5. The Riesz representation theorem. (Bauer, Chapter IV)

In this chapter, all top. spaces are assumed to be Hausdorff,

because: compact sets will play an important role in the construction of measures.
In a Hausdorff space, all cpt subsets are closed, hence Borel sets.

5.1. Borel and Radon measures.

Definition 5.1. X top. space,

$\mathcal{B}(X) :=$ Borel σ-algebra on X

$:=$ σ-algebra on X generated by all open sets in X. □

Observation. Other generator of $\mathcal{B}(X)$: $\{A \subseteq X \mid A \text{ closed}\}$

Example. (Extended real line)

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, with convention: $\forall a \in \mathbb{R}$ $a < \infty$, $a > -\infty$.

let topology on $\overline{\mathbb{R}}$ be generated by $\{(a,b), (a,\infty], [-\infty, b) \mid a,b \in \mathbb{R}\}$

$\leadsto$ subspace topology on $\mathbb{R}$ = usual topology on $\mathbb{R}$.

$\mathcal{B}(\overline{\mathbb{R}}) = \{M, M \cup \{+\infty\}, M \cup \{-\infty\}, M \cup \{+\infty, -\infty\} \mid M \in \mathcal{B}(\mathbb{R})\}$.

Definition 5.2. X Hausdorff space. A measure $\mu : \mathcal{B}(X) \to [0,\infty]$ is called

i) Borel measure on X if $\mu(K) < \infty$ for all cpt sets $K \subseteq X$
ii) locally finite if every $x_0 \in X$ has a neighborhood V_{x_0} with $\mu(V_{x_0}) < \infty$
iii) inner regular if $\forall B \in \mathcal{B}(X)$ $\mu(B) = \sup\{\mu(K) \mid K \subseteq B, K \text{ cpt}\}$ (*)
iv) outer regular if $\forall B \in \mathcal{B}(X)$ $\mu(B) = \inf\{\mu(V) \mid V \supseteq B, V \text{ open}\}$ (**)
v) regular if it is inner regular and outer regular.

A set $B \in \mathcal{B}(X)$ is called inner regular / outer regular, if (*) / (**) holds.

Elstrodt: $\mu : \mathcal{B}(X) \to [0,\infty]$ Borel $:\Leftrightarrow$ μ locally finite.
$\mu : \mathcal{B}(X) \to [0,\infty]$ Radon measure $:\Leftrightarrow$ μ locally finite and inner regular

Rudin: • $B \in \mathcal{B}(X)$ inner regular $:\Leftrightarrow$ ($\mu(B) < \infty$ or B open) and (*) holds.

• Borel measure only defined for locally cpt. top spaces
$\leadsto$ every measure $\mu : \mathcal{B}(X) \to [0,\infty]$ is called a Borel measure.

Observations 5.3.

a) i) – v) also make sense for measures defined on a σ-algebra $\mathcal{O}\!\mathcal{l}$ with $\mathcal{O}\!\mathcal{l} \supseteq \mathcal{B}(X)$.

b) μ finite measure on $\mathcal{B}(X)$ $\Rightarrow$ μ Borel measure.

c) μ locally finite measure $\Rightarrow$ μ Borel measure

Proof. $K \subseteq X$ cpt. $\forall x \in K$ choose open neighbourhood V_x with $\mu(V_x) < \infty$.
Since K cpt and $K \subseteq \bigcup_{x \in K} V_x$ $\exists$ $x_1, \ldots, x_n \in K$ st. $K \subseteq \bigcup_{m=1}^{n} V_{x_m}$.
$\Rightarrow \mu(K) \le \sum_{m=1}^{n} \mu(V_{x_m}) < \infty$. □

d) μ locally finite $\Rightarrow$ $\forall K \subseteq X$ cpt. $\exists V \subseteq X$ open with $K \subseteq V$ and $\mu(V) < \infty$
Proof. as above. (If in addition, μ inner regular $\to$ see Prop 5.7)

e) X locally compact. (that is: every $x_0 \in X$ has a compact neighbourhood)
Then: μ is locally finite $\Leftrightarrow$ $\mu(K) < \infty$ (K cpt).

Examples.

i) Dirac measure: X Hausdorff space, $a \in X$; $\mu_a(B) := \begin{cases} 1, & a \in B \\ 0, & a \notin B. \end{cases}$
$\leadsto$ μ_a is locally finite, inner regular and outer regular.

ii) $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ with μ = counting measure.

- μ is not a Borel measure because it is not locally finite (every open set has measure ∞!)
- μ is inner regular
- μ is not outer regular (let $x_0 \in \mathbb{R}$ $\Rightarrow$ $\mu(\{x_0\}) = 1$, but $\mu(V) = \infty$ for every neighbourhood of x_0).

iii) It can be shown: Every Borel measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is regular,
in particular β^n is regular.
(Bauer, Thm 26.3. On a polish space X every locally finite Borel measure is a σ-finite Radon measure.
$\mathbb{R}$ is Polish and every Borel measure is locally finite (because $\mathbb{R}$ is locally cpt)
$\leadsto$ every Borel measure on $\mathbb{R}$ is a Radon measure.
By Cor. 26.4, every Radon measure on a Polish space is outer regular.
In summary: Every Borel measure on $\mathcal{B}(\mathbb{R}^n)$ is regular.)

Definition 5.4. X Hausdorff space, $\mu: \mathcal{B}(X) \to [0,\infty]$ measure.

μ is a Radon measure on X $:\Longleftrightarrow$ μ is locally finite and inner regular.

(If X additionally locally cpt, then a Radon measure is an inner regular Borel measure)

Recall. X top space is first countable

$:\Longleftrightarrow$ every $a \in X$ has a countable neighbourhood basis

$:\Longleftrightarrow$ $\exists (V_j)_{j=1}^\infty$ sequence of neighbourhoods of a s.t. for every neighbourhood W of a $\exists j_0$ s.t. $V_{j_0} \subseteq W$.

Obviously:
- the V_j can be chosen to be open since every neighbourhood V_j contains an open set W_j with $a \in W_j \subseteq V_j$
- the V_j can be chosen such that $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ (take $V_1' = V_1$, $V_2' := V_1 \cap V_2$, $V_3' = V_1 \cap V_2 \cap V_3 \dots$)

Lemma 5.5. X Hausdorff space, first countable; $\mu: \mathcal{B}(X) \to [0,\infty]$ Borel measure.
If μ is inner regular, then μ is locally finite (hence a Radon measure).
(It suffices to assume that every open set in X is inner regular).

Proof Assume μ is not locally finite.

$\Rightarrow \exists x \in X$ s.t. $\forall V$ neighbourhood of x $\mu(V) = \infty$.

Let $(V_j)_{j=1}^\infty$ be an ~~the~~ open neighbourhood basis of x. Without restriction $V_1 \supseteq V_2 \supseteq \dots$

Then: $\bigcap_{j=1}^\infty V_j = \{x\}$ ("$\subseteq$" because $x \in V_j$ for all $j \in \mathbb{N}$.
"$\supseteq$" because: assume $\exists y \neq x$, $y \in \bigcap_{j=1}^\infty V_j$.
X Hausdorff $\Rightarrow \exists W_x, W_y$ open, s.t. $x \in W_x$, $y \in W_y$, $W_x \cap W_y = \emptyset$.
W_x neighbourhood of x $\Rightarrow \exists j_0$: $V_{j_0} \subseteq W_x$
$\Rightarrow V_{j_0} \cap W_y = \emptyset \Rightarrow y \notin V_{j_0} \Rightarrow y \notin \bigcap_{j=1}^\infty V_j$ ↯)

By assumption: $\forall n \in \mathbb{N}$ $\mu(V_n) = \infty$

$\Rightarrow \forall n \in \mathbb{N}$ $\exists K_n$ cpt. s.t. $K_n \subseteq V_n$, $\mu(K_n) > n$.

Let $K := \bigcup_{n=1}^\infty K_n \cup \{x\}$.

Then K is cpt. because: Let $\mathcal{U} = (U_\lambda)_{\lambda \in \Lambda}$ an open cover of K.

$\Rightarrow \exists \lambda_0$ s.t. $x \in U_{\lambda_0}$

$\Rightarrow \exists j_0 \in \mathbb{N}$ s.t. $V_{j_0} \subseteq U_{\lambda_0}$, in particular $\bigcup_{j \geq j_0} K_j \subseteq U_{\lambda_0}$.

Since $\bigcup_{j=1}^{j_0-1} K_j$ is compact, there ex. $\lambda_1, \dots, \lambda_m$ s.t. $\bigcup_{j=1}^{j_0-1} K_j \subseteq \bigcup_{j=1}^m U_{\lambda_j}$.

$\Rightarrow K \subseteq \bigcup_{j=0}^m U_{\lambda_j}$ $\Rightarrow$ K cpt.

$\Rightarrow \mu(K) < \infty$.

On the other hand: $\forall n: K_n \subseteq K \Rightarrow \forall n$ $\mu(K) \geq \mu(K_n) > n$ ↯.

Definition 5.6 X top space, $A \subseteq X$ is called σ-compact
$:\Longleftrightarrow \exists (K_n)_n$, $K_n \subseteq X$ cpt s.t. $A = \bigcup_{n=1}^\infty K_n$

Proposition 5.7. X top space, X Hausdorff, $\mu: \mathcal{B}(X) \to [0,\infty]$ measure.

i) $A \subseteq X$ σ-cpt $\Rightarrow$ A inner regular

ii) μ finite Radon measure $\Rightarrow$ μ regular

iii) μ Radon measure, $K \subseteq X$ cpt $\Rightarrow$ K outer regular.
In particular: $\forall \varepsilon > 0$ $\exists U$ open s.t. $K \subseteq U$ and $\mu(U) \leq \mu(K) + \varepsilon$

Proof

i) Let K_n $(n \in \mathbb{N})$ cpt. sets s.t. $A = \bigcup_{n=1}^\infty K_n$. Since finite unions of cpt sets are cpt, we can assume $K_1 \subseteq K_2 \subseteq \dots$

$\Rightarrow \mu(A) = \lim_{n\to\infty} \mu(K_n)$.

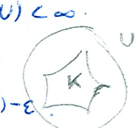
$\Rightarrow \mu(A) \leq \sup\{\mu(K) \mid K \text{ cpt}, K \subseteq A\}$; "$\geq$" is clear.

ii) $A \in \mathcal{B}(X)$. $\Rightarrow X \setminus A \in \mathcal{B}(X) \Rightarrow \exists K_1, K_2, \dots,$ cpt, $\subseteq X \setminus A$, s.t.
$\mu(X\setminus A) = \lim_{n\to\infty} \mu(K_n)$

$\Rightarrow \mu(A) \underset{\mu \text{ finite}}{=} \mu(X) - \mu(X \setminus A) = \mu(X) - \lim_{n\to\infty} \mu(K_n) = \lim_{n\to\infty} \mu(X \setminus K_n)$.

$\Rightarrow$ A outer regular because: $\forall n$ $X \setminus K_n$ open and $A \subseteq X \setminus K_n$.

iii) μ Radon measure and K cpt. ($\Rightarrow$ μ locally finite).
By Observation 5.3. d): $\exists U \subseteq X$ open, s.t. $K \subseteq U$, $\mu(U) < \infty$.
$U \setminus K$ open & μ inner regular.



$\Rightarrow \forall \varepsilon > 0$ $\exists L_\varepsilon \subseteq U \setminus K$, L_ε cpt. s.t. $\mu(L_\varepsilon) \geq \mu(U \setminus K) - \varepsilon$

$\Rightarrow U \setminus L_\varepsilon$ is open, $K \subseteq U \setminus L_\varepsilon$, and $\mu(U \setminus L_\varepsilon) = \mu(U) - \mu(L_\varepsilon)$
$\leq \mu(U) - \mu(U \setminus K) + \varepsilon$
$= \mu(K) + \varepsilon$ □

$\mu^*: \mathcal{P}(X) \to [0,\infty]$, $\mu^*(\emptyset) = 0$, $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$ and $\mu^*(\bigcup A_j) \le \sum_{j=1}^{\infty} \mu^*(A_j)$

Lemma 5.8. X Hausdorff space, μ^* outer measure on X.
Assume that:

i) $\forall A \subseteq X$ $\quad \mu^*(A) = \inf\{\mu^*(U) \mid A \subseteq U,\ U \text{ open}\}$

ii) $\forall U \subseteq X$, U open. $\quad \mu^*(U) = \sup\{\mu^*(K) \mid K \subseteq U,\ K \text{ cpt}\}$

iii) $\forall K_1, K_2 \subseteq X$ cpt with $K_1 \cap K_2 = \emptyset$: $\quad \mu^*(K_1 \cup K_2) = \mu^*(K_1) + \mu^*(K_2)$

$\Rightarrow$ $\mu^*|_{\mathcal{B}(X)}$ is a measure.

Proof. Let $\mathcal{A}^* :=$ σ-algebra of all μ^*-measurable sets, i.e.

$$A \in \mathcal{A}^* \iff \forall M \subseteq X \quad \mu^*(M) = \mu^*(M \cap A) + \mu^*(M \setminus A).$$
$$\iff \forall M \subseteq X \quad \mu^*(M) \ge \mu^*(M \cap A) + \mu^*(M \setminus A) \qquad (*)$$

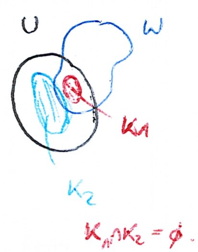
because "$\le$" follows from subadditivity of μ^*.

We know: $\mu^*|_{\mathcal{A}^*}$ is a measure, so we only have to show: $\mathcal{B}(X) \subseteq \mathcal{A}^*$.
Since the open sets generate $\mathcal{B}(X)$, we only have to show:
Every open set U belongs to $\mathcal{A}^*$ (i.e. satisfies $(*)$).
This will be done in two steps.

① $A \in \mathcal{A}^* \iff \forall U \subseteq X$, U open $\quad \mu^*(U) \ge \mu^*(U \setminus A) + \mu^*(M \cap A)$.

Proof. "$\Rightarrow$" is clear.
"$\Leftarrow$" Assume $(*)$ holds for all open sets. Let $M \subseteq X$ arbitrary.
By monotonicity of μ^*, we have for all $U \subseteq X$ open with $M \subseteq U$:
$$\underbrace{\mu^*(U) \ge}_{\text{assumption}} \mu^*(U \setminus A) + \mu^*(U \cap A) \ge \mu^*(M \setminus A) + \mu^*(M \cap A)$$
Taking the infimum over all open sets U containing M, we obtain by i):
$$\mu^*(M) \ge \mu^*(M \setminus A) + \mu^*(M \cap A).$$
□

② $W \subseteq X$ open $\Rightarrow W \in \mathcal{A}^*$.



Proof. Let $W \subseteq X$ open; by ① we have to show:
$\forall U \subseteq X$ open: $\quad \mu^*(U) \ge \mu^*(U \cap W) + \mu^*(U \setminus W)$
Choose $K_1 \subseteq U \cap W$ cpt and $K_2 \subseteq \underbrace{U \setminus K_1}_{\text{open!}}$ cpt.

$$\Rightarrow \underbrace{\mu^*(U) \ge}_{\text{isot. of } \mu^*} \mu^*(K_1 \cup K_2) \overset{iii)}{=} \mu^*(K_1) + \mu^*(K_2) \qquad (**)$$

Since $\sup\{\mu^*(K_2) \mid K_2 \text{ cpt},\ K_2 \subseteq U \setminus K_1\} = \mu^*(U \setminus K_1)$ by ii)
and $\mu^*(U \setminus K_1) \ge \mu^*(U \setminus W)$ (because $U \setminus K_1 \supseteq U \setminus W$)

$(**)$ $\Rightarrow$ $\mu^*(U) \ge \mu^*(K_1) + \mu^*(U \setminus W)$

Taking the supremum over all $K_1 \subseteq U \cap W$, K_1 cpt, ii) gives
$$\mu^*(U) \ge \mu^*(U \cap W) + \mu^*(U \setminus W).$$
□

~~5.2. Locally compact Hausdorff spaces~~