Let $u = \sum_{j=1}^{m} \alpha_j \chi_{A_j} \in \mathcal{E}^+(X,\mathcal{B})$ with $\alpha_1,\dots,\alpha_m > 0$, $A_1,\dots,A_m \in \mathcal{B}$, pairwise disjoint and $u \le f$.

$\Rightarrow \forall j \ \ \mu(A_j) \le \mu(\underbrace{\operatorname{supp}(f)}_{\in \mathcal{K}}) < \infty$. Let $\varepsilon > 0$ with $\varepsilon < \min\{\alpha_1,\dots,\alpha_n\}$

$\Rightarrow \exists K_j \in \mathcal{K}$ s.t. $\mu(A_j) \le \mu(K_j) + \varepsilon$ (inner regularity of μ)

Since the K_j are pairwise disjoint $\exists U_j \subseteq X$ open, pairwise disjoint, such that $K_j \subseteq U_j$.

Without restriction, $U_j \subseteq \underbrace{\{x \in X \mid f(x) > \alpha_j - \varepsilon\}}_{\text{open, and } \supseteq K_j}$

$\forall j \in \{1,\dots,m\}$ choose $\varphi_j \in C_c(X)$ with $\chi_{K_j} \le \varphi_j \le \chi_{U_j}$

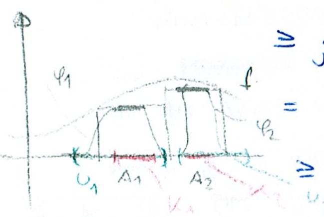
$\Rightarrow g := \sum_{j=1}^{m} (\alpha_j - \varepsilon)\cdot \varphi_j \in C_c(X)$ and $0 \le g \le f$.

$$\Rightarrow J(f) \overset{J \text{ pos}}{\ge} J(g) = \sum_{j=1}^{m} (\alpha_j - \varepsilon) J(\varphi_j) \underset{\text{Def of } \mu}{\ge} \sum_{j=1}^{m} (\alpha_j - \varepsilon)\mu(K_j)$$

$$\ge \sum_{j=1}^{m} (\alpha_j - \varepsilon)(\mu(A_j) - \varepsilon)$$

$$= \int_X u \, d\mu - \underbrace{\varepsilon \sum_{j=1}^{m} (\underbrace{\mu(A_j) - \varepsilon + \alpha_j}_{\ge 0})}_{\ge 0}$$

$$\ge \int_X u \, d\mu .$$



(2) $f \in C_c(X), f \ge 0 \Rightarrow J(f) = \int_X f \, d\mu$.

Without restriction: $0 \le f \le 1$ (J and $\int_X \cdot \, d\mu$ are linear)

Let $\varepsilon > 0$.

$\Rightarrow \exists U$ open s.t. $\operatorname{supp} f =: K \subseteq U$, $\bar{U}$ cpt and $\mu(U) \le \mu(K) + \varepsilon$

Partition of unity $\Rightarrow \exists \varphi \in C_c(X)$, s.t. $0 \le \varphi \le 1$, $\varphi|_K = 1$, $\operatorname{supp} \varphi \subseteq U$.

$\Rightarrow \varphi - f \ge 0$, $\varphi - f \in C_c(X)$

$$\overset{(1)}{\Rightarrow} J(\varphi) - J(f) = J(\varphi - f) \ge \int_X (\varphi - f)\, d\mu = \int_X \varphi \, d\mu - \int_X f \, d\mu$$

$$\Rightarrow 0 \overset{(1)}{\le} J(f) - \int_X f \, d\mu \le J(\varphi) - \underbrace{\int_X \varphi \, d\mu}_{\ge \mu(\operatorname{supp}\varphi) \ge \mu(K)}$$

$$\underset{(1)}{\le} \mu(\operatorname{supp}\varphi) - \mu(K)$$

$$\le \mu(U) - \mu(K) < \varepsilon .$$

True for every $\varepsilon > 0$ $\Rightarrow J(f) - \int_X f \, d\mu = 0$.

□

- Other versions of the representation theorem: see Elstrodt, VIII §2.

  For example: $C_0(X) \cong$ finite Borel measures on $\mathcal{B}$

  (X locally cpt. Hausdorff space, $C_0(X) := \{f \in C(X) \mid \forall \varepsilon > 0 \ \exists K \subseteq X$ cpt. s.t. $|f(x)| < \varepsilon,\ x \in X \setminus K\}$)

- Other measure associated to a given pos. linear form $J : C_c(X) \to \mathbb{R}$:

  As before:

  $\forall K \in \mathcal{K} \quad \mu_*(K) := \inf\{J(f) \mid f \in C_c(X),\ f \ge \chi_K\}$

  $\forall A \in \mathcal{B} \quad \mu_*(A) := \sup\{\mu_*(K) \mid K \in \mathcal{K},\ K \subseteq A\}$

  $\forall A \in \mathcal{B} \quad \mu^*(A) = \inf\{\mu_*(U) \mid U \text{ open},\ A \subseteq U\}$

  $\rightsquigarrow \mu_*(U) = \mu^*(U)$, $\mu_*(K) = \mu^*(K)$ for all U open, $K \in \mathcal{K}$.

  It can be shown:

  - μ^* is an outer measure on $(X, \mathcal{P}X)$, $\mu^*|_{\mathcal{B}}$ is a measure on $(X, \mathcal{B})$; it is a Borel measure!
  - μ_* is a Borel measure on $(X, \mathcal{B})$.
  - $\mu^*(A) = \mu_*(A)$ for all $A \in \mathcal{B}(X)$ with σ-finite μ^*-measure

  $\longrightarrow$ $\mu_\circ := \mu_*|_{\mathcal{B}(X)} =:$ essential measure (is a Radon measure)

  $\mu^\circ := \mu^*|_{\mathcal{B}(X)} =$ principal measure (is outer regular)

  $\longrightarrow$ Both $\mu_\circ$ and μ° represent J. In general: $\mu_\circ \ne \mu^\circ$, but:

  for every μ that represents J we have:

  $\mu(K) \le \mu_\circ(K)$ ($K \in \mathcal{K}$)

  $\mu^\circ(U) \le \mu(U)$ (U open)

  In general: $\mu_\circ \ne \mu^\circ$. But: if X is σ-cpt, then $\mu_\circ = \mu^\circ$.

Application. (Construction of the Borel measure on $\mathbb{R}^n$)

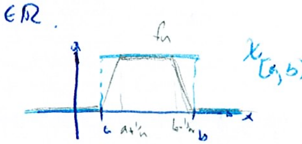
The Riemann integral gives a well defined pos. linear form on $C_c(X)$:

$$J: C_c(\mathbb{R}) \to \mathbb{R} \qquad f \mapsto \int_{-\infty}^{\infty} f(x)\,dx$$

By the Riesz representation theorem exists a unique Radon measure μ on $\mathcal{B}(\mathbb{R})$ s.t. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{\mathbb{R}} f\,d\mu$, $f \in C_c(X)$

To show that $\mu = \beta$ = Borel measure it suffices to show that $\mu([a,b)) = \beta([a,b)) = b-a$ for all $a<b \in \mathbb{R}$.

To this end, fix $a<b \in \mathbb{R}$



$\Longrightarrow \exists (f_n)_n \subseteq C_c(X)$ s.t.

$$0 \le f_n \le \chi_{[a,b)},\ f_n \nearrow \chi_{[a,b)} \ \& \ \int_{-\infty}^{\infty} \chi_{[a,b)}(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

$$\Longrightarrow \mu([a,b)) = \int_{\mathbb{R}} \chi_{[a,b)}\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,d\mu$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} \chi_{[a,b)}(x)\,dx = b-a.$$

$\square$

6. Representation of measures and the Radon-Nikodym theorem.

6.1. Decomposition theorems.

Definition 6.1. $\mathcal{A}$ σ-algebra, $\nu: \mathcal{A} \to \mathbb{R}$ is called a signed measure if

i) $\nu(\emptyset) = 0$

ii) $\nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \nu(A_j)$ for $A_j \in \mathcal{A}$, pairwise disjoint.

$\nu: \mathcal{A} \to \mathbb{C}$ is called a complex measure if i) and ii) hold.

Observations.

i) The series in (ii) is abs. convergent because it does not depend on the order of summation (l.h.s. does not depend on ordering of the A_j's)

ii) Sometimes: a signed measure is allowed to have values in $[-\infty,\infty)$ or $(-\infty,\infty]$.

For us: a signed/complex measure ~~take~~ is finite valued.

Lemma 6.2 μ signed or complex measure on a σ-algebra $\mathcal{A}$.

$\Longrightarrow \exists M \in \mathbb{R}$ s.t. $\forall A \in \mathcal{A}$ $|\mu(A)| < M$.

Proof. Assume μ is not finite $\Longrightarrow$ Either real or imaginary part of μ is not bdd.
Without restriction: real part of μ is not bdd from above.

$\Longrightarrow \exists (A_n)_n \in \mathcal{A}$ and $0 < a_1 < b_1 < a_2 < b_2 < \dots$

with $a_n > b_{n-1} + \dots + b_2 + b_1 + 1$

and $a_n < \mathrm{Re}\,\mu(A_n) < b_n$.

Define $B_n := A_n \setminus (A_1 \cup \dots \cup A_{n-1})$

$$\Longrightarrow \mathrm{Re}(\mu(B_n)) \ge \mathrm{Re}(\mu(A_n)) - \sum_{j=1}^{n-1} \mathrm{Re}(\mu(A_j)) \ge a_n - \sum_{j=1}^{n-1} b_j \ge 1.$$

$$\Longrightarrow \mathrm{Re}\left(\mu\left(\bigcup_{n=1}^{\infty} B_n\right)\right) \ge \sum_{n=1}^{\infty} 1 = \infty$$

B_n's pairwise disjoint.

On the other hand

$$\mathrm{Re}\Big(\mu\big(\underbrace{\textstyle\bigcup_{n=1}^{\infty} B_n}_{\in \mathcal{A}}\big)\Big) < \infty$$