emptyset) = 0$, $\mu(X) = \infty$, $\nu(X) = 1$.

$\Rightarrow$ $\nu \ll \mu$, but ν has no μ-density.

ii) $X = [0,1]$, μ counting measure, ν = Borel measure on $[0,1]$.

$\Rightarrow$ $\nu \ll \mu$, but ν has no μ-density.

Proof. Assume $\exists f \in \mathcal{L}_1(\mu)$ s.t. $\nu(A) = \int_A f\, d\mu$, $A \in \mathcal{B}([0,1])$.

Assume $\exists x_0 \in [0,1]$ with $f(x_0) \neq 0$.

$\Rightarrow$ $0 = \nu(\{x_0\}) = \int_{\{x_0\}} f\, d\mu = f(x_0) \neq 0$ ↯

□

Corollary 6.11. μ σ-finite, $\nu \ll \mu$, $g \in \mathcal{L}_1(\nu)$.

$$\Rightarrow \int g\, d\nu = \int g f \, d\mu \qquad \text{where } \nu = f \odot \mu . \qquad (*)$$

Proof. $(*)$ holds for characteristic functions, hence, by linearity, for $g \in \mathcal{E}^+(X, \mathcal{A})$.

By monotone conv.: $(*)$ holds for $g \geq 0$, $g \in \mathcal{L}_1(\nu)$

$\Rightarrow$ $(*)$ true for all $g \in \mathcal{L}_1(\nu)$.

Theorem 6.12. (Lebesgue's decomposition theorem)

μ σ-finite measure, ν (σ-finite) signed measure on a σ-algebra $\mathcal{A}$.
if we allow $\nu(\cdot) \in (-\infty, \infty]$ or $[-\infty, \infty)$

$\Rightarrow$ $\exists$ ~~measures~~ signed measures ρ, σ on $\mathcal{A}$ s.t.

$$\nu = \rho + \sigma, \qquad \rho \ll \mu, \ \sigma \perp \mu.$$

ρ and σ are σ-finite, and finite if and only if ν is finite ← clear if ν "signed measure" according to def. 6.1.

Proof.

Existence. Without restriction: ν pos. measure (use Jordan decomposition thm)

Let $\tau := \mu + \nu$

$\Rightarrow$ τ σ-finite measure and $\mu \ll \tau$.

Let $g \in \mathcal{L}_1(X, \tau)$ (or g τ-quasiintegrable if μ is not finite), $g \geq 0$,
s.t. $\mu = g \odot \tau$.

$\Rightarrow$ $\forall A \in \mathcal{A}$: $\quad \mu(A) = \int_A g \, d\mu + \int_A g \, d\nu$

Let $N := \{x \in X \mid g(x) = 0\}$

$\Rightarrow$ $\mu(N) = 0$

Let $\quad \rho(A) := \nu(A \setminus N), \quad \sigma = \nu(A \cap N)$

$\Rightarrow$ ρ, σ are σ-finite and

- $\sigma \perp \mu$ because: $\mu(N) = 0, \quad \sigma(X \setminus N) = \nu((X \setminus N) \cap N) = 0$.
- $\rho \ll \mu$ because: Let $A \in \mathcal{A}$ with $\underbrace{\mu(A) = 0}_{\Rightarrow\ \mu(A \setminus N) = 0 \text{ because}}$

$$\Rightarrow 0 = \mu(A \setminus N) = \int_{A \setminus N} g \, d\mu + \int_{A \setminus N} g \, d\nu$$
$$= \int_{A \setminus N} g \, d\nu.$$

Since $g > 0$ on $A \setminus N$ $\Rightarrow$ $\nu(A \setminus N) = 0$

$\Rightarrow$ $\rho(A) = \nu(A \setminus N) = 0$

Uniqueness. Let $\nu = \rho + \sigma = \rho' + \sigma'$ with measures ρ', σ' s.t. $\rho' \ll \mu$, $\sigma' \perp \mu$.

$\Rightarrow$ $\exists N, N'$ μ-zero sets s.t. $|\sigma|(X \setminus N) = 0, \ |\sigma'|(X \setminus N') = 0$

$\Rightarrow$ $\forall A \in \mathcal{A}$:

$$\rho(A) = \rho(A \setminus (N \cup N')) + \overbrace{\rho(N \cup N')}^{= 0 \text{ because } \rho \ll \mu}$$

$$\overset{\rho = \nu - \sigma}{=} \nu(A \setminus (N \cup N')) - \underbrace{\sigma(A \setminus (N \cup N'))}_{= 0 \text{ because } A \setminus (N \cup N') \subseteq X \setminus N}$$

$$= \nu(A \setminus (N \cup N')) - \sigma'(A \setminus (N \cup N')) = \rho'(A \setminus (N \cup N')) + \rho'(N \cup N') = \rho'(A).$$

$$\sigma(A) = \sigma(A \cap N) + \overbrace{\sigma(A \setminus N)}^{= 0} = \nu(A \cap N)$$

$$= \nu(A \cap N \cap N') + \underbrace{\nu((A \cap N) \setminus N')}_{= 0 \text{ because: }}$$

$$\nu((A \cap N) \setminus N') = \rho'((A \cap N) \setminus N') + \sigma'((A \cap N) \setminus N')$$
$$\overset{\rho = \rho'}{=} \rho((A \cap N) \setminus N') + \sigma'((A \cap N) \setminus N') = 0$$

Analogously: $\quad \sigma'(A) = \nu(A \cap N \cap N')$.

$\square$

Example.

- β Lebesgue measure on $\mathbb{R}$, ~~=~~ $\nu(A) = \begin{cases} 0, & 0 \notin A \\ 1, & 0 \in A \end{cases}$ $\quad (A \in \mathcal{B})$

  $\Rightarrow$ ν measure on $\mathcal{B}$, and $\nu \perp \beta$ $\quad$ (Because: $\begin{cases} \nu(\mathbb{R} \setminus \{0\}) = 0 \\ \beta(\{0\}) = 0 \end{cases}$)

- β Lebesgue measure on $[0,1]$.

  ν = measure induced by the Cantor function Ψ

  (Ψ constant on intervals E_n^k, where E_n^k = kth interval from the left of length $(1/3)^n$ $(k = 1, \dots, 2^{n-1})$,

  $\Psi(x) = \dfrac{2k-1}{2^n}$ for $x \in E_n^k$ $\quad$ ⤳ Ψ defined on $[0,1] \setminus C$.

  ↑ Cantor set.

  Extend Ψ continuously to $[0,1]$ (possible because Ψ increasing on $[0,1] \setminus C$ and range(Ψ) is dense in $[0,1]$)

  $\Rightarrow$ Ψ monotonic, cont. of total variation 1,

  and: $\quad \mu_\Psi(C) = \quad , \ \mu_\Psi([0,1] \setminus C) = 0$

  $\quad \beta(C) = 0, \ \beta([0,1] \setminus C) = 1$

  $\Rightarrow$ $\mu_\Psi \perp \beta$.

$\square$