3. Product measures and Fubini-Tonelli

Definition 3.1. $(X_j, \mathcal{A}_j)$ measure spaces ($j=1,\dots,n$).

Then $\bigotimes_{j=1}^{n} \mathcal{A}_j := \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ is the smallest σ-algebra on $X_1 \times \dots \times X_n$ such that for all $j=1,\dots,n$ the projections

$$pr_j : X_1 \times \dots \times X_n \longrightarrow X_j, \quad pr_j((x_1,\dots,x_n)) = x_j$$

are measurable. $\otimes \mathcal{A}_j$ is called the product σ-algebra.

Theorem 3.2. $(X_j, \mathcal{A}_j)$ measure spaces ($j=1,\dots,n$).

Assume that $\forall j=1,\dots,n$ exists a family $\mathcal{E}_j \subseteq \mathcal{A}_j$ such that

- $\mathcal{E}_j$ generates $\mathcal{A}_j$ (i.e. $\mathcal{A}_j$ is the smallest σ-algebra containing $\mathcal{E}_j$)
- $\mathcal{E}_j$ contains a sequence $(B_{jk})_{k\in\mathbb{N}}$ s.t.
  $B_{j1} \subseteq B_{j2} \subseteq B_{j3} \subseteq \dots$ and $\bigcup_{k=1}^{\infty} B_{jk} = X_j$.

Then $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ is generated by the sets

$$B_1 \times \dots \times B_n \quad \text{where } B_j \in \mathcal{E}_j \quad (j=1,\dots,n). \qquad (*)$$

Proof. Let $\tilde{\mathcal{A}}$ be the σ-algebra on $X_1 \times \dots \times X_n$ generated by $(*)$

① $\tilde{\mathcal{A}} \subseteq \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$:

To show: $B_1 \times \dots \times B_n \in \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ for all $B_j \in \mathcal{E}_j$ ($j=1,\dots,n$).

This is true because:

$$B_1 \times \dots \times B_n = \bigcap_{j=1}^{n} X_1 \times \dots \times B_j \times \dots \times X_n = \bigcap_{j=1}^{n} pr_j^{-1}(B_j) \in \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$$

because all pr_j are $\bigotimes_{j=1}^{n}\mathcal{A}_j$-$\mathcal{A}_j$-mb

② $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n \subseteq \tilde{\mathcal{A}}$

To show: all pr_j are $\tilde{\mathcal{A}}$-$\mathcal{A}_j$ mb.

Since $\mathcal{E}_j$ generates $\mathcal{A}_j$, we only have to show: $\forall B_j \in \mathcal{E}_j$: $pr_j^{-1}(B_j) \in \tilde{\mathcal{A}}$

This follows from

$$pr_j^{-1}(B_j) = X_1 \times \dots \times B_j \times \dots \times X_n = \bigcup_{k=1}^{\infty} \underbrace{B_{1k} \times \dots \times B_{j-1,k} \times B_j \times \dots \times B_{nk}}_{\in \tilde{\mathcal{A}}} \in \tilde{\mathcal{A}}.$$

□

Corollary 3.3. $\bigotimes_{j=1}^{n}\mathcal{A}_j$ is generated by the sets $A_1 \times \dots \times A_n$ where $A_j \in \mathcal{A}_j$ ($j=1,\dots,n$). □

Example 3.4.

$\mathcal{J} := \{[a,b[\subseteq \mathbb{R} \mid a \le b\}$, $\mathcal{B} :=$ Borel-σ-algebra on $\mathbb{R}$.

$\mathcal{J}^s := \{[a,b[\subseteq \mathbb{R}^s \mid a \le b\}$.

$\mathcal{J}$ generates $\mathcal{B}$ and $\mathcal{J}$ satisfies the hypothesis of thm 3.2

$\Longrightarrow$ $\mathcal{J}^s$ generates $\mathcal{B}^s := \bigotimes_{j=1}^{s}\mathcal{B}$.

Moreover: $\boxed{\mathcal{B}^s = \mathcal{B}(\mathbb{R}^s)}$ (= Borel σ-algebra on $\mathbb{R}^s$)

Proof. $\mathcal{B}(\mathbb{R}^s) \subseteq \mathcal{B}^s$ because by Übungsblatt 1 $\mathcal{J}^s$ contains all open sets (and the open sets generate $\mathcal{B}(\mathbb{R}^s)$)

$\mathcal{B}^s \subseteq \mathcal{B}(\mathbb{R}^s)$ because $\mathcal{B}^s$ is generated by sets of the type $[a_1,b_1[\times \dots \times [a_s,b_s[$, each of which belong to $\mathcal{B}(\mathbb{R}^s)$

(observe: $[a,b[= \bigcap_{n\in\mathbb{N}} (a-\frac{1}{n}, b) \in \mathcal{B}(\mathbb{R}^s)$). □

Notation. $(X,\mathcal{A})$, $(Y,\mathcal{B})$ measure spaces, $A \subseteq X \times Y$; $x \in X, y \in Y$.

Then: $A_x := \{y \in Y \mid (x,y) \in A\} \subseteq Y$

$A_y := \{x \in X \mid (x,y) \in A\} \subseteq X$.

Note. X, Y top. spaces. Then $\mathcal{B}(X) \otimes \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$. $(*)$

Strict inclusion is possible (Example: $\mathcal{B}(\mathbb{R}^2, \sigma^2) \neq \mathcal{B}(\mathbb{R},\sigma) \otimes \mathcal{B}(\mathbb{R},\sigma)$ where $\sigma = \{[a,b[\mid a \le b \in \mathbb{R}\}$)

Equality holds in $(*)$, if e.g. has a countable basis of topology (Elstrodt, III. §5)

Theorem 3.5. $(X,\mathcal{A})$, $(Y,\mathcal{B})$ measurable spaces.

$$\left.\begin{array}{l} M\subseteq X\times Y, \\ M\in\mathcal{A}\otimes\mathcal{B} \end{array}\right\}, x\in X,\ y\in Y. \implies M_x\in\mathcal{B},\ M_y\in\mathcal{A}.$$

Proof.

Let $\Sigma := \{S\subseteq X\times Y \mid S_y\in\mathcal{A}\}$

We will show: Σ is a σ-algebra and Σ contains all sets of the form $A\times B$ with $A\in\mathcal{A}$, $B\in\mathcal{B}$ (*)

Because then:

$\mathcal{A}\otimes\mathcal{B}\subseteq\Sigma$ by corollary 3.3, in particular:

$\forall M\in\mathcal{A}\otimes\mathcal{B}$ $M_y\in\mathcal{A}$.

Analogously $M_x\in\mathcal{A}$ can be proved.

Now we prove (*):

① $\{A\times B \mid A\in\mathcal{A},\ B\in\mathcal{B}\}\subseteq\Sigma$ is obvious because

$$(A\times B)_y = \left\{\begin{array}{ll} A & \text{if } y\in B \\ \emptyset & \text{if } y\notin B\end{array}\right\}\in\mathcal{A}.$$

② Σ is a σ-algebra:

- Let $(M_j)_j\subseteq\Sigma$. $\implies \left(\bigcup_{j=1}^{\infty} M_j\right)_y = \bigcup_{j=1}^{\infty}\overbrace{(M_j)_y}^{\in\mathcal{A}}\in\mathcal{A}$

  $\implies \bigcup_{j=1}^{\infty} M_j\in\Sigma$

- $M\in\Sigma \implies (X\setminus M)_y = X_1\setminus M_y\in\mathcal{A} \implies X\setminus M\in\Sigma$

  $X = X_1\times X_2$

- $(X_1\times X_2)_y = X_1\in\mathcal{A} \implies X_1\times X_2\in\Sigma$.

□

Theorem 3.6. $(X,\mathcal{A},\mu)$, $(Y,\mathcal{B},\nu)$ σ-finite measure spaces; $M\in\mathcal{A}\otimes\mathcal{B}$.

$$\implies \left\{\begin{array}{l} X\longrightarrow[0,\infty],\ x\longmapsto\nu(M_x) \\ Y\longrightarrow[0,\infty],\ y\longmapsto\mu(M_y)\end{array}\right\} \text{ are measurable and}$$

$$\int_X \nu(M_x)\,d\mu = \int_Y \mu(M_y)\,d\nu.$$

Proof. Step 1 Assume $M=\bigcup_{j=1}^{n} A_j\times B_j$ with $A_j\in\mathcal{A}$, $B_j\in\mathcal{B}$ $(j=1,\dots,n)$.

Without restriction: the union is disjoint.

$$\implies \chi_M(x,y) = \sum_{j=1}^{n}\chi_{A_j}(x)\,\chi_{B_j}(y)$$

$$\chi_{M_x}(y) = \sum_{j=1}^{n}\chi_{A_j}(x)\,\chi_{B_j}(y)$$

$$\implies \nu(M_x) = \int_Y \chi_{M_x}(y)\,d\nu = \sum_{j=1}^{n}\chi_{A_j}(x)\cdot\nu(B_j)$$

In particular: $x\longmapsto\nu(M_x)$ is measurable (it's a simple fct)

and $\int_X \nu(M_x)\,d\mu = \int_X \sum_{j=1}^{n}\chi_{A_j}(x)\,\nu(B_j)\,d\mu = \sum_{j=1}^{n}\mu(A_j)\,\nu(B_j)$.

The statement with μ and ν exchanged follows analogously.

Step 2.

Let $\Sigma := \{M\subseteq X\times Y \mid$ the statement is true for $M\}$.

We will show: Σ is a monotone family. (*)

Then: $\Sigma \overset{\text{Step 1}}{\supseteq} m(\{A\times B \mid A\in\mathcal{A},\ B\in\mathcal{B}\})$

$= \sigma(\{A\times B \mid A\in\mathcal{A},\ B\in\mathcal{B}\})$ (Thm. 1.?)

$= \mathcal{A}\otimes\mathcal{B}$ (Thm. 3.2),

Hence the statement is true for every $M\in\mathcal{A}\otimes\mathcal{B}$

Proof of (*)

① Let $(M_n)_n\subseteq\Sigma$, $M_1\subseteq M_2\subseteq\dots\subseteq M := \bigcup_{j=1}^{\infty} M_j$

$\implies (M_1)_x\subseteq(M_2)_x\subseteq\dots\subseteq M_x = \bigcup_{j=1}^{\infty}(M_j)_x$

all $(M_j)_x\in\mathcal{B} \implies M_x\in\mathcal{B}$. and $\nu(M_x) = \lim_{j\to\infty}\nu((M_j)_x)$

By assumption $\forall j\in\mathbb{N}$ $x\longmapsto\nu((M_j)_x)$ mb.

$\implies x\longmapsto\nu(M_x)$ mb as limit of mb functions.

Obviously: $\nu((M_j)_x)$ is increasing in j.

Analogously: $y\longmapsto\mu(M_y)$ mb and $\mu((M_j)_y)\nearrow\mu(M_y)$.

$$\implies \int_X \nu(M_x)\,d\mu = \int_X \lim_{n\to\infty}\nu((M_n)_x)\,d\mu$$

$$\overset{\text{monotone conv}}{=} \lim_{n\to\infty}\int_X \nu((M_n)_x)\,d\mu \overset{\text{statement holds for } M_n}{=} \lim_{n\to\infty}\int_Y \mu((M_n)_y)\,d\nu$$

$$\overset{\text{monotone conv}}{=} \int_Y \lim_{n\to\infty}\mu((M_n)_y)\,d\nu$$

$$= \int_Y \mu(M_y)\,d\nu.$$

② Now let $(M_j)_j \subseteq \Sigma$ with $M_1 \supseteq M_2 \supseteq \dots \supseteq M := \bigcap_{j=1}^\infty M_j$.

~~If $\mu(M_1)<\infty$, then μ~~

If $\mu(X) < \infty$ and $\nu(Y)<\infty$, then $\mu(M_y) = \lim_{n\to\infty}\mu((M_n)_y)$ and $\nu(M_x) = \lim_{n\to\infty} \nu((M_n)_x)$. and the argument from before works.

(Take complements: one shows that $X\setminus M = \bigcup_{j=1}^\infty (X\setminus M_j) \in \Sigma$)

③ In the general case: Choose $(S_k)_k \subseteq \mathcal{A}$, $(T_k)_k \subseteq \mathcal{B}$ s.t.

$\bigcup_k S_k = X$, $\bigcup_k T_k = Y$, $\mu(S_k) < \infty$, $\nu(T_k) < \infty$ $(k \in \mathbb{N})$.

$\Rightarrow \forall k \in \mathbb{N}$ $M \cap (S_k \times T_k) \in \Sigma$ by ②

$\Rightarrow$ $M = \bigcup_{k=1}^\infty M \cap (S_k \times T_k) \in \Sigma$ by ① □

Remark. σ-finiteness is a necessary condition in the theorem.

Example: $(\mathbb{R}, \mathcal{M}, \lambda)$ $\mathbb{R}$ with Lebesgue measure

$(\mathbb{R}, P\mathbb{R}, \mu)$ $\mathbb{R}$ with counting measure (not σ-finite)

Let $\Delta = \{(x,x) \mid x \in \mathbb{R}\} \in \mathcal{M}\otimes\mathcal{M} \subseteq \mathcal{M} \otimes P\mathbb{R}$.
↖ Δ closed

And: $\int_{\mathbb{R}} \overbrace{\lambda(\Delta_x)}^{=0}\, d\mu = \int_{\mathbb{R}} 0 \, d\mu = 0$

$\int_{\mathbb{R}} \overbrace{\mu(\Delta_x)}^{=1}\, d\lambda = \int_{\mathbb{R}} 1 \, d\mu = \infty$. □

Theorem & Definition 3.7. $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ σ-finite measure spaces.

$\Rightarrow$ $\exists!$ measure $\mu\otimes\nu : \mathcal{A}\otimes\mathcal{B} \to [0,\infty]$ s.t.

$\mu\otimes\nu(A\times B) = \mu(A)\cdot\mu(B)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

$\mu\otimes\nu$ is called the product measure of μ and ν and

$$\mu\otimes\nu(M) = \int_X \nu(M_x)\, d\mu = \int_Y \mu(M_y)\, d\nu, \quad M \in \mathcal{A}\otimes\mathcal{B}.$$

$\mu\otimes\nu$ is σ-finite.

Proof. Existence and the explicit formula for $\mu\otimes\nu$ follows from Theorem 3.6.

- $\mu\otimes\nu(\emptyset) = 0$ ✓
- $\mu\otimes\nu(M) \geq 0$, $M \in \mathcal{A}\otimes\mathcal{B}$
- $(M_j)_j \subseteq \mathcal{A}\otimes\mathcal{B}$, pairwise disjoint.

$$\Rightarrow \mu\otimes\nu(\biguplus M_j) = \int_X \nu(\biguplus M_j)_x\, d\mu = \int_X \nu(\biguplus (M_j)_x)\, d\mu = \int_X \sum_{j=1}^\infty \nu(M_j)_x\, d\mu \overset{\text{monotone convergence}}{=} \sum_{j=1}^\infty \int_X \nu(M_j)_x\, d\mu = \sum_{j=1}^\infty (\mu\otimes\nu)(M_j).$$

Obviously $\mu\otimes\nu$ is σ-finite because $X \times Y = \bigcup A_k \times B_k$ and $\mu\otimes\nu(A_k\times B_k) < \infty$ if $X = \bigcup_k A_k$, $Y = \bigcup_{k=1}^\infty B_k$ with $\mu(A_k) < \infty$, $\nu(B_k) < \infty$.

Uniqueness: $H := \{A\times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ is a semiring on $X\times Y$ and $\sigma : H \to [0,\infty]$, $\sigma(A\times B) = \mu(A)\mu(B)$ is a σ-finite premeasure on H.

$\Rightarrow$ There exists a unique extension of σ to a measure on the σ-algebra generated by H, which is $\mathcal{A}\otimes\mathcal{B}$. □

By induction:

$(X_j, \mathcal{A}_j, \mu_j)$ σ-finite measure spaces.

$\Rightarrow$ $\exists!$ measure $\mu := \bigotimes_{j=1}^n \mu_j = \mu_1 \otimes \dots \otimes \mu_n : \bigotimes_{j=1}^n \mathcal{A}_j \to [0,\infty]$

s.t. $\mu(A_1 \times \dots \times A_n) = \prod_{j=1}^n \mu_j(A_j)$ $(A_j \in \mathcal{A}_j,\ j = 1, \dots, n)$.

μ is σ-finite and it is called the product measure of $\mu_1, \dots, \mu_n$.

For every $M \in X$ fixed, the map $x_j \mapsto \mu_1 \otimes \dots \otimes \mu_{j-1} \otimes \mu_{j+1} \otimes \dots \otimes \mu_n(M$

is measurable and $\bigotimes_{k=1}^n \mu_k(M) = \int_{X_j} \left(\bigotimes_{\substack{k=1 \\ k\neq j}}^n \mu_k\right)(M_{x_j})\, d\mu_j$

and $(\mu_1 \otimes \dots \otimes \mu_{n-1}) \otimes \mu_n = \mu_1 \otimes \dots \otimes \mu_n$. □

Observation. μ, ν complete $\not\Rightarrow$ $\mu \otimes \nu$ complete.

Example: $\mu = \nu = \lambda$ = Lebesgue measure on $\mathbb{R}$, $\mathfrak{M}$ = Lebesgue mb. sets

It can be show that $\{(x,0) \mid x \in \mathbb{R}\}$ has Lebesgue measure 0 in $\mathbb{R}^2$.

If λ^2 ~~are~~ were complete, then also every subset must ~~have~~ be in $\mathfrak{B}^2$ $\mathfrak{M}^2$

In particular: $\{(x,0) \mid x \in M\} \in \mathfrak{B}^2$ (M = non-Lebesgue measurable set)

By Thm 3.5 then $M = \{(x,0) \mid x \in M\}_0 \in \mathfrak{M}$ ↯.

Theorem 3.7. (Fubini) Tonelli?

$(X, \mathfrak{A}, \mu)$, $(Y, \mathfrak{B}, \nu)$ σ-finite measure spaces.

$f: X \times Y \longrightarrow [0, \infty]$ $\quad \mathfrak{A} \otimes \mathfrak{B}$-mb

$$\Longrightarrow \begin{cases} X \to [0,\infty], & x \mapsto \int_Y f(x,y)\, d\nu \\ Y \to [0,\infty], & y \mapsto \int_X f(x,y)\, d\mu \end{cases} \text{ are measurable and}$$

$$\int_{X \times Y} f(x,y)\, d(\mu \otimes \nu) = \int_X \left(\int_Y f(x,y)\, d\nu \right) d\mu = \int_Y \left(\int_X f(x,y)\, d\mu \right) d\nu$$

Proof.

First note that for fixed $x \in X$ the map $y \mapsto f(x,y)$ is measurable (because: $A \subseteq \mathbb{R}$ mb $\Rightarrow$ ~~$f(x,y)$~~ $f(x,\cdot)^{-1}(A) = (\overbrace{f^{-1}(A)}^{\in \mathfrak{A} \otimes \mathfrak{B}})_x$ mb (Thm 3.5)), same for $x \mapsto f(x,y)$, so the integrals in the claim make sense.

The statement is true for characteristic functions:

~~let $A \in \mathfrak{A}\otimes$~~ let $M \in \mathfrak{A} \otimes \mathfrak{B}$; $f := \chi_M$; ~~fix $y \in Y$~~

$\Longrightarrow \int_Y f(x,y)\, d\nu = \int_Y \chi_M(x,y)\, d\nu = \nu(M_x)$ mb. by Thm. 3.5 and

$$\int_{X \times Y} \overbrace{f(x,y)}^{= \chi_M(x,y)}\, d(\mu \otimes \nu) = \mu \otimes \nu(M) \overset{\text{Thm } 3.7}{=} \int_X \nu(M_x)\, d\mu = \int_X \left(\int_Y f(x,y)\, d\nu \right) d\mu.$$

Analogously for μ, ν exchanged.

$\Longrightarrow$ The statement is true for simple fcts $\in E^+(\mathfrak{A} \otimes \mathfrak{B}, X \times Y)$.

For arbitrary f as in the hypothesis: Choose sequence $(s_n)_n \in E^+(\mathfrak{A} \otimes \mathfrak{B})$ st. $s_1 \leq s_2 \leq \ldots$ and $s_n \nearrow f$.

For fixed $x \in X$: $s_n(x, \cdot) \nearrow f(x, \cdot)$ and all $s_n(x,\cdot) \in E^+(\mathfrak{B}, Y)$.

$$\Longrightarrow \int_Y s_n(x,y)\, d\nu \nearrow \int_Y f(x,y)\, d\nu \qquad (*)$$

In particular $x \mapsto \int_Y f(x,y)\, d\nu$ is mb because it is limit of mb fct's.

$$\Longrightarrow \int_{X \times Y} f\, d(\mu \otimes \nu) = \lim_{n \to \infty} \int_{X \times Y} s_n(x,y)\, d(\mu \otimes \nu)$$

$$\overset{s_n \in E^+}{=} \lim_{n \to \infty} \int_X \left(\int_Y s_n(x,y)\, d\nu \right) d\mu$$

$$\overset{\text{mon. convergence}}{=} \int_X \left(\lim_{n \to \infty} \int_Y s_n(x,y)\, d\nu \right) d\mu$$

$$\overset{(*)}{=} \int_X \left(\int_Y f(x,y)\, d\nu \right) d\mu.$$

Analogously for μ and ν exchanged. □

Theorem 3.8. (Fubini)

$(X, \mathfrak{A}, \mu)$, $(Y, \mathfrak{B}, \nu)$ σ-finite measure spaces.

i) $f: X \times Y \to \mathbb{K}$ $\quad \mu \otimes \nu$-integrable

$\Longrightarrow$ $\begin{cases} f(x,\cdot) \text{ is } \nu\text{-integrable for } \mu\text{-a.a. } x \in X \text{ and} \\ A := \{x \in X \mid f(x,\cdot) \text{ is not } \nu\text{-integrable}\} \in \mathfrak{A}. \end{cases}$

and

$\begin{cases} f(\cdot,y) \text{ is } \mu\text{-integrable for } \nu\text{-a.a. } y \in Y \text{ and} \\ B := \{y \in Y \mid f(\cdot,y) \text{ is not } \mu\text{-integrable}\} \in \mathfrak{B}. \end{cases}$

The fcts $x \mapsto \int_Y f(x,y)\, d\nu$ and $y \mapsto \int_X f(x,y)\, d\mu$ are integrable on $X \setminus A$ and $Y \setminus B$, resp., and

$$\int_{X \times Y} f(x,y)\, d(\mu \otimes \nu) = \int_{X \setminus A} \left(\int_Y f(x,y)\, d\nu \right) d\mu = \int_{Y \setminus B} \left(\int_X f(x,y)\, d\mu \right) d\nu.$$

ii) $f: X\times Y \longrightarrow \mathbb{K}$ $\quad \mathcal{A}\otimes\mathcal{B}$-mb.

If at least one of the following integrals

$$\int_{X\times Y} |f|\, d(\mu\otimes\nu), \quad \int_X \left(\int_Y |f(x,y)|\, d\nu\right) d\mu, \quad \int_Y \left(\int_X |f(x,y)|\, d\mu\right) d\nu$$

is finite then all of them are finite and equal, f is $\mu\otimes\nu$-integrable and the claims in i) hold.

Proof.

i) f integrable $\Longrightarrow |f|$ integrable

Note: $\forall x\in X$ $f(x,\cdot)$ is mb.

$$\Longrightarrow A = \{x\in X \mid \int_Y |f(x,y)|\, d\nu = \infty\}$$

By Thm. 3.7:

$$\int_X \left(\int_Y |f(x,y)|\, d\nu\right) d\mu = \int_{X\times Y} f\, d(\mu\otimes\nu) < \infty$$

$$\Longrightarrow \mu(A) = \mu\{x\in X \mid \int_Y |f(x,y)|\, d\nu < \infty\} = 0$$

Now: $\forall g\in \{(\mathrm{Re}\, f)_\pm, (\mathrm{Im}\, f)_\pm\}$ the fct $x\longmapsto \int g(x,y)\, d\nu$ is integrable because $\int_X \left(\int g(x,y)\, d\nu\right) d\mu \le \int_X \left(\int_Y |f(x,y)|\, d\nu\right) d\mu < \infty$.

$\Longrightarrow \int f(x,y)\, d\nu$ is integrable on $X\setminus A$ and

$$\int_{X\setminus A} \left(\int_Y f(x,y)\, d\nu\right) d\mu = \int_{X\setminus A} \left(\int_Y (\mathrm{Re}\, f)_+\, d\nu\right) d\mu - \int_{X\setminus A} \left(\int_Y (\mathrm{Re}\, f)_-\, d\nu\right) d\mu$$
$$+ i\int_{X\setminus A} \left(\int_Y (\mathrm{Im}\, f)_+\, d\nu\right) d\mu - i\int_{X\setminus A} \left(\int_Y (\mathrm{Im}\, f)_-\, d\nu\right) d\mu$$

$$\overset{\mu(A)=0}{=} \int_X \left(\int_Y (\mathrm{Re}\, f)_+\, d\nu\right) d\mu + \dots$$

$$\overset{\text{Thm } 3.7.}{=} \int_{X\times Y} (\mathrm{Re}\, f)_+\, d(\mu\otimes\nu) + \dots$$

$$= \int_{X\times Y} f\, d(\mu\otimes\nu).$$

Analogously with μ and ν exchanged.

ii) Follows from Thm 3.7. and i). $\square$

Notation. Since we can change $\int_Y f(x,y)\, dy$ on A without changing the value of $\int_X\int_Y f(x,y)\, dy$ we write

$$\int_{X\times Y} f(x,y)\, d(\mu\otimes\nu) = \int_X \int_Y f(x,y)\, d\nu\, d\mu.$$

Similar for B.

Corollary 3.9.

Let $(a_{jk})_{j,k} \subseteq \mathbb{C}$. Then:

$$\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} a_{jk} = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk}\right) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{jk}\right)$$

if at least one of the series converges when a_{jk} is replaced by $|a_{jk}|$.

Proof. Apply Fubini's thm to $(\mathbb{N}, \mathcal{P}\mathbb{N}, \mu)$ $\quad \mu$ = counting measure. $\square$