Remark. Theorem 2.35 is not true for improper Riemann-integrals.

Example. $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{\sin x}{x}$.

Then: $R-\int_0^\infty f(x)dx$ exists, but $f \notin \mathcal{L}_1((0,\infty),\mathbb{R})$.

Without proof:

Theorem $J \subseteq \mathbb{R}$ interval, $f: J \to \mathbb{K}$ R-integrable on every cpt. subset of J. Then:

f L-integrable $\iff$ $|f|$ improper R-integrable.

in this case: $\int_J f\, d\lambda = R-\int_J f(x)dx$.

Proof. Elstrodt, Satz 6.3 (S.153). □

2.4. Convergence.

Recall $f_n: X \to \mathbb{K}$ (measurable) where $(X, \mathcal{A}, \mu)$ is a measure space.

Then: $f_n \to f$ pointwise, if $f_n(x) \to f(x)$ for all $x \in X$

$f_n \to f$ μ-a.e., if $\exists N \in \mathcal{A}$ s.t. $\mu(N) = 0$ and $f_n(x) \to f(x)$ for all $x \in X \setminus N$.

Proposition 2.36. $f_n, f: X \to \mathbb{K}$ measurable.

i) $f_n \to f$ μ-a.e. $\iff$ $\forall \varepsilon > 0$ $\mu\left(\bigcap_{n=1}^\infty \bigcup_{k \geq n} \{|f_n - f| \geq \varepsilon\}\right) = 0$

ii) If $\mu(X) < \infty$, then:

$f_n \to f$ μ-a.e. $\iff$ $\forall \varepsilon > 0$ $\lim_{n\to\infty} \mu\left(\bigcup_{k \geq n} \{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0$

Proof.

i) $f_n \to f$ μ-a.e. $\iff$ $\forall \varepsilon > 0: \mu\{x \in X \mid \forall n \geq 1\ \exists k \geq n: |f_n(x) - f(x)| > \varepsilon\} = 0$

ii) If $\mu(X) < \infty$, then $\mu\left(\bigcup_{k \in \mathbb{N}} \{|f_n - f| \geq \varepsilon\}\right) < \infty$

$\Rightarrow \mu\left(\bigcap_{n=1}^\infty \ldots\right) = \lim_{n\to\infty} \mu(\ldots)$ □

Observation: If $\mu(X) = \infty$, then "$\Leftarrow$" is still true.

"$\Rightarrow$" in general does not hold. Example: $f_n := \chi_{[n,\infty)}$

Then: $f_n \to 0$ everywhere, but $\mu\left\{\bigcup_{k \geq n} \{x \in X \mid |f_n(x) - 0| \geq \tfrac{1}{2}\}\right\} = \infty$ for all $n \in \mathbb{N}$.

Definition 2.37 $f_n, f : X \to \mathbb{K}$ measurable.

Then: $f_n \to f$ ~~quasi uniformly~~ almost uniformly

$:\iff \forall \delta > 0 \ \exists A \in \mathcal{A}$ s.t. $\mu(A) < \delta$ and $(f_n|_{X \setminus A})_n$ conv. uniformly.

Now we show that such a sequence has a limit function.

Lemma 2.38. $f_n : X \to \mathbb{K}$, ~~almost uniformly~~ measurable and $(f_n)_n$ conv. almost uniformly. Then there ex. a measurable function f, s.t. $f_n \to f$ μ-a.e.

Proof. $\forall k \in \mathbb{N}$ choose $A_k \in \mathcal{A}$ s.t. $\mu(A_k) < 1/k$ and $(f_n|_{X \setminus A_k})_n$ conv. uniformly.

Let $A = \bigcap_{k=1}^{\infty} A_k \Rightarrow \mu(A) = 0$ and for all $x \in X \setminus A$: $f_n(x)$ conv.

Let $f := \limsup_{n \to \infty} f_n \Rightarrow f$ is mb and $f_n \to f$ on $X \setminus A$. $\square$

If $\mu(X) < \infty$, then the converse is also true:

Theorem 2.39. (Egoroff) $(X, \mathcal{A}, \mu)$ measure space.

Assume $\mu(X) < \infty$ and $f_n, f : X \to \mathbb{K}$ mb s.t. $f_n \to f$ μ-a.e.

Then: $f_n \to f$ almost uniformly.

Proof. Let $\delta > 0$.

To show: $\exists A \in \mathcal{A}$ s.t. $\mu(A) < \delta$ and $\forall k \in \mathbb{N} \ \exists n_k \in \mathbb{N}: \forall n \geq n_k \ \forall x \in X \setminus A: |f_n(x) - f(x)| < 1/k$

By assumption: $\lim_{n \to \infty} \mu\left(\bigcup_{k \geq n} \{x \in X \mid |f_k(x) - f(x)| \geq \varepsilon\}\right) = 0$ (Prop. 2.36)

For every $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ such that $\mu(B_{n_k}) < \delta/2^k$, where

$B_{n_k} := \bigcup_{m \geq n_k} \{x \in X \mid |f_m(x) - f(x)| \geq 1/k\}$

Let $A := \bigcup_{k=1}^{\infty} B_{n_k} \Rightarrow A \in \mathcal{A}$, $\mu(A) < \delta$ and if $x \in X \setminus A$,

then: $\forall k \ x \notin B_{n_k} \Rightarrow \forall k \quad |f_m(x) - f(x)| < 1/k$ for all $m \geq n_k$. $\square$

Example. $X = [0, 1[$, $f_n : X \to \mathbb{R}$, $f_n(x) = x^n$, $f : X \to \mathbb{R}$, $f(x) = 0$

Then: $f_n \to f$ ~~pointwise~~ everywhere.

$f_n \to f$ not uniformly, but almost uniformly because

$\forall \delta > 0$: $f_n|_{[0, 1-\delta]} \to f|_{[0, 1-\delta]}$ uniformly.

Definition 2.40 $f_n, f : X \to \mathbb{K}$ measurable.

$f_n \to f$ in measure $:\iff \forall \varepsilon > 0 \quad \lim_{n \to \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) = 0$

Observe: The limit function is not unique.

But: Assume $f_n \to f$ and $f_n \to g$ in measure.

Then: $f = g$ μ-a.e.

Proof.

$\forall k \in \mathbb{N} \quad \mu(\underbrace{\{x \in X \mid |f(x) - g(x)| \geq 1/k\}}_{\subseteq \{|f_n - f| \geq \frac{1}{2k}\} \cup \{|f_n - g| \geq \frac{1}{2k}\}})$ $\qquad 1/k \leq |f(x) - g(x)| \leq |f_n(x) - f(x)| + |f_n(x) - g(x)|$

$\leq \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \frac{1}{2k}\}) + \mu(\{x \in X \mid |f_n(x) - g(x)| \geq \frac{1}{2k}\})$

$\to 0, \quad n \to \infty$.

$\Rightarrow \forall k \in \mathbb{N} \quad \mu(\{x \in X \mid |f(x) - g(x)| \geq \frac{1}{k}\}) = 0$

$\Rightarrow \mu(\{x \in X \mid f(x) \neq g(x)\}) = \mu\left(\bigcup_{k \in \mathbb{N}} \{x \in X \mid |f(x) - g(x)| \geq 1/k\}\right) = 0$.

Theorem 2.41. $f, f_n : X \to \mathbb{K}$ mb.

i) $f_n \to f$ almost unif $\Rightarrow f_n \to f$ in measure

ii) $f_n \to f$ μ-almost everywhere and $\mu(X) < \infty \Rightarrow f_n \to f$ in measure.

Proof.

i) Let $\varepsilon > 0$, and $\delta > 0$. We show: $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N \quad \mu(|f_n - f| \geq \varepsilon) < \delta$ (*)

By assumption ex. $A \in \mathcal{A}$ s.t. $\mu(A) < \delta$ and $f|_{X \setminus A} \to f|_{X \setminus A}$ unif.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \ \forall x \in X \setminus A$: $|f_n(x) - f(x)| < \varepsilon$.

Hence (*) holds.