4. The transformation theorem for Lebesgue integrals.

4.1. Integration with respect to an image measure

Theorem & Definition 4.1.

$(X, \mathcal{A}, \mu)$ measure space, $(Y, \mathcal{B})$ measurable space; $T: X \to Y$ mb.

Define $\nu: \mathcal{B} \to [0,\infty]$, $\nu(B) := \mu(T^{-1}(B))$

Then ν is a measure on $(Y, \mathcal{B})$, called the image of μ under the mapping T, denoted by $\nu =: T(\mu)$.

Proof. $\forall B \in \mathcal{B}$, $T^{-1}(B) \in \mathcal{A}$, hence ν is well-defined.

Obviously: $\cdot$ $\forall B \in \mathcal{B}$: $\nu(B) = \mu(T^{-1}(B)) \geq 0$,

$\cdot$ $\nu(\emptyset) = \mu(\emptyset) = 0$

$\cdot$ σ-additivity: $(B_j)_j \subseteq \mathcal{B}$, pairwise disjoint

$\Rightarrow (T^{-1}(B_j))_j \in \mathcal{A}$, $-$ " $-$

and $\nu\left(\dot{\bigcup}_{j=1}^{\infty} B_j\right) = \mu\left(\dot{\bigcup}_{j=1}^{\infty} T^{-1}(B_j)\right) = \sum_{j=1}^{\infty} \mu(T^{-1}(B_j)) = \sum_{j=1}^{\infty} \nu(B_j)$ $\square$

Important Example: Affine mappings on $\mathbb{R}^n$

Recall: $f: \mathbb{R}^n \to \mathbb{R}^n$ affine $\iff$ $\exists c \in \mathbb{R}^n$, $A: \mathbb{R}^n \to \mathbb{R}^n$ linear s.t. $f(x) = Ax + c$.

- f bijective $\iff$ A bijective $\iff$ $\det A =: \det f \neq 0$
- The inverse of a bijective affine map is again a bijective affine map.
  $\left(f(x) = Ax + c =: y \iff x = A^{-1}y - A^{-1}c = f^{-1}(x). \right)$

Notation: $\mathcal{J}^n := \{ [a,b[\mid a,b \in \mathbb{R}^n, a \leq b \}$ semiring on $\mathbb{R}^n$

$\mathcal{B}^n :=$ Borel σ-algebra on $\mathbb{R}^n$ (Recall: $\mathcal{J}^n$ generates $\mathcal{B}^n$)

$\beta^n :=$ Borel-Lebesgue measure on $\mathcal{B}^n$

$\mathcal{L}^n :=$ Lebesgue σ-algebra on $\mathbb{R}^n$ = completion of $\mathcal{B}^n$

$\lambda^n :=$ Lebesgue measure on $\mathcal{L}^n$ = completion of β^n.

Translation invariance of λ^n and β^n: $\forall c \in \mathbb{R}^n$: $T_c: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + c$

Theorem 4.2. $c \in \mathbb{R}^n$.

i) T_c is $\mathcal{B}^n$-$\mathcal{B}^n$-mb and β^n is translation invariant (i.e. $\forall c$: $T_c(\beta^n) = \beta^n$)

ii) T_c is $\mathcal{L}^n$-$\mathcal{L}^n$-mb and λ^n is translation invariant.

iii) μ translation invariant measure on $\mathcal{B}^n$ (or $\mathcal{L}^n$) with $\mu([0,1[^n) = 1$, then $\mu = \beta^n$ (or $\mu = \lambda^n$).

Proof.

i) T_c cont $\Rightarrow$ T_c $\mathcal{B}^n$-$\mathcal{B}^n$-mb. $\Rightarrow$ $T_c\beta^n$ is a measure on $\mathcal{B}^n$

Let $a, b \in \mathbb{R}^n$, $a \leq b$.

$\Rightarrow T_c(\beta^n)([a,b[) = \beta^n(T_c^{-1}[a,b[) = \beta^n([a-c, b-c[) = \beta^n([a,b[)$.

$\Rightarrow T_c(\beta^n)|_{\mathcal{J}^n} = \beta^n|_{\mathcal{J}^n}$

Since $\mathcal{J}^n$ generates $\mathcal{B}^n$ and $T_c(\beta^n)$ and β^n are σ-finite, their extensions to $\mathcal{B}^n$, $T_c(\beta^n)$ and β^n, coincide on $\mathcal{B}^n$. $\square$

ii) Let $M \in \mathcal{L}^n$. To show: $T_c^{-1}(M) \in \mathcal{L}^n$.

Choose $A, B \in \mathcal{B}^n$, $N \in \mathcal{L}^n$ st. $M = A \cup N$, $N \subseteq B$, $\beta^n(B) = 0$.

$\Rightarrow T_c^{-1}(M) = \underbrace{T_c^{-1}(A)}_{\in \mathcal{B}^n} \cup T_c^{-1}(N)$

Since $T_c^{-1}(N) \subseteq T_c^{-1}(B)$ and $\beta^n(T_c^{-1}(B)) = (T_c\beta^n)(B) \overset{i)}{=} \beta^n(B) = 0$, we have that $T_c^{-1}(N) \in \mathcal{L}^n$, implying that $T_c^{-1}(M) \in \mathcal{L}^n$, and

$\lambda^n(T_c^{-1}(M)) = \lambda^n(T_c^{-1}(A)) + \underbrace{\lambda^n(T_c^{-1}(N))}_{=0}$

$= \beta^n(T_c^{-1}(A)) \overset{i)}{=} \beta^n(A) = \lambda^n(A \cup N) = \lambda^n(M)$. $\square$

iii) Let μ as in the claim and $d_1, \dots, d_n \in \mathbb{N}$.

$\Rightarrow [0,1[^n = \dot{\bigcup}_{0 \leq k_1 < d_1} \cdots \dot{\bigcup}_{0 \leq k_n < d_n} \left[\underbrace{\left([0,\tfrac{1}{d_1}[\times \dots \times [0,\tfrac{1}{d_n}[\right)}_{=: Q} + \left(\tfrac{k_1}{d_1}, \dots, \tfrac{k_n}{d_n}\right) \right]$

Translation invariance of μ gives:

$1 = \mu([0,1[^n) = d_1 \cdot \ldots \cdot d_n \cdot \mu(Q)$

$\Rightarrow \mu(Q) = \frac{1}{d_1 \cdot \ldots \cdot d_n} = \beta^n(Q) = \lambda^n(Q)$.

$\Rightarrow \mu = \beta^n = \lambda^n$ on $\mathcal{J}^n_{\mathbb{Q}} = \{ [a,b[\in \mathcal{J}^n \mid a,b \in \mathbb{Q}^n \}$

Since $\mathcal{J}^n_{\mathbb{Q}}$ generates $\mathcal{B}^n$ $\Rightarrow$ $\mu = \beta^n$ on $\mathcal{B}^n$. Completion gives: $\mu = \lambda^n$ on $\mathcal{L}^n$.

Corollary 4.3. μ translation invariant measure on $\mathcal{B}^n$ (or $\mathcal{L}^n$), and $\mu([0,1[^n) = \alpha < \infty$.

Then: $\mu = \alpha\beta^n$ (or $\mu = \alpha\lambda^n$).

Proof. Case 1. $\alpha = 0$ $\Rightarrow$ $\mu(\mathbb{R}) = \mu\left(\bigcup_{q\in\mathbb{Z}^n} [0,1[^n + q\right) = \sum_{q\in\mathbb{Z}^n} 0 = 0$

$\Rightarrow \mu \equiv 0$.

Case 2. $\alpha \in (0,\infty)$: $\alpha\mu$ is translation invariant and $\alpha\mu([0,1[^n) = 1$.

$\Rightarrow \alpha\mu = \beta^n$ (or $\alpha\mu = \lambda^n$) by Thm 4.2 iii). ☑

Note $\alpha \neq \infty$ is necessary, because: μ counting measure on $\mathbb{R}^n$.

$\Rightarrow \mu([0,1[^n) = \infty$; and μ is translation invariant,

But: $\mu \neq \infty\cdot\beta^n$ (because: $\mu(\{0\}) = 1 \neq \underbrace{\infty\cdot\beta^n(\{0\})}_{\text{not defined}}$)

Note. In the corollary 4.3 and Thm 4.2. we can also take $[0,1]^n$ etc. instead of $[0,1[^n$.

Theorem 4.4. $f: \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = Ax + c$ affine map with $\det(A) \neq 0$.

$\Rightarrow$ f is $\mathcal{B}^n$-$\mathcal{B}^n$ and $\mathcal{L}^n$-$\mathcal{L}^n$-mb.

and $f(\beta^n) = |\det f|^{-1}\beta^n$

$f(\lambda^n) = |\det f|^{-1}\lambda^n$.

Proof. We prove the thm. only for $\mathcal{B}^n$ and β^n. The claim for $\mathcal{L}^n$ and λ^n follows then as in the proof of Thm. 4.2 ii).

f continuous $\Rightarrow$ f is $\mathcal{B}^n$-$\mathcal{B}^n$-mb.

Note: $f = T_c \circ A$, so by thm. 4.2 we can assume $c = 0$, so $f(x) = Ax$, $x \in \mathbb{R}^n$.

i) $f(\beta^n)$ is translation invariant:

Let $\gamma \in \mathbb{R}^n$, $B \in \mathcal{B}^n$

$$\Rightarrow T_\gamma(f(\beta^n))(B) = f(\beta^n)(T_\gamma^{-1}B) = f(\beta^n)(B-\gamma)$$
$$= \beta^n(A^{-1}B - A^{-1}\gamma)$$
$$\xrightarrow{\beta^n \text{ transl. inv.}} = \beta^n(A^{-1}B)$$
$$= f(\beta^n)(B).$$

ii) $f(\beta^n)([0,1[^n) < \infty$:

$$f(\beta^n)([0,1[^n) \leq f(\beta^n)([0,1]^n) = \beta^n\Big(\overbrace{\underbrace{A^{-1}}_{\text{cont}}(\underbrace{[0,1]^n}_{\text{cpt}})}^{\text{cpt, hence bdd}}\Big) < \infty$$

(or: $f(\beta^n)([0,1[) = \beta^n(\underbrace{A^{-1}[0,1[}_{\subseteq [0,\|A^{-1}\|[\text{ bdd}}) < \infty$

By i), ii) and Corollary 4.3: $\exists\, \alpha \in \mathbb{R}$ s.t. $\boxed{f(\beta^n) = \alpha\beta^n}$ (*)

To show: $\alpha = |\det f|^{-1}$.

Case 1 f is orthogonal ($\Leftrightarrow AA^* = A^*A = 1$)

$\Rightarrow K_1(0) = f^{-1}(K_1(0))$.

$\Rightarrow \alpha\beta^n(K_1(0)) = f(\beta^n)(K_1(0)) = \beta^n(f^{-1}(K_1(0))) = \beta^n(K_1(0))$

$\Rightarrow \alpha = 1$ $(\beta^n(K_1(0)) \neq 0\,!)$

$= |\det f|^{-1}$.

Case 2 $A = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$ w.r.t. standard basis $e_1, \dots, e_n$. Note: all $\lambda_j \neq 0$ because $\det A \neq 0$.

$$\Rightarrow \alpha = \alpha\cdot\beta^n([0,1[^n) = f(\beta^n)([0,1[^n) = \beta^n(A^{-1}[0,1[^n)$$
$$= \beta^n\left(\left[0,\tfrac{1}{|\lambda_1|}\right[\times \dots \times \left[0,\tfrac{1}{|\lambda_n|}\right[\right) = |\lambda_1\cdots\lambda_n|^{-1}$$
$$= |\det A|^{-1}.$$

Case 3. $A \in GL(n,\mathbb{R})$ arbitrary

$\Rightarrow AA^*$ symm. & positive.

$\Rightarrow \exists\, V$ orthogonal, D diagonal s.t. $AA^* = VD^2V^*$.

$\Rightarrow W := D^{-1}V^*A$ is orthogonal $(WW^* = D^{-1}V^*AA^*VD^{-1} = D^{-1}\overbrace{V^*V}^{=1}D^2\overbrace{V^*V}^{=1}D^{-1} = 1$, $W^*W = \dots = 1$.$)$

$\Rightarrow A = VDW$

$\Rightarrow \forall B \in \mathcal{B}^n$:

$$f(\beta^n)(B) = (VDW)(\beta^n)(B) = W(\beta^n)((VD)^{-1}(B))$$
$$\xrightarrow{\text{Case 1}} \beta^n((VD)^{-1}(B)) = (D\beta^n)(V^{-1}(B))$$
$$\text{Case 2} \longrightarrow = |\det D|^{-1}\beta^n(V^{-1}(B))$$
$$\text{Case 1.} \longrightarrow = |\det D|^{-1}\beta^n(B)$$
$$= |\det A|^{-1}\beta^n(B)$$

$(\det A = \det(VDW) = \det V \det D \det W = \det D.)$

Corollary 4.5. $f: \mathbb{R}^n \to \mathbb{R}^n$ bijective and affine.

$\Rightarrow \forall A \in \mathcal{B}^n$: $f(A) \in \mathcal{B}^n$ and $\beta^n(f(A)) = |\det f|\, \beta^n(A)$

$\forall B \in \mathcal{L}^n$: $f(B) \in \mathcal{L}^n$ and $\lambda^n(f(A)) = |\det f|\, \lambda^n(B)$.

Proof. Apply thm 4.4. to the affine bijection f^{-1}.

4.2. Transformation Formula.

Notation $X \in \mathcal{B}^n \Rightarrow \mathcal{B}^n_X := \mathcal{B}^n|_X$, $\beta^n_X := \beta^n|_X$

$X \in \mathcal{L}^n \Rightarrow \mathcal{L}^n_X := \mathcal{L}^n|_X$, $\lambda^n_X := \lambda^n|_X$.

Recall. $X, Y \subseteq \mathbb{R}^n$ open.

$\varphi: X \to Y$ is called C^1-diffeomorphism

$:\Leftrightarrow$ φ is bijection and φ and φ^{-1} are C^1 (differentiable with cont. derivative)

Lemma 4.6. $X, Y \subseteq \mathbb{R}^n$ open, $\varphi: X \to Y$ C^1-diffeom. (φ homeomorphism is sufficient)

Then: $\mathcal{B}^n_Y = \varphi(\mathcal{B}^n_X) := \{\varphi(A) \mid A \in \mathcal{B}^n_X\}$.

Proof. By assumption, φ and φ^{-1} are mb (they are cont!)

- $B \in \mathcal{B}^n_Y \Rightarrow A := \varphi^{-1}(B) \in \mathcal{B}^n_X \Rightarrow B = \varphi(A) \in \varphi(\mathcal{B}^n_X)$
- $B \in \varphi(\mathcal{B}^n_X) \Rightarrow \exists A \in \mathcal{B}^n_X$ st. $B = \varphi(A) \Rightarrow B = \underbrace{(\varphi^{-1})}_{mb!}{}^{-1}(A) \in \mathcal{B}^n_Y$.

Recall. (~~Intermediate~~ Mean value thm)

$\varphi: X \to Y$ differentiable, $a, b \in X$ st. the line connecting a and b lies in X.
$X, Y \subseteq \mathbb{R}^n$ open

$\Rightarrow \|\varphi(x) - \varphi(y)\| \leq \|x - y\| \sup\{\|D\varphi(x + \lambda(y-x))\| \mid 0 \leq \lambda \leq 1\}$.

Proof. $\varphi(y) - \varphi(x) = \int_0^1 \frac{d}{dt}\varphi(x + t(y-x))\,dt = \int_0^1 D\varphi(x + t(y-x))(y-x)\,dt$

$\Rightarrow \|\varphi(y) - \varphi(x)\| \leq \int_0^1 \|D\varphi(x + t(y-x))\|\,\|y-x\|\,dt$

$\leq \|y-x\| \cdot \sup\{\|D\varphi(x + t(y-x))\| \mid t \in [0,1]\}$.

□

Theorem 4.7. (Transformation Theorem)

$X, Y \subseteq \mathbb{R}^n$ open. $\varphi: X \to Y$ C^1-diffeomorphism.

Then. i) $\forall A \in \mathfrak{B}_X^n$ $\quad \beta^n(\varphi(A)) = \int_A |\operatorname{Det} D\varphi| \, d\beta^n$

ii) $f: Y \to [0,\infty]$ mb $\Rightarrow \int_Y f \, d\beta^n = \int_X f\circ\varphi \, |\operatorname{Det} D\varphi| \, d\beta^n$

iii) $f: Y \to \mathbb{C}$ or $\mathbb{R}$. Then:

f is β^n integrable over Y $\Leftrightarrow$ $f\circ\varphi \cdot |\det D\varphi|$ is β^n-integrable over X

In this case: $\int_Y f \, d\beta^n = \int_X f\circ\varphi \, |\det D\varphi| \, d\beta^n$.

Proof

Let $H = \{ [a,b[\subseteq \mathbb{R}^n \mid a \le b,\ a,b \in \bigcup_{k=1}^{\infty} 2^{-k}\mathbb{Z}^n,\ [a,b] \subseteq X \}$

Then: • H is a semiring

• Every open subset of X is countable union of members of X

$\Rightarrow$ $\mathfrak{B}_X^n$ is generated by H.

$\boxed{\text{Step 1} \quad \forall J \in H: \ \beta^n(\varphi(J)) \le \int_J |\det D\varphi| \, d\beta^n}$

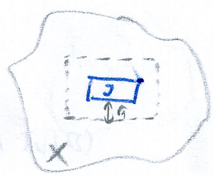
Proof. $J \subseteq H \Rightarrow \bar{J} \subseteq X$. $\boxed{\text{Let } \varepsilon > 0}$

$D\varphi$, $\underbrace{(D\varphi)^{-1}}_{\text{matrix inverse}}$ cont in X $\Rightarrow$ unif. cont. in $\bar{J}$.

Let $M := \sup\{ \| [(D\varphi)(x)]^{-1} \| \mid x \in \bar{J} \} < \infty$.

• $\bar{J} \subseteq X \Rightarrow \exists r_1 > 0$ tq. $\forall a \in J$: $\overline{B_{r_1}(a)} \subseteq X$.

by

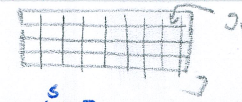


• $D\varphi$ unif cont on compact sets.

$\Rightarrow \exists r_2 > 0$ tq. $\forall a \in \bar{J}$ $\overline{B_{r_2}(a)} \subseteq X$ and

$\forall x \in \overline{B_{r_2}(a)}$: $\quad \| D\varphi(a) - D\varphi(x) \| \le \frac{\varepsilon}{M\sqrt{n}}$. $\quad (*)$

Let $r = \min\{r_1, r_2\}$. (Can choose $r = r_2$).



Now: Choose $J_\nu \in H$ s.t. $J = \dot{\bigcup}_{\nu=1}^{s} J_\nu$ such that edge lengths δ of all J_ν are equal and $\delta < r/\sqrt{n}$

Now for any ν and any $b \in \bar{J}_\nu$: $\quad \bar{J}_\nu \subseteq \overline{B_r(b)} \subseteq X$ (Choice of r)

$\forall a \in \bar{J}_\nu: \|a-b\| = \sqrt{|a_1-b_1|^2 + \dots} \le \sqrt{n\delta^2} = \sqrt{n}\,\delta < r$

For every ν choose $a_\nu \in \bar{J}_\nu$ s.t.

$|\det D\varphi(a_\nu)| = \min\{ |\det D\varphi(a)| \mid a \in \bar{J}_\nu \}$ $\quad$ [a_ν ex. because $|\det D\varphi(\cdot)|$ is cont & $\bar{J}_\nu$ closed)

Let $x \in J_\nu \Rightarrow \|x - a_\nu\| < \sqrt{n}\,\delta$.

Apply mean value theorem to $J_\nu \to \mathbb{R}^n$, $x \mapsto \varphi(x) - D\varphi(a_\nu)x$.

$\Rightarrow \| \varphi(x) - D\varphi(a_\nu)x - \varphi(a_\nu) + D\varphi(a_\nu)a_\nu \|$

$\le \sup\{ \|D\varphi(y) - D\varphi(a_\nu)\| \mid y \in \bar{J}_\nu \} \, \|x - a_\nu\|$

$\overset{(*)}{\le} \frac{\varepsilon}{M\sqrt{n}} \|x - a_\nu\| \le \varepsilon\delta/M$.

$\Rightarrow \varphi(x) \in \varphi(a_\nu) + D\varphi(a_\nu)(x - a_\nu) + K_{\varepsilon\delta/M}(0)$

$= \varphi(a_\nu) + D\varphi(a_\nu)\Big(x - a_\nu + \overbrace{\underbrace{(D\varphi(a_\nu))^{-1}}_{\|\cdot\| \le M} K_{\varepsilon\delta/M}(0)}^{\subseteq K_{\varepsilon\delta}(0)} \Big)$

$\subseteq \varphi(a_\nu) + D\varphi(a_\nu)\big(J_\nu - a_\nu + K_{\varepsilon\delta}(0) \big)$

$= \varphi(a_\nu) - D\varphi(a_\nu)a_\nu + D\varphi(a_\nu)\big(\underbrace{J_\nu + K_{\varepsilon\delta}(0)}_{} \big)$

$J_\nu + K_{\varepsilon\delta}(0) \subseteq$ Cube with edge length $\le \delta + 2\varepsilon\delta$.

$\Rightarrow J_\nu \subseteq \varphi(a_\nu) - D\varphi(a_\nu)a_\nu + D\varphi(a_\nu)\{\text{cube with edge} \le \delta + 2\varepsilon\delta\}$

$\Rightarrow \beta^n(J_\nu) \le \beta^n\big(\underbrace{D\varphi(a_\nu)}_{\in GL(n,\mathbb{R})} \{\dots\} \big) \le |\det D\varphi(a_\nu)| \, (\delta + 2\varepsilon\delta)^n$

$= |\det D\varphi(a_\nu)| \underbrace{\beta^n(J_\nu)}_{= \delta^n} \cdot (1+2\varepsilon)^n \le \int_{J_\nu} |\det D\varphi| \, d\beta^n \, (1+2\varepsilon)^n$

$\Rightarrow \beta^n(J) = \sum_{\nu=1}^{s} \beta^n(J_\nu) = (1+2\varepsilon)^n \int_J |\det D\varphi| \, d\beta^n$.

Letting $\varepsilon \to 0$ proves the claim. $\square$

$\boxed{\textbf{Step 2.}\ \forall A \in \mathcal{B}_X^n:\ \beta^n(\varphi(A)) \le \int_A |\det D\varphi|\, d\beta^n}$

Proof. True for $A \in H$ by step 1.

$$\left.\begin{array}{l} \mu_1 : H \longrightarrow [0,\infty],\ A \longmapsto \beta^n(\varphi(A)) \\ \mu_2 : H \longrightarrow [0,\infty],\ A \longmapsto \int_A |\det D\varphi|\, d\beta^n \end{array}\right\} \text{ are } \sigma\text{-finite measures on } H.$$

$\rightarrow$ μ_1 and μ_2 have unique extensions to $\sigma(H) = \mathcal{B}_X^n$. Since $\mu_1^* \le \mu_2^*$, the inequality remains true for the extensions. □

$\boxed{\textbf{Step 3.}\ f: Y \longrightarrow [0,\infty] \text{ mb} \Rightarrow \int_Y f\, d\beta^n \le \int_X f\circ\varphi\, |\det D\varphi|\, d\beta^n.}$ $(**)$

Proof. True for charact function $f = \chi_B$ with $B \in \mathcal{B}_Y^n$ by Step 2:

$$\int_Y f\, d\beta^n = \int_Y \chi_B\, d\beta^n = \beta^n(B) = \beta^n(\varphi(\varphi^{-1}(B))) \le \int_{\varphi^{-1}(B)} |\det D\varphi|\, d\beta^n$$
$$= \int_X \chi_{\varphi^{-1}(B)}\, |\det D\varphi|\, d\beta^n = \int_X \chi_B\circ\varphi\, |\det D\varphi|\, d\beta^n.$$

$\rightarrow$ $(**)$ is true for all $f \in E^+(Y, \mathcal{B}_Y^n)$.

Now let $f: Y \longrightarrow [0,\infty]$ mb and $(s_n)_n \subseteq E^+(Y, \mathcal{B}_Y^n)$ st.
$0 \le s_1 \le s_2 \le \ldots \le f$ and $s_n \longrightarrow f$ pointwise.

$$\Rightarrow \int_Y f\, d\beta^n \overset{\text{mon. conv.}}{=} \lim_{n\to\infty} \int_Y s_n\, d\beta^n = \lim_{n\to\infty} \int_X s_n\circ\varphi\, |\det D\varphi|\, d\beta^n$$
$$\overset{\text{mon. conv.}}{=} \int_X \lim_{n\to\infty} s_n\circ\varphi\, |\det D\varphi|\, d\beta^n = \int_X f\circ\varphi\, |\det D\varphi|\, d\beta^n.$$

$\boxed{\textbf{Step 4.}\ f: Y \longrightarrow [0,\infty] \text{ mb} \Rightarrow \int_Y f\, d\beta^n = \int_X f\circ\varphi\, |\det D\varphi|\, d\beta^n}$

Apply Step 3 to $\underbrace{\varphi^{-1}}_{C^1\text{-diffeo}}$ instead of φ and $\underbrace{f\circ\varphi\, |\det D\varphi|}_{\text{mb},\ \ge 0}$ instead of f

$$\Rightarrow \int_X f\circ\varphi\, |\det D\varphi|\, d\beta^n \le \int_Y f\circ\varphi\circ\varphi^{-1}\, |\det (D\varphi\circ\varphi^{-1})|\, |\det D\varphi^{-1}|\, d\beta^n \overset{(*)}{=} \int_Y f\, d\beta^n.$$ □

$(*)$ $1 = \det(D\mathbb{1}) = \det(D(\varphi\circ\varphi^{-1})) = \det((D\varphi)\circ\varphi^{-1} \cdot D\varphi^{-1})$
$= \det((D\varphi)\circ\varphi^{-1}) \cdot \det(D\varphi^{-1})$

Now all statements are clear:

ii) is proved in Step 4

i) is the special case $f = \chi_{\varphi(A)}$

iii) follows from ii) applied to $(\mathrm{Re} f)_\pm, (\mathrm{Im} f)_\pm$. □

Corollary 4.8. $X, Y \subseteq \mathbb{R}^n$ open, $\varphi: X \longrightarrow Y$ C^1-diffeo, $A \in \mathcal{B}_X^n$.

$\Rightarrow$ $\int_{\varphi(A)} f\, d\beta^n = \int_A f\circ\varphi\, |\det D\varphi|\, d\beta^n$. if $\left\{\begin{array}{l} f: \varphi(A) \longrightarrow \mathbb{R} \text{ or } \mathbb{C} \text{ integrable} \\ \text{or } f: \varphi(A) \longrightarrow [0,\infty]. \end{array}\right.$

Proof. Apply the theorem to $g: Y \longrightarrow \mathbb{K}$, $g = \chi_{\varphi(A)} f$

Corollary 4.9. The transformation formula is also true for Lebesgue sets, Lebesgue mb fct's (instead of Borel).

Proof. By thm 4.7. i): $\beta^n(A) = 0 \Leftrightarrow \beta^n(\varphi(A)) = 0$ for all $A \in \mathcal{B}_X^n$.

$\Rightarrow$ φ defines a bijection $\mathcal{L}_X^n \longrightarrow \mathcal{L}_Y^n$ which maps zero sets in zero sets.

Since λ_X^n, λ_Y^n are the completions of β_X^n, β_Y^n, the corollary is proved. □

Question. What if $\varphi: X \longrightarrow Y$ C^1, but not everywhere $\det D\varphi \ne 0$?

Theorem (Sard) $X \subseteq \mathbb{R}^n$ open, $\varphi: X \longrightarrow \mathbb{R}^n$. C^1.

$C := \{x \in X \mid \mathrm{rang}(D\varphi(x)) < n\}$ =: set of critical points of φ

$\Rightarrow$ $\beta^n(\varphi(C)) = 0$ $\longleftarrow$ closed!

Application: $X \subseteq \mathbb{R}^n$ open; $\varphi: X \longrightarrow \mathbb{R}^n$, C_1. $C := \{x \in X \mid \mathrm{rang}\, D\varphi(x) < n\}$

Suppose $\varphi|_{X\setminus C}$ injective

$\Rightarrow$ ($f: \varphi(X) \longrightarrow \mathbb{K}$ is $\lambda^n_{\varphi(X)}$-integrable $\Leftrightarrow$ $f\circ\varphi\, |\det D\varphi|$ is λ_X^n-integrable

In this case: $\int_{\varphi(X)} f\, d\lambda^n = \int_X f\circ\varphi\, |\det D\varphi|\, d\lambda^n$. □

Application of the transformation formula: Polar coordinates in $\mathbb{R}^2$

$X := (0,\infty) \times (0, 2\pi)$

$Y := \mathbb{R}^2 \setminus \{(x,0) \mid x \in \mathbb{R}\}$

$\phi : X \to Y, \quad \phi(r,\theta) = (r\cos\theta,\ r\sin\theta)$

$D\phi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$, $\det D\phi(r,\theta) = r$.

$\Rightarrow$ $f : \mathbb{R}^2 \to [0,\infty]$ Borel-measurable (or Lebesgue-mb)

Then: $\int_{\mathbb{R}^2} f \, d\lambda^2 = \int_{\mathbb{R}^2 \setminus \{(x,0) \mid x \in \mathbb{R}\}} f \, d\lambda^2 = \int_X f(r\cos\theta, r\sin\theta)\, r \, d\lambda^2(r,\theta)$ $(*)$

If $f : \mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{C}$ Borel measurable (or Lebesgue-mb)

Then: f is λ^2-integrable in $\mathbb{R}^2$

$\iff$ $(r,\theta) \mapsto r f(r\cos\theta, r\sin\theta)$ is λ^2-integrable in X.

In this case: $(*)$ is true.

Example. $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$

Proof $\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy$

$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dx\, dy \underset{\text{Fubini}}{=} \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\,dy$

$= \int_{(0,\infty)\times(0,2\pi)} e^{-r^2} r\, dr\, d\theta \underset{\text{Fubini}}{=} \int_0^{2\pi} \Big(\underbrace{\int_0^\infty e^{-r^2} r\, dr}_{= \frac{1}{2} e^{-r^2} \big|_0^\infty = \frac{1}{2}} \Big) d\theta$

$= \frac{1}{2} \int_0^{2\pi} d\theta = \pi.$ $\square$