## 5.2 Locally compact Hausdorff spaces



**Recall**. $X$ Hausdorff space. $\rightsquigarrow$

$X$ locally cpt $\Leftrightarrow$ every $a \in X$ has a compact neighbourhood.

**Observation**. Every locally compact space is subspace of a compact space.

Proof. $X$ locally cpt, $\omega_0 \notin X$.

Let $X' := X \cup \{\omega_0\}$ and $U \subseteq X'$ open $:\Leftrightarrow \begin{cases} U \subseteq X \text{ open} & \text{if } \omega_0 \notin U \\ \text{or } X\setminus U \text{ cpt} & \text{if } \omega_0 \in U. \end{cases}$

Easy to check: $X'$ is a top. space, and its top. restricted to $X$ gives the original topology on $X$.

$X'$ is called the Alexandroff (or one-point) compactification of $X$.

**Definition 5.9.** $X$ top. space, $f: X \to \mathbb{R}$ (or $\mathbb{C}$ or any vector space)

$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}} =:$ support of $f$

$C(X) := \{f: X \to \mathbb{K} \mid f \text{ continuous}\}$

$C_b(X) := \{f \in C(X) \mid f \text{ bounded}\}$

$C_c(X) := \{f \in C(X) \mid \operatorname{supp} f \text{ is compact}\}$.

**Obviously**: $C_c(X) \subseteq C_b(X) \subseteq C(X)$ with equality if $X$ is cpt.

**Note**. $(C_b, \|\cdot\|_\infty)$ is a Banach space.

**Important theorems** (without proof)

**Urysohn's lemma**. $X$ locally compact, $U \subseteq X$ open, $K \subseteq U$, $K$ cpt


$\Rightarrow \exists f \in C_c(X)$ st. $0 \le f \le 1$, $f|_K = 1$, $f|_{X\setminus U} = 0$. $(*)$

In particular: $\operatorname{supp} f$ is a compact neighbourhood of $K$.

Since $X$ locally compact, there ex. $V \subseteq X$ st. $V$ open and $K \subseteq V \subseteq \overline{V} \subseteq U$ (Rudin, Thm. 2.7)

$\Rightarrow$ $f$ can be chosen such that $\operatorname{supp} f \subseteq U$ (apply $(*)$ to $K$ and $V$ instead of $K$ and $U$)

**Theorem** (Partition of unity)

$X$ locally compact, $K \subseteq X$ cpt, $U_1, \dots, U_n \subseteq X$ open, st. $K \subseteq \bigcup_{j=1}^n U_j$

$\Rightarrow \exists f_1, \dots, f_n \in C_c(X)$ st.

i) $\forall j$ $f_j \ge 0$ and $\operatorname{supp} f_j \subseteq U_j$

ii) $\forall x \in K$: $\sum_{j=1}^n f_j(x) = 1$

iii) $\forall x \in X$: $\sum^n f_j(x) \le 1$



**Corollary**. $X$ locally compact, $K \subseteq X$ cpt, $U_1, \dots, U_n \subseteq X$ open with $K \subseteq \bigcup_{j=1}^n U_j$

$\Rightarrow \exists K_j$ cpt. st. $K_j \subseteq U_j$ and $K = \bigcup_{j=1}^n K_j$.

Proof. Choose $f_1, \dots, f_n$ ~~as in the theorem~~ partition of unity subordinate to $U_1, \dots, U_n$. and define $K_j := \operatorname{supp} f_j \cap K$.

$\Rightarrow$ all $K_j$ are compact and $\bigcup_{j=1}^n K_j \subseteq K$.

On the other hand, for $x \in K$ $\exists j_x$ st. $f_{j_x}(x) \neq 0$ because $\sum_{j=1}^n f_j(x) = 1$

$\Rightarrow x \in \underbrace{K \cap \operatorname{supp} f_{j_x}}_{= K_j} \subseteq \bigcup_{j=1}^n K_j$ □

locally finite & inner regular

**Now** $X$ locally compact Hausdorff space, $\mu$ Radon measure on $X$

$\Rightarrow \forall K \subseteq X$ cpt. $\mu(K) < \infty$ ($\mu$ locally finite & $X$ locally cpt $\Rightarrow \mu(K) < \infty$, $K \subseteq X$, $K$ cpt)

$\Rightarrow \boxed{C_c(X) \subseteq L_1(X)}$

**Prueba** $u \in C_c(X) \Rightarrow \|u\|_\infty < \infty$ and $u$ measurable (because continuous)

$\Rightarrow \int_X |u| \, d\mu \le \int_X \|u\|_\infty \chi_{\operatorname{supp}(u)} \, d\mu = \|u\|_\infty \, \mu(\underbrace{\operatorname{supp} u}_{\text{cpt}}) < \infty$

$\Rightarrow$ Every Radon measure on $X$ induces a positive linear form on $C_c(X)$ (☆)

**Where**: $Y$ $\mathbb{K}$-VS, $J: Y \to \mathbb{K}$ is called a linear form if ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

$J(\alpha x + y) = \alpha J(x) + J(y)$, $\alpha \in \mathbb{K}$, $x, y \in Y$.

If $Y$ is a space of $\mathbb{K}$-valued functions (e.g. $Y = C_c(X)$), then $J$ is called a positive linear form on $Y$ if $J$ is a lin. form on $Y$ and $J(u) \ge 0$ whenever $u \ge 0$

**Question**: Is the converse of (☆) true?

That is: Given a positive linear form on $C_c(X)$, can it be represented by a Radon measure $\mu$?

## 5.3 The Riesz representation theorem: (Elstrodt, VIII §2)

Here always: $X$ locally cpt Hausdorff space,

$J : C_c(X) \to \mathbb{C}$ positive linear form.

We will show: there exist a unique Radon measure $\mu$ on $X$ s.t.

$$\boxed{\forall u \in C_c(X) \quad J(u) = \int_X u \, d\mu.} \qquad (*)$$

Two steps:

① Define

$$\mu_0(K) := \inf\{ J(f) \mid f \in C_c(X),\ f \geq \chi_K \}, \qquad K \subseteq X \text{ cpt} \qquad (\triangledown)$$

and show that $\mu_0$ can be extended to a measure on $\mathcal{B}(X)$

② Show that $\mu_0$ represents $J$, that is, satisfies $(*)$.

Notation. $\mathfrak{K} := \{ K \subseteq X \mid K \text{ cpt} \}$.

**Lemma 5.10** $X$ locally cpt Hausdorff space, $J : C_c(X) \to \mathbb{K}$ pos. lin. form, $\mu_0$ as in $(\triangledown)$. Then:

i) For all $K, L \in \mathfrak{K}$:

(K1) $K \subseteq L \Rightarrow 0 \leq \mu_0(K) \leq \mu_0(L) < \infty$

(K2) $\mu_0(K \cup L) \leq \mu_0(K) + \mu_0(L)$

(K3) $K \cap L = \emptyset \Rightarrow \mu_0(K \cup L) = \mu_0(K) + \mu_0(L)$.

ii) $K \in \mathfrak{K},\ \varepsilon > 0 \Rightarrow \exists U \subseteq X$ open s.t. $\forall L \subseteq U$, $L$ cpt $\mu_0(L) \leq \mu_0(K) + \varepsilon$.

Proof

i) (K1): $J$ positive and $\chi_L \geq \chi_K \Rightarrow 0 \leq \mu_0(K) \leq \mu_0(L)$.

By partition of unity $\exists f \in C_c(X)$ s.t. $f \geq \chi_L$

$\Rightarrow \mu_0(L) \leq J(f) < \infty$

(K2) $\underbrace{\forall f, g \in C_c(X),\ f \geq \chi_K,\ g \geq \chi_L}_{(*)} \Rightarrow f+g \in C_c(X),\ f+g \geq \chi_K + \chi_L \geq \chi_{K \cup L}$

$\Rightarrow \mu_0(K \cup L) \leq J(f+g) = J(f) + J(g)$

Taking the infimum of all $f, g$ with $(*)$ yields: $\mu_0(K \cup L) \leq \mu_0(K) + \mu_0(L)$.



(K3) Only to show: $\mu_0(K \cup L) \geq \mu_0(K) + \mu_0(L)$.

Let $h \in C_c(X)$, $h \geq \chi_{K \cup L}$.

Choose $\varphi \in C_c(X)$ s.t. $\varphi|_K = 1$, $\varphi|_L = 0$ (possible by partition of unity because $X \setminus L$ is an open neighbourhood of $K$)

and set $f_h := \varphi \cdot h$, $g_h := (1-\varphi) h$.

Then: $f_h, g_h \in C_c(X)$, $f_h \geq \chi_K$, $g_h \geq \chi_L$ and $f_h + g_h = h$.

$$\Rightarrow \mu_0(K \cup L) = \inf\{ \underbrace{J(h)}_{= J(f_h) + J(g_h)} \mid h \in C_c(X),\ h \geq \chi_{K \cup L} \}$$

$$\geq \inf\{ J(f) \mid f \in C_c(X),\ f \geq \chi_K \} + \inf\{ J(g) \mid \dots \}$$

$$= \mu_0(K) + \mu_0(L).$$

ii) Let $K \subseteq X$ and $\delta > 0$. $\Rightarrow \exists f \in C_c(X)$, $f \geq \chi_K$ s.t. $\mu_0(K) + \delta \geq J(f)$.

Since $f$ cont $\Rightarrow U_\delta := \{ x \in X \mid f(x) > \frac{1}{1+\delta} \}$ is an open neighbourhood of $K$

Choose $\delta$ small enough s.t. $\delta \cdot (\mu_0(K) + \delta + 1) < \varepsilon$ (possible because $\mu_0(K) <$

then for all $L \subseteq U_\delta$, $L$ cpt:

$$\mu_0(L) \underset{\substack{\uparrow \\ (1+\delta) f \geq \chi_L,\ \text{cpt. support}}}{\leq} (1+\delta) J(f) \leq (1+\delta)(\mu_0(K) + \delta) \leq \mu_0(K) + \varepsilon.$$

**Lemma 5.11** $X$ Hausdorff space, $\mu_0 : \mathfrak{K} \to [0, \infty)$ s.t. (K1)-(K2) and ii) of Lemma 5.12 hold. Then:

(S) $K, L \in \mathfrak{K}$ with $K \subseteq L$

$$\Rightarrow \mu_0(L) - \mu_0(K) = \inf\{ \mu_0(C) \mid C \in \mathfrak{K},\ C \subseteq L \setminus K \}$$

Proof. "$\geq$" Let $C \in \mathfrak{K}$, $C \subseteq L \setminus K \Rightarrow C \dot\cup K \subseteq L$

$$\Rightarrow \mu_0(L) \underset{(K1)}{\geq} \mu_0(C \dot\cup K) \underset{(K3)}{=} \mu_0(C) + \mu_0(K)$$

Taking the infimum over all such $C$ shows $\mu_0(L) - \mu_0(K) \geq \inf\{\dots\}$

"$\leq$" Let $\varepsilon > 0$ [(*)] and choose $U \subseteq X$, $U$ open, $U \supseteq K$ such that

$H \in \mathfrak{K}$, $H \subseteq U \Rightarrow \mu_0(H) \leq \mu_0(K) + \varepsilon$.


$L \setminus U$, and $K$ cpt, $X$ Hausdorff

$\Rightarrow \exists$ open sets $W, V$ s.t. $K \subseteq V$, $L \setminus U \subseteq W$, $V \cap W = \emptyset$.

Let $C := L \setminus V$, $D := L \setminus W$.

$\Rightarrow C$ and $D$ are cpt, $C \subseteq L \setminus K$ and $\mu_0(D) \leq \mu_0(K) + \varepsilon$ because $D \subseteq U$.

and $C \cup D = (L \setminus V) \cup (L \setminus W) = L \setminus \underbrace{(V \cap W)}_{= \emptyset} = L$

$\Rightarrow \mu_0(L) \leq \mu_0(C) + \mu_0(D) \leq \mu_0(C) + \mu_0(K) + \varepsilon$

$\Rightarrow \mu_0(L) - \mu_0(K) \leq \mu_0(C) + \varepsilon$.

(*) We have shown: $\exists C \subseteq L \setminus K$ cpt $\wedge$ $\mu_0(C) + \varepsilon \geq \mu_0(L) - \mu_0(K)$

## Theorem 5.12. (Extension theorem)

Let $X$ Hausdorff space, $\mu_0 : \mathcal{K} \to [0,\infty)$ s.t. (K1)–(K3) and (S) hold.

$\Longrightarrow$ exists a unique measure $\mu : \mathcal{B}(X) \to [0,\infty]$ s.t. $\mu$ is inner regular and $\mu|_{\mathcal{K}} = \mu_0$.

This measure is given by

$$\mu(A) = \sup\{\mu_0(K) \mid K \in \mathcal{K},\ K \subseteq A\}, \qquad A \in \mathcal{B}(X). \qquad (*)$$

### Proof

**Uniqueness** If $\mu_0$ can be extended to an inner regular measure $\mu$ on $\mathcal{B}(X)$, then it must be given by $(*)$ (by inner regularity).

**Existence** (Similar to the proof that $\mu^*$ is an outer measure and a measure on $\sigma(H)$, see Carathéodory's thm)

Define $\mu : \mathcal{P}X \to [0,\infty]$ by $(*)$.

By (K1): $\mu(K) = \mu_0(K)$ $(K \in \mathcal{K})$ and obviously: $\mu(A) \le \mu(B)$ for $A \subseteq B$.

For $K \in \mathcal{K}$ define

$$\mathcal{A}_K := \{A \subseteq X \mid \mu(K) \le \mu(K \setminus A) + \mu(K \cap A)\}$$

Note "$\ge$" is always true

and $\mathcal{A} := \bigcap_{K \in \mathcal{K}} \mathcal{A}_K$.

We will show: $\mathcal{A}$ is a $\sigma$-algebra, $\mu|_{\mathcal{A}}$ is a measure and $\mathcal{B}(X) \subseteq \mathcal{A}$.

① $\mathcal{A}$ closed under taking complements & $\emptyset \in \mathcal{A}$

Obviously: $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$.

$\emptyset$ cpt $\Rightarrow \mu(\emptyset) = \mu_0(\emptyset) \overset{(K3)}{=} \mu_0(\emptyset) + \mu_0(\emptyset) \Rightarrow \mu_0(\emptyset) = 0$

② $(A_j)_j \subseteq \mathcal{P}X$ pairwise disjoint $\Longrightarrow \sum_{k=1}^{\infty} \mu(A_k) \le \mu\left(\bigcup_{j=1}^{\infty} A_j\right)$

Let $A, B \subseteq X$ with $A \cap B = \emptyset$.

Then: $\mu(A) + \mu(B) \le \mu(A \cup B)$ because:

- clear if $\mu(A) = \infty$ or $\mu(B) = \infty$ since then also $\mu(A \cup B) = \infty$
- if $\mu(A) < \infty$ and $\mu(B) < \infty$: Let $\varepsilon > 0$ and choose $K, L \in \mathcal{K}$ with $K \subseteq A$, $L \subseteq B$ and $\mu(A) + \mu(B) - \varepsilon \le \mu(K) + \mu(L) = \mu(K \dot\cup L) \le \mu(A \cup B)$

Now let $(A_j)_j \subseteq \mathcal{P}X$, pairwise disjoint.

By induction: $\sum_{j=1}^{N} \mu(A_j) \le \mu\left(\bigcup_{j=1}^{N} A_j\right) \le \mu\left(\bigcup_{j=1}^{\infty} A_j\right)$

True for every $N$

$\Rightarrow \sum_{j=1}^{\infty} \mu(A_j) \le \mu\left(\bigcup_{j=1}^{\infty} A_j\right)$.



③ Subadditivity & $\mathcal{A}$ is $\sigma$-algebra

Let $(M_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}$. We will show: $M := \bigcup_{j=1}^{\infty} M_j \in \mathcal{A}$ and $\mu(M) \le \sum_{j=1}^{\infty} \mu(M_j)$

Together with ② $\rightsquigarrow$ $\mu$ is $\sigma$-additiv.

Fix $K \in \mathcal{K}$ and $\varepsilon > 0$ (To show: $M \in \mathcal{A}_K$, that is $\forall \varepsilon > 0$ $\mu(K) \le \mu(K \cap M) + \mu(K \setminus M) + \varepsilon$

Since $M_n \in \mathcal{A}$, we can choose $A_n, B_n \in \mathcal{K}$ with $\mu_0(K) \le \mu_0(A_n) + \mu_0(B_n) + \varepsilon/2^n$
and $A_n \subseteq K \cap M_n$ and $B_n \subseteq K \setminus M_n$ (1)

$\forall n \in \mathbb{N}$: $A_n' := (A_1 \cup \ldots \cup A_{n-1}) \cap A_n \subseteq K \cap \left(\bigcup_{j=1}^{\infty} M_j\right) \cap M_n \subseteq A_n$

$n \ge 2$: $B_n' := (B_1 \cap \ldots \cap B_{n-1}) \cup B_n \subseteq K \setminus \left(\bigcup_{j=1}^{n-1} M_j\right) \cup (K \setminus M_n)$

$B_1' := B_1 \subseteq K \setminus M_1$

$\Longrightarrow$ $A_n', B_n'$ are compact and $A_n' \cap B_n' = \emptyset$.

$\Longrightarrow -\varepsilon/2^n \overset{(1)}{\le} \mu_0(A_n) + \mu_0(B_n) - \mu_0(K)$ $\quad$ ($K \supseteq A_n' \dot\cup B_n' \overset{(K1),(K3)}{\Longrightarrow} \mu_0(K) \ge \mu_0(A_n') + \mu_0(B_n')$)

$\overset{(K1),(K3)}{\le} \mu_0(A_n) - \mu(A_n') + \mu_0(B_n) - \mu_0(B_n')$

$\overset{(S)}{=} \sup\{\mu_0(C) \mid C \in \mathcal{K},\ C \subseteq A_n \setminus A_n'\} - \sup\{\mu_0(C) \mid C \in \mathcal{K},\ C \subseteq B_n' \setminus B_n\}$

$\overset{\text{def. of } \mu}{=} \mu(A_n \setminus A_n') - \mu(B_n' \setminus B_n)$

$= \mu((A_1 \cup \ldots \cup A_n) \setminus (A_1 \cup \ldots \cup A_{n-1})) - \mu((B_1 \cap \ldots \cap B_{n-1}) \setminus (B_1 \cap \ldots \cap B_n))$

$\overset{\text{(S) and def of } \mu}{=} \mu(A_1 \cup \ldots \cup A_n) - \mu(A_1 \cup \ldots \cup A_{n-1}) - \mu(B_1 \cap \ldots \cap B_{n-1}) + \mu(B_1 \cap \ldots \cap B_n)$

$\Longrightarrow \sum_{n=2}^{N} -\varepsilon/2^n \le \sum_{n=2}^{N} (\ldots) \overset{\text{telescopic sum!}}{=} \mu(A_1 \cup \ldots \cup A_N) + \mu(B_1 \cap \ldots \cap B_N) - \mu(A_1) - \mu(B_1)$

$\Longrightarrow \mu(A_1 \cup \ldots \cup A_N) + \mu(B_1 \cap \ldots \cap B_N) \ge \underbrace{\mu(A_1) + \mu(B_1)}_{\ge \mu(K) - \varepsilon/2} - \varepsilon \sum_{n=2}^{N} \frac{1}{2^n}$

$\ge \mu(K) - \varepsilon.$ (2)

Note: $\mu(\underbrace{K \cap M}_{\supseteq A_1 \cup \ldots \cup A_n}) + \mu(K \setminus M) \ge \mu(A_1 \cup \ldots \cup A_n) + \mu\left(\bigcap_{j=1}^{\infty} B_j\right)$

$= \mu(A_1 \cup \ldots \cup A_n) + \mu\left(\bigcap_{j=1}^{n} B_j\right) + \mu\left(\bigcap_{j=1}^{\infty} B_j\right) - \mu\left(\bigcap_{j=1}^{n} B_j\right)$

$\ge \mu(K) - \varepsilon + \mu\left(\bigcap^{\infty} B_j\right) - \mu\left(\bigcap^{n} B_j\right)$,

so we have to show that $\mu\left(\bigcap^{n} B_j\right) - \mu\left(\bigcap_{j=1}^{\infty} B_j\right)$ is small for $n$ large.

Observe that $\bigcap_{j=1}^{\infty} B_j$ is compact (because it is a closed subset of the cpt set $B_1$)

By (S) $\exists D \in \mathcal{K}$, $D \subseteq B_1 \setminus \bigcap_{j=1}^{\infty} B_j$ s.t. $\mu_0(D) \ge \mu_0(B_1) - \mu_0\left(\bigcap_{j=1}^{\infty} B_j\right) - \varepsilon$ (3)

By construction: $D \cap \left(\bigcap_{j=1}^{\infty} B_j\right) = \emptyset$

$\Longrightarrow \exists N_0 \in \mathbb{N}$ s.t. $\forall N \ge N_0$ $D \cap \bigcap_{j=1}^{N} B_j = \emptyset$

$\Longrightarrow \forall N \ge N_0$ $\mu_0(D) + \mu_0\left(\bigcap_{j=1}^{N} B_j\right) = \mu_0\Big(\underbrace{D \dot\cup \bigcap_{j=1}^{N} B_j}_{\subseteq B_1}\Big) \le \mu_0(B_1)$

$\overset{(3)}{\le} \mu_0(D) + \mu_0\left(\bigcap_{j=1}^{\infty} B_j\right) + \varepsilon$

$\Longrightarrow$ $\forall N \ge N_0$ $\mu_0\left(\bigcap_{j=1}^{N} B_j\right) - \mu_0\left(\bigcap_{j=1}^{\infty} B_j\right) < \varepsilon.$ (4)

$\mu_0(D) \neq 0$

$$\Rightarrow \mu(\overbrace{K\cap M}^{\supseteq A_1\cup\dots\cup A_N}) + \mu(\underbrace{K\setminus M}_{\supseteq \bigcap_{j=1}^{\infty} B_j}) \geq \mu\Big(\bigcup_{n=1}^{N} A_n\Big) + \mu\Big(\bigcap_{j=1}^{\infty} B_j\Big) \geq \mu\Big(\bigcup_{n=1}^{N} A_n\Big) + \mu\Big(\bigcap_{j=1}^{N} B_j\Big) - \varepsilon$$

$$\geq \mu\Big(\bigcup_{n=1}^{N} A_n\Big) + \mu(K\setminus M) \geq \dots$$

$$\overset{(2)}{\geq} \mu(K) - 2\varepsilon \,. \quad \text{for } N \geq N_0$$

True for every $\varepsilon > 0$ $\Rightarrow$ $\mu(K\cap M) + \mu(K\setminus M) = \mu(K)$ $\Rightarrow$ $M \in \mathfrak{A}_K \overset{K\in\mathcal{K} \text{ arbitrary}}{\Rightarrow} M \in \mathfrak{A}$

In addition:

$\forall N \geq N_0$ $\displaystyle\sum_{n=1}^{N} \mu(\underbrace{M_n \cap K}_{\supseteq A_n}) + \mu(K\setminus M) \geq \sum_{n=1}^{N} \mu(A_n) + \mu(K\setminus M)$

$$\geq \mu\Big(\bigcup_{n=1}^{N} A_n\Big) + \mu(K\setminus M) \geq \mu(K) - 2\varepsilon$$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(M_n\cap K) + \mu(K\setminus M) \geq \mu(K) - 2\varepsilon$$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(M_n\cap K) + \mu(K\setminus M) \geq \mu(K) \quad \text{since } \varepsilon > 0 \text{ arbitrary} \qquad (5)$$

$$\Rightarrow \mu(M) = \sup\{\underbrace{\mu_0(K)}_{\overset{(5)}{\leq} (\sum_{n=1}^{\infty}\mu(M_n\cap K)) + \mu(\underbrace{K\setminus M}_{=\emptyset})} \mid K\in\mathcal{K},\ K\subseteq M\} \leq \sum_{n=1}^{\infty} \mu(M_n\cap K) \qquad (6)$$

(6) together with (2) shows that $\mu$ is a measure.

(4) $\mathcal{B} \subseteq \mathfrak{A}$: Since $\mathfrak{A}$ is a $\sigma$-algebra and the set of all closed subsets of $X$ generate $\mathcal{B}$, it suffices to prove: $A \subseteq X$ closed $\Rightarrow$ $A \in \mathfrak{A}$.

So, fix $A \subseteq X$, closed, $K \in \mathcal{K}$.

$\Rightarrow$ ~~$K\setminus A$ and~~ $K\cap A$ are compact, and, by (S) and the def. of $\mu$:

$$\mu_0(K) - \mu_0(K\cap A) \overset{(S)}{=} \sup\{\mu_0(C) \mid C\in\mathcal{K},\ C\subseteq K\setminus(K\cap A)\}$$

$$= \sup\{\mu_0(C) \mid C\in\mathcal{K},\ C\subseteq K\setminus A\}$$

$$= \mu(K\setminus A)$$

$\Rightarrow$ $\mu_0(K) = \mu(K\cap A) + \mu(K\setminus A)$ $\Rightarrow$ $A \in \mathfrak{A}_K$ $\Rightarrow$ $A \in \mathfrak{A}$ since $K\in\mathcal{K}$ arbit.

$\square$



**Theorem 5.13.** (Riesz representation theorem)

$X$ locally compact Hausdorff space, $J: C_c(X) \to \mathbb{K}$ pos. linear form.

$\Rightarrow$ $\exists!\ \mu: \mathcal{B}(X) \to [0,\infty]$ Radon measure (locally finite & inner regular)

such that $J(f) = \int_X f\,d\mu$, $f\in C_c(X)$. $\qquad (*)$

The following holds:

$\forall K\subseteq X$, $K$ cpt $\qquad \mu(K) = \inf\{J(f) \mid f\in C_c(X),\ f\geq \chi_K\}$ $\qquad (1)$

$\forall A\in \mathcal{B}(X)$ $\qquad \mu(A) = \sup\{\mu(K) \mid K\in\mathcal{K},\ K\subseteq A\}$ $\qquad (2)$

Proof.

Uniqueness. Let $\mu$ be a Radon measure on $X$ which satisfies $(*)$.

$\mu$ inner regular $\Rightarrow$ $\mu$ determined by its values on compact subsets

$\Rightarrow$ we only have to show (1).

"$\leq$" $f\in C_c(X)$ with $f\geq \chi_K$

$$\Rightarrow J(f) \overset{(*)}{=} \int_X f\,d\mu \geq \int_X \chi_K\,d\mu = \mu(K)$$

"$\geq$" Fix $K\in\mathcal{K}$ and $\varepsilon > 0$

$\Rightarrow$ $\exists U \supseteq K$ open, st. $\mu(U) \leq \mu(K) + \varepsilon$. (Prop 5.7 iii))

Partition of unity $\to$ $\exists f\in C_c(X)$ s.t. $\chi_K \leq f \leq \chi_U$.

$$\Rightarrow J(f) \doteq \int_X f\,d\mu \leq \int_X \chi_U\,d\mu = \mu(U) \leq \mu(K) + \varepsilon.$$

Existence. Define $\mu: \mathcal{B}(X) \to [0,\infty]$ by (1) and (2).

By the extension theorem: $\mu$ is an inner regular measure.

$\mu$ is locally finite because $X$ ~~Hausdorff~~ locally cpt & $\mu(K) < \infty$ $(K\in\mathcal{K})$. (Observ. 5.3.e)

$\rightsquigarrow$ We only have to prove $(*)$.

Since $f\in C_c(X) \subseteq L_1(X)$, we can assume $f\geq 0$ ($J$ and $\int\cdot\,d\mu$ are linear)

(1) $f\in C_c(X)$, $f\geq 0$ $\Rightarrow$ $J(f) \geq \int_X f\,d\mu$

By def. of $\int_X f\,d\mu$, it suffices to show $\int_X u\,d\mu \leq J(f)$ for all $u\in E^+(X,\mathcal{B})$ with $u\leq f$.