2.3. Convergence theorems.

- Fatou's lemma
- Satz von Levi (Monotone convergence thm)
- Satz von Lebesgue (Dominated convergence thm)

Rudin: mon. conv ⇒ Fatou ⇒ Lebesgue

Theorem 2.20. (Fatou's lemma)

$(X, \mathcal{A}, \mu)$ measure space. $(f_n)_n$ family of functions with:

$\forall n \in \mathbb{N}$ $f_n : X \longrightarrow [0,\infty]$, mb.

$$\Rightarrow \int_X \overbrace{\liminf_{n\to\infty} f_n}^{=:f} \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Proof. Observe: $\liminf_{n\to\infty} f_n =: f$ exists and is measurable by Prop 2.10}.

Since $\int_X f \, d\mu = \sup\{ \int \varphi d\mu \mid \varphi \in E^+(X,\mathcal{A}),\ \varphi \le f\}$ it suffices to show:

$\varphi \in E^+(X,\mathcal{A})$ with $\varphi \le f \Rightarrow \int \varphi \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu$.

So, fix $\varphi \in E^+(X,\mathcal{A})$ with $\varphi \le f$.

Case 1. $\int_X \varphi d\mu = \infty$ (to show: $\lim_{n\to\infty} \int_X f_n d\mu = \infty$)

$\Rightarrow \int f d\mu = \infty$

$\Rightarrow \exists A \in \mathcal{A}$ and $a > 0$ s.t. $\mu(A) = \infty$ and $\varphi|_A > a > 0$.

$\forall n \in \mathbb{N}$ $A_n := \{x \in X \mid \forall k \ge n \ f_k(x) \ge a\}$

$= \bigcap_{k \ge n} f_k^{-1}([a,\infty]) \in \mathcal{A}$

Obviously: $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

Also: $A \subseteq \bigcup_{j=1}^\infty A_j$ because: $x \in A \Rightarrow a < \varphi(x) \le \liminf_{n\to\infty} f_n(x)$ (Def of a; Hip. on φ.)

$\Rightarrow \exists n \in \mathbb{N}: \forall k \ge n \ f_k(x) > a.$

$\Rightarrow x \in A_n \subseteq \bigcup_{j=1}^\infty A_j$.

$\Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{j=1}^\infty A_j) \ge \mu(A) = \infty.$

Note that $\forall n \in \mathbb{N}$ $f_n \ge a \cdot \chi_{A_n}$

$\Rightarrow \int_X f_n d\mu \ge \int_X a \cdot \chi_{A_n} d\mu = a \cdot \mu(A_n) \longrightarrow \infty.$

$\Rightarrow \liminf_{n\to\infty} \int_X f_n d\mu = \infty = \int_X f d\mu.$

Case 2. $\int \varphi d\mu < \infty$.

Let $A := \{x \in X \mid \varphi(x) > 0\}$ and $M := \max\{\varphi(x) \mid x \in X\}$

Observe: $\mu(A) < \infty$. (because $\infty > \int_X \varphi d\mu \ge \mu(A) \cdot \min\{\varphi(x) \mid x \in X\}$ and $\mu(A) = 0$ if $\min\{\varphi(x) \mid x \in X\} = 0$.)

$M = 0 \Rightarrow \varphi \equiv 0 \Rightarrow \int \varphi d\mu = 0 \le \liminf_{n\to\infty} \int_X f_n d\mu$

$M \ne 0$ Fix $\varepsilon > 0$ and define $\varepsilon' := \min\{1, \frac{\varepsilon}{M + \int \varphi d\mu}\}$

$\forall n \in \mathbb{N}$ $A_n := \{x \in X \mid \forall k \ge n \ f_k(x) > (1-\varepsilon')\varphi(x)\}$

$\Rightarrow \forall n \in \mathbb{N}$ $A_n \in \mathcal{A}$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

Also: $A \subseteq \bigcup_{n=1}^\infty A_n$ because: $x \in A \Rightarrow 0 < (1-\varepsilon')\varphi(x) < f(x) = \liminf_{n\to\infty} f_n(x)$

$\Rightarrow \exists n \in \mathbb{N}: \forall k \ge n \ f_k(x) \ge (1-\varepsilon')\varphi(x)$

$\Rightarrow x \in A_n$.

Now: $\forall n \in \mathbb{N}$ $B_n := A \setminus A_n$

$\Rightarrow B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $\bigcap_{n=1}^\infty B_n = \emptyset$.

$\Rightarrow \lim_{n\to\infty} \mu(B_n) = 0$ (Thm. ? because $B_1 \subseteq A$, hence $\mu(B_1) \le \mu(A) < \infty$)

$\Rightarrow \exists N \in \mathbb{N}: \mu(B_k) = \mu(A \setminus A_k) < \varepsilon'$ $(k \ge N)$.

$\Rightarrow \forall k \ge N$

$$\int_X f_k d\mu \ge \int \chi_{A_k} f_k d\mu \ge (1-\varepsilon') \int \chi_{A_k} \varphi d\mu$$

$$= (1-\varepsilon') \int \varphi - \underbrace{\chi_{X\setminus A_k} \varphi}_{=\chi_{A\setminus A_k}\varphi} d\mu$$

$$= \int \varphi d\mu - \varepsilon' \int \varphi d\mu - \overbrace{(1-\varepsilon')}^{\le 1} \int \chi_{A \setminus A_k} \overbrace{\varphi}^{\le M} d\mu$$

$$\ge \int \varphi d\mu - \varepsilon' \int \varphi d\mu - M \underbrace{\mu(A \setminus A_k)}_{< \varepsilon'}$$

$$\ge \int \varphi d\mu - \varepsilon' \left(\int \varphi d\mu + M \right)$$

$$\ge \int \varphi d\mu - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary $\Rightarrow \int_X f_k d\mu \ge \int \varphi d\mu$ for $k \ge n$

$\Rightarrow \liminf_{k\to\infty} \int f_k d\mu \ge \int \varphi d\mu$

□

Observation Strict inequality in Fatou's lemma is possible:

Example 1 $f_n : [-1,1] \to \mathbb{R}$, $f_n := \begin{cases} \chi_{[-1,0]}, & n \text{ even} \\ \chi_{[0,1]}, & n \text{ odd} \end{cases}$ $\Rightarrow \int \liminf_{n\to\infty} f_n d\mu = 0 < 1 = \liminf_{n\to\infty} \int_X f_n d\mu$

Example 2 $f_n : \mathbb{R} \to \mathbb{R}$, $f_n = \chi_{[n,\infty)}$

$\Rightarrow \int_{\mathbb{R}} \liminf f_n d\mu = 0 < \infty = \liminf_{n\to\infty} \int_{\mathbb{R}} f_n d\mu$

(although: $f_n \longrightarrow f$ pointwise)

Theorem 2.21 (Monotone convergence; Satz von Levi)

$(X,\mathcal{A},\mu)$ measure space, $f_n : X \to \mathbb{R}\cup\{+\infty\}$ mb $(n\in\mathbb{N})$

s.t. $0 \le f_1 \le f_2 \le \dots \le \lim_{n\to\infty} f_n =: f$. (f mb because it's limit of mb fct's)

$$\Rightarrow \int_X \underbrace{\lim f_n}_{f}\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$$

Proof.

- $\forall n\in\mathbb{N}$ $f_n \le f$ $\Rightarrow$ $\forall n\in\mathbb{N}$ $\int_X f\,d\mu \ge \int_X f_n\,d\mu$
  $\Rightarrow \int_X f\,d\mu \ge \limsup_{n\to\infty} \int_X f_n\,d\mu$ (*)
- Fatou's lemma: $\int_X f\,d\mu \le \liminf_{n\to\infty} \int_X f_n\,d\mu$ (**)

(*) & (**): $\limsup \dots = \liminf \dots = \lim \dots$ and

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu$$ □

Corollary 2.22. $(X,\mathcal{A},\mu)$ measure space, $f,g : X \to [0,\infty]$ mb.

$\Rightarrow$ i) $\int f+g\,d\mu = \int f\,d\mu + \int g\,d\mu$

ii) $\int f\,d\mu = 0 \iff f = 0$ μ-a.e.

iii) $\int f\,d\mu < \infty \Rightarrow f < \infty$ μ-a.e.

Proof.

i) By theorem 2.15 $\exists\ (s_n)_n, (t_n)_n \subseteq E^+(X,\mathcal{A})$ s.t.

$$\begin{cases} 0 \le s_1 \le s_2 \le \dots \le f, & s_n \to f \\ 0 \le t_1 \le t_2 \le \dots \le g, & t_n \to g \end{cases} \quad (n\to\infty)$$

$\Rightarrow \int s_n\,d\mu \to \int f\,d\mu$, $\int t_n\,d\mu \to \int g\,d\mu$ (Monotone conv. thm)

Also: $0 \le s_1+t_1 \le s_2+t_2 \le \dots \le f+g$; $s_n+t_n \to f+g$ $(n\to\infty)$

and $\int s_n+t_n\,d\mu \to \int f+g\,d\mu$.

$$\Rightarrow \int f+g\,d\mu = \lim_{n\to\infty}\int s_n+t_n\,d\mu = \lim_{n\to\infty}\left(\int s_n\,d\mu + \int t_n\,d\mu\right) = \int f\,d\mu + \int g\,d\mu.$$

ii) Let $N := \{x\in X \mid f(x) > 0\}$. To show: $\int f\,d\mu = 0 \iff \mu(N) = 0$.

"$\Leftarrow$" Let $\varphi \in E^+(X,\mathcal{A})$ with $0 \le \varphi \le f$;

Let $A_1,\dots,A_n \in \mathcal{A}$ pairwise disjoint, and $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ s.t.

$\varphi = \sum_{j=1}^n \alpha_j \cdot \chi_{A_j}$. By assumption: $\alpha_j > 0 \Rightarrow f|_{A_j} \ge \alpha_j > 0 \Rightarrow A_j \subseteq N$

$$\Rightarrow \int \varphi\,d\mu = \sum_{\substack{j=1\\ \alpha_j\neq 0}}^n \alpha_j \underbrace{\mu(A_j)}_{\le \mu(N)=0} = 0$$

"$\Rightarrow$" Let $A_n := \{x\in X \mid f(x) > \frac{1}{n}\}$

$\Rightarrow N = \bigcup_{n\in\mathbb{N}} A_n$ and $\forall n\in\mathbb{N}$ $\overbrace{\frac{1}{n}\chi_{A_n}}^{\in E^+(X,\mathcal{A})} \le f$

$\Rightarrow 0 = \int f\,d\mu \ge \int \frac{1}{n}\chi_{A_n}\,d\mu = \frac{1}{n}\mu(A_n)$

$\Rightarrow \forall n\in\mathbb{N}$ $\mu(A_n) = 0$

$\Rightarrow \mu(N) = 0$.

iii) Übungen (Blatt 4, Aufg. 3).

⌈Entweder mit charakteristischer Fkt. oder:

$B_n := \{x\in X \mid f(x) > n\}$ $\Rightarrow$ $M := \{x\in X \mid f(x) = \infty\} = \bigcap_{n\in\mathbb{N}} B_n$

$\forall n\in\mathbb{N}$: $J := \int f\,d\mu \ge \int n\cdot\chi_{B_n}\,d\mu = n\cdot\mu(B_n)$

$\Rightarrow \forall n\in\mathbb{N}$ $\mu(B_n) \le J/n$. In particular: $\mu(B_1) < \infty$

$\Rightarrow \mu(M) = \lim_{n\to\infty}\mu(B_n) \le \lim_{n\to\infty} J/n \to 0$, $n\to\infty$ ⌋

Corollary 2.23. $(X,\mathcal{A},\mu)$ measure space, $f_n : X \to [0,\infty]$, mb. $(n\in\mathbb{N})$

$$\Rightarrow \int_X \sum_{n=1}^\infty f_n\,d\mu = \sum_{n=1}^\infty \int_X f_n\,d\mu.$$

Corollary 2.24. $(X,\mathcal{A},\mu)$ measure space, $f,g : X \to [0,\infty]$, mb.

and $f = g$ μ-a.e.

$\Rightarrow \int f\,d\mu = \int g\,d\mu$.

Proof. Let $A := \{x\in X \mid f(x) = g(x)\}$ $\Rightarrow \mu(X\setminus A) = 0$.

Since $f = \chi_A f + \chi_{X\setminus A} f$:

$$\int_X f\,d\mu = \int_X \underbrace{\chi_A f}_{=\chi_A g}\,d\mu + \underbrace{\int_X \overbrace{f\cdot\chi_{X\setminus A}}^{=0\ \mu\text{-a.e.}}\,d\mu}_{=0 \text{ by Cor 2.22}} = \int_X \chi_A\cdot g\,d\mu = \dots = \int_X g\,d\mu.$$ □

Corollary 2.25. $(X, \mathcal{A}, \mu)$ measure space; $f_n : X \to [0,\infty]$, mb, $(n \in \mathbb{N})$

Assume $f(x) = \lim_{n\to\infty} f_n(x)$ exists μ-a.e.

$$\Rightarrow \int f(x)\,dx \leq \liminf_{n\to\infty} \int f_n\,d\mu.$$

Proof. Let $\tilde{f}(x) := \liminf_{n\to\infty} f_n(x)$.

$\Rightarrow f = \tilde{f}$ μ-a.e. and

$$\int_X f\,d\mu \underset{\text{Cor. 2.24}}{=} \int_X \tilde{f}\,d\mu \underset{\text{Fatou}}{\leq} \liminf_{n\to\infty} \int_X f_n\,d\mu \qquad \square$$

Note. Strict inequality is possible.

Ex. $f_n := \chi_{[n,\infty)} : \mathbb{R} \to \mathbb{R}$.

$\leadsto f_n(x) \to 0$, $x \in \mathbb{R}$, but $\int 0\,d\mu = 0 < \infty = \liminf_{n\to\infty} \int_X f_n\,d\mu$ $\square$

So far: Theorems and Corollaries valid for functions with values in $[0,\infty]$.

Now: Integration rules for functions with values in $\mathbb{R} \cup \{\pm\infty\}$ (or $\mathbb{C}$).

Recall. $f : X \to \overline{\mathbb{R}}$ mb. Then f integrable $\Leftrightarrow$ f_+ and f_- are integrable and $\int f\,d\mu = \int f_+\,d\mu - \int f_-\,d\mu$ in the case of integrability.

Now we show that we can also choose other representations of f as difference of non-neg. fct's to obtain $\int f\,d\mu$.

Proposition 2.26.

$(X, \mathcal{A}, \mu)$ measure space, $f_1, f_2 : X \to [0,\infty]$ mb with $\mathcal{I}(f_1), \mathcal{I}(f_2) < \infty$. Let $f := f_1 - f_2$.

$\Rightarrow$ f mb. and $\int f\,d\mu = \int f_1\,d\mu - \int f_2\,d\mu$.

Proof. f mb by Cor. 2.7.

$\forall x \in X$:

- $f(x) \geq 0 \Rightarrow f_+(x) = f(x) \leq f_1(x)$
- $f(x) < 0 \Rightarrow f_+(x) = 0 \leq f_1(x)$

$\Big\} \Rightarrow f_+(x) \leq f_1(x)$

$$\Rightarrow \int_X f_+(x)\,dx \leq \int_X f_1(x)\,dx < \infty \Rightarrow f_+ \in \mathcal{L}_1(X, \overline{\mathbb{R}})$$

Analogously: $f_-(x) \leq f_2(x) \Rightarrow f_- \in \mathcal{L}_1(X, \overline{\mathbb{R}})$

$\Rightarrow f \in \mathcal{L}_1(X, \overline{\mathbb{R}})$ by def. of $\mathcal{L}_1(X, \overline{\mathbb{R}})$.

Now: $f_+ - f_- = f = f_1 - f_2 \Rightarrow f_+ + f_2 = f_1 + f_-$

By linearity of $\int \cdot\,d\mu$ for non-neg. fcts ((Cor 2.22):

$$\int f_+\,d\mu + \int f_2\,d\mu = \int f_1\,d\mu + \int f_-\,d\mu$$

$$\Rightarrow \int f\,d\mu = \int f_+\,d\mu - \int f_-\,d\mu = \int f_1\,d\mu - \int f_2\,d\mu.$$

$\square$

Theorem 2.27. (Integration rules).

$(X, \mathcal{A}, \mu)$ measure space; $f, g : X \to \mathbb{K}$, $\alpha \in \mathbb{K}$ $(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$

$\Rightarrow$ i) $f \in \mathcal{L}_1(X, \mathbb{K}) \Rightarrow \alpha f \in \mathcal{L}_1(X, \mathbb{K})$ and $\int \alpha f\,d\mu = \alpha \int f\,d\mu$

(doesn't make sense for $\overline{\mathbb{R}}$-valued fct's)

ii) $f, g \in \mathcal{L}_1(X, \mathbb{K}) \Rightarrow f + g \in \mathcal{L}_1(X, \mathbb{K})$ and $\int f + g\,d\mu = \int f\,d\mu + \int g\,d\mu$.

iii) $f \in \mathcal{L}_1(X, \mathbb{K})$ and $f = g$ μ-a.e. $\Rightarrow g \in \mathcal{L}_1(X, \mathbb{K})$ and $\int f\,d\mu = \int g\,d\mu$.

iv) $\mathbb{K} = \mathbb{R}$, $f, g \in \mathcal{L}_1(X, \mathbb{R})$, $f \leq g \Rightarrow \int f\,d\mu \leq \int g\,d\mu$.

Proof. only for $\mathbb{K} = \mathbb{R}$ (the case $\mathbb{K} = \mathbb{C}$ follows easily by separation of f in real and imaginary part).

i) $\alpha = 0$: clear.

$\alpha > 0$: $(\alpha f)_+ = \alpha f_+$, $(\alpha f)_- = \alpha f_- \Rightarrow (\alpha f)_\pm \in \mathcal{L}_1(X, \mathbb{R})$ and

$$\Rightarrow \int (\alpha f)\,d\mu = \int (\alpha f)_+\,d\mu - \int (\alpha f)_-\,d\mu$$
$$= \alpha \int f_+\,d\mu - \alpha \int f_-\,d\mu$$
$$= \alpha \left(\int f_+\,d\mu - \int f_-\,d\mu \right) = \alpha \int f\,d\mu.$$

$\alpha = -1$: $(-f)_+ = f_-$, $(-f)_- = f_+$

$$\Rightarrow \int -f\,d\mu = \int f_-\,d\mu - \int f_+\,d\mu = -\int f\,d\mu.$$

ii) $f+g = \overbrace{(f_+ + g_+)}^{\geq 0} - \overbrace{(f_- + g_-)}^{\geq 0}$ (Note: in general $(f+g)_+ \neq f_+ + g_+$)

$$\begin{cases} \int f_+ + g_+ \, d\mu = \int f_+ \, d\mu + \int g_+ \, d\mu < \infty \\ \int f_- + g_- \, d\mu = \int f_- \, d\mu + \int g_- \, d\mu < \infty \end{cases}$$

$\xRightarrow{\text{Prop 2.26}}$ $f+g \in \mathcal{L}_1(X, \mathbb{R})$ and

$$\int f+g \, d\mu = \int (f_+ + g_+) \, d\mu - \int (f_- + g_-) d\mu$$
$$= \int f_+ \, d\mu + \int g_+ \, d\mu - \int f_- \, d\mu - \int g_- \, d\mu$$
$$= \int f \, d\mu + \int g \, d\mu.$$

$\square$

iii) $A_\pm := \{x \in X \mid g_\pm(x) \neq f_\pm(x)\} \subseteq \{x \in X \mid g(x) \neq f(x)\} =: A$.

By corollary 2.24, applied to f_+, g_+ and f_-, g_-, the assertion follows.

iv) $g \geq f \Rightarrow g - f \geq 0 \Rightarrow J(g) - J(f) = J(g-f) \geq 0$.

$\square$

Corollary. $(\mathcal{L}_1(X,\mu), \|\cdot\|_1)$ is a seminormed space.

Proposition 2.28

$(X, \mathcal{A}, \mu)$ measure space, $f: X \to \mathbb{K}$ mb.

$\Rightarrow$ i) $f \in \mathcal{L}_1(X, \mathbb{K}) \iff |f| \in \mathcal{L}_1(X, \mathbb{K})$

ii) $f \in \mathcal{L}_1(X, \mathbb{K}) \Rightarrow |\int f \, d\mu| \leq \int |f| \, d\mu$.

Proof. Observe: • f mb $\Rightarrow |f|$ mb. (Cor. 2.7)

• f, g mb, $0 \leq f \leq g$ and $\int g \, d\mu < \infty \Rightarrow \int f \, d\mu < \infty$.

i) "$\Rightarrow$" $f \in \mathcal{L}_1(X, \mathbb{R}) \xRightarrow{\text{Def}} (\mathrm{Re} f)_\pm, (\mathrm{Im} f)_\pm \in \mathcal{L}_1(X, \mathbb{R})$

$\Rightarrow |f| \leq (\mathrm{Re} f)_+ + (\mathrm{Re} f)_- + (\mathrm{Im} f)_+ + (\mathrm{Im} f)_- \in \mathcal{L}_1$

"$\Leftarrow$" $|\mathrm{Re} f|, |\mathrm{Im} f| \leq |f|$

$\Rightarrow (\mathrm{Re} f)_\pm, (\mathrm{Im} f)_\pm \in \mathcal{L}_1(X, \mathbb{R})$.

ii) $\mathbb{K} = \mathbb{R}$: $\int |f| \, d\mu = \overbrace{\int f_+ \, d\mu}^{\geq 0} + \overbrace{\int f_- \, d\mu}^{\geq 0} \geq |\int f_+ \, d\mu - \int f_- \, d\mu|$

$\mathbb{K} = \mathbb{C}$. Let $R \in [0, \infty)$, $\varphi \in \mathbb{R}$ s.t. $\int_X f \, d\mu = R e^{i\varphi}$

$$\Rightarrow \underbrace{|\int_X f \, d\mu|}_{\in \mathbb{R}} = R = \int_X e^{-i\varphi} f \, d\mu = \overbrace{\int_X \mathrm{Re}(e^{-i\varphi} f) d\mu}^{\in \mathbb{R}} + i \overbrace{\underbrace{\int_X \mathrm{Im}(e^{-i\varphi} f) d\mu}_{\in \mathbb{R}}}^{= 0}$$

$$= \int_X \underbrace{\mathrm{Re}(e^{-i\varphi} f)}_{\in \mathbb{R}} d\mu \underset{\text{Case } \mathbb{K} = \mathbb{R}}{\leq} \int_X |e^{-i\varphi} f| \, d\mu$$

$$= \int_X |f| \, d\mu.$$

$\square$

Theorem 2.29 (Lebesgue's dominated convergence theorem)

$(X, \mathcal{A}, \mu)$ measure space.

Let $s \in \mathcal{L}_1(X, \mathbb{R})$, $s \geq 0$ and

$(f_n)_n$ family of functions and ~~f~~ $f: X \to \mathbb{R} \cup \{\pm\infty\}$ s.t.

$\forall n \in \mathbb{N}$ $f_n \in \mathcal{L}_1(X, \mathbb{R})$ and $|f_n| < s$ μ-a.e.

and $f_n \longrightarrow f$ μ-a.e.

Then: $f \in \mathcal{L}_1(X, \mathbb{R})$ and $\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$.

Proof.

Case 1 $\mathbb{K} = \mathbb{R}$.

By assumption: $f_n(x) \leq s(x)$, $-f_n(x) \leq s(x)$ μ-f.ü.

Let $A_n^+ := \{x \in X \mid s(x) < f_n(x)\} \in \mathcal{A}$ $\leadsto \mu(A_n^\pm) = 0$

$A_n^- := \{x \in X \mid s(x) < -f_n(x)\} \in \mathcal{A}$

$$\Rightarrow \left.\begin{cases} A^+ := \bigcup_{n \in \mathbb{N}} A_n^+ \cup \{x \in X \mid f_n(x) \not\to f(x)\} \\ A^- := \underbrace{\bigcup_{n \in \mathbb{N}} A_n^-}_{\in \mathcal{A}} \cup \underbrace{\{x \in X \mid f_n(x) \not\to f(x)\}}_{\in \mathcal{A} \text{ because: } (\limsup f_n, \liminf f_n \text{ mb!})} \end{cases}\right\} \text{ are zero sets}$$

$= A_\infty \cup B$

donde $A_\infty := \{x \in X \mid \limsup_{n\to\infty} f_n(x) = \pm\infty$ o $\liminf_{n\to\infty} f_n = \pm\infty\}$

$B = \{x \in X \mid \limsup_{n\to\infty} f_n(x) - \liminf_{n \to\infty} f_n(x) \neq 0\}$

A_∞, B mb

Define

$$\varphi_n(x) := \begin{cases} 0, & x \in A_n^+ \\ s(x) - f_n(x), & x \notin A_n^+ \end{cases} \qquad \psi_n(x) := \begin{cases} 0, & x \in A_n^- \\ s(x) + f_n(x), & x \notin A_n^- \end{cases}.$$

$\to$ $\forall n \in \mathbb{N}$. φ_n, ψ_n are measurable and ≥ 0.

$\forall x \in X \setminus A_+$: $\varphi_n(x) = s(x) - f_n(x) \longrightarrow s(x) - f(x) \geq 0$

$\forall x \in X \setminus A_-$: $\psi_n(x) = s(x) + f_n(x) \longrightarrow s(x) + f(x) \geq 0$

$\Rightarrow$ $\forall x \in X \setminus (A_+ \cup A_-)$: $|f(x)| \leq s(x)$.

Since $A_+ \cup A_-$ zero set $\Rightarrow$ $f \in \mathcal{L}_1(X, \mathbb{R})$.

By corollary 2.25 (Fatou):

(1) $\underbrace{\int s(x) - f(x)\, d\mu}_{= \int s\, d\mu - \int f\, d\mu} \leq \liminf_{n\to\infty} \int s(x) - f_n(x)\, d\mu = \int s(x) dx - \limsup_{n\to\infty} \int f_n(x) d\mu$

(2) $\underbrace{\int s(x) + f(x)\, d\mu}_{= \int s(x)\, d\mu + \int f\, d\mu} \leq \liminf_{n\to\infty} \int s(x) + f_n(x)\, d\mu = \int s(x) dx + \liminf_{n\to\infty} \int f_n(x) d\mu$

$\Rightarrow$ $\int f(x) d\mu \overset{(2)}{\leq} \liminf \int f_n(x) d\mu \overset{(1)}{\leq} \limsup \int f_n(x) d\mu \leq \int f(x) d\mu$. □

Case 2. $\mathbb{K} = \mathbb{C}$.

Let $g_n := \mathrm{Re}(f_n)$, $h_n := \mathrm{Im} f_n$.

$\Rightarrow$ $g_n, h_n \in \mathcal{L}_1(X, \mathbb{R})$, $|g_n|, |h_n| \leq s$ μ-a.e.

and $g_n \longrightarrow \mathrm{Re}(f)$, $h_n \longrightarrow \mathrm{Im}(f)$ μ-a.e.

$\Rightarrow$ g_n, h_n satisfy the hypothesis of Case 1. □

Observe: Existence of the dominating fct ϕ is crucial.

Example: $f_n: \mathbb{R} \to \mathbb{R}$, $f(x) := \frac{1}{2n} \chi_{[0,n]}$

$\Rightarrow$ $f_n \to 0$ uniformly.

nevertheless: $\int_{\mathbb{R}} f_n d\mu = 1 \not\to 0 = \int_{\mathbb{R}} 0\, d\mu$

Integration over subspaces: (Blatt 4, Aufgabe 2)

$(X, \mathfrak{A}, \mu)$ measure space; $B \subseteq X$ measurable; $B \neq \emptyset$.

$\to$ induced measure space $(B, \mathfrak{A}_B, \mu_B)$ where

$\mathfrak{A}_B := \{A \cap B \mid A \in \mathfrak{A}\}$, $\mu_B := \mu|_{\mathfrak{A}_B}$.

Let $f: X \to \mathbb{R} \cup \{\pm\infty\}$.

$\Rightarrow$ $f|_B$ is $\mathfrak{A}_B$ measurable $\Leftrightarrow$ $f \cdot \chi_B$ is $\mathfrak{A}$-mb.

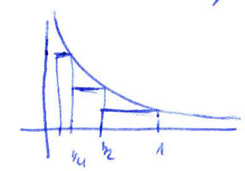
If $f \geq 0$, then: $\int_B f|_B d\mu_B = \int_X f \chi_B \cdot d\mu$.

If $f \in \mathcal{L}_1(X, \mathfrak{A}, \mu)$ $\Rightarrow$ $f|_B \in \mathcal{L}_1(B, \mathfrak{A}_B, \mu_B)$ and

$$\int_X \chi_B \cdot f\, d\mu = \int_B f|_B\, d\mu_B.$$

Examples

- $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ is measurable because it is continuous.
  It is not integrable because $\int_{\mathbb{R}} f\, d\lambda = \int_{\mathbb{R}} \chi_{\mathbb{R}} d\mu = 1 \cdot \mu(\mathbb{R}) = \infty$.
- $f: (0,1] \to \mathbb{R}$, $f(x) = 1/x$ is measurable (because it is continuous)
  It is not integrable because:
  $\forall n \in \mathbb{N}$ $\varphi_n := \sum_{m=1}^{n} 2^{m-1} \chi_{(2^{-m}, 2^{-m+1}]}$

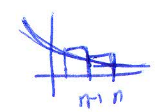


  $\Rightarrow$ $\forall x \in (0,1]$: $\forall n \in \mathbb{N}$ $\varphi_n(x) \leq f(x)$.

  $\overset{\text{Def. of the integral}}{\Longrightarrow}$ $\int_{(0,1]} f\, d\mu \geq \int_{(0,1]} \varphi_n d\mu = \sum_{m=1}^{n} 2^{m-1} \cdot 2^{-m} = n/2$ for all $n \in \mathbb{N}$

  $\Rightarrow$ $\int_{(0,1]} f\, d\mu = \infty$

- $f: [0, \infty) \to \mathbb{R}$, $f(x) = e^{-x}$ is measurable (because continuous) and integrable.
  Because: Obviously $f \leq g := \sum_{n=0}^{\infty} e^{-n} \chi_{[n, n+1)}$



  $\forall n \in \mathbb{N}$ let: $\varphi_m := \sum_{n=0}^{\infty} e^{-n} \chi_{[n, n+1)}$

Then: $0 \le \varphi_1 \le \varphi_2 \le \dots$ and $\varphi_n \to g$ pointwise.

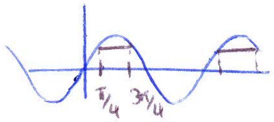
$$\Rightarrow \int g\,d\mu \underset{\text{mon. convergence}}{=} \lim_{n\to\infty}\int \varphi_n\,d\mu = \lim_{n\to\infty}\sum_{k=0}^{n} e^{-k} = \sum_{k=0}^{\infty} e^{-k} = \frac{e}{e-1} < \infty$$

$$\Rightarrow g \in \mathcal{L}_1([0,\infty),\mathbb{R}) \underset{0\le f\le g}{\Longrightarrow} f \in \mathcal{L}_1([0,\infty),\mathbb{R})$$

- $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \sin x$ is not integrable.

Proof. It suffices to show that $f_+:\mathbb{R}\to\mathbb{R}$, $f_+(x) := \max\{0, f(x)\}$ is not integrable.

Note: $f_+ \geq g := \frac{1}{2}\sqrt{2}\,\chi_A \geq 0$



where $A = \bigcup_{n\in\mathbb{Z}} [2n\pi + \pi/4,\ 2n\pi + 3\pi/4]$

Since $\int_{\mathbb{R}} g\,d\lambda = \frac{1}{2}\sqrt{2}\,\mu(A) = \infty$, $f_+ \notin \mathcal{L}_1([0,\infty),\mathbb{R})$.

Application of convergence theorems to integrals which depend on a parameter.

First observe:

Lemma 2.30.

$(X,\mathfrak{A},\mu)$ measure space, $f_n, f: X\to\mathbb{R}$ measurable.

Then: $\{x\in X \mid f_n \cancel{(x)} \not\to f(x)\}$

$$= \bigcup_{k\in\mathbb{N}} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x\in X \mid |f_n(x) - f(x)| \geq 1/k\} \in \mathfrak{A}.$$

Proof. Let $x_0\in X$. Then:

$f_n(x_0) \not\to f(x_0)$

$\Leftrightarrow \exists k\in\mathbb{N}$ s.t. $\forall N\in\mathbb{N}\ \exists n\geq N$ with $|f_n(x_0) - f(x_0)| \geq 1/k$

$\Leftrightarrow \exists k\in\mathbb{N}$ s.t. $\forall N\in\mathbb{N}$ $x_0 \in \bigcup_{n\geq N}\{x\in X \mid |f_n(x)-f(x)| \geq 1/k\}$

$\Leftrightarrow \exists k\in\mathbb{N}$ s.t. $x_0 \in \bigcap_{N\in\mathbb{N}}\bigcup_{n\geq N}\{x\in X \mid |f_n(x)-f(x)| \geq 1/k\}$

$\Leftrightarrow x_0 \in \bigcup_{k\in\mathbb{N}}\bigcap_{N\in\mathbb{N}}\bigcup_{n\geq N}\{x\in X \mid |f_n(x)-f(x)| \geq 1/k\}$ □

Theorem 2.31. (Continuity)

T metric space, $(X,\mathfrak{A},\mu)$ measure space, $t_0\in T$.

Let $f: T\times X \to \mathbb{K}$ s.t.

i) $\forall t\in T$ $f(t,\cdot) \in \mathcal{L}_1(X,\mathbb{K})$

ii) for μ-a.e. $x\in X$ $f(\cdot,x): T\to\mathbb{K}$ is continuous in t_0

iii) $\exists U$ neighbourhood of t_0 and $g\in\mathcal{L}_1(X,\mathbb{K})$, $g\geq 0$, s.t. $|f(t,x)| \leq g(x)$ for all $t\in U$ and μ-a.e. $x\in X$.

Then the maps

$F: T\to\mathbb{K}$, $F(t) = \int_X f(t,x)\,d\mu$

$\Phi: T \to L_1(X)$, $\Phi(t) = f(t,\cdot)$

are continuous in t_0.

Proof. Let $(t_n)_n \subseteq U$ with $t_n \to t_0$ $(n\to\infty)$.

Then $f(t_n,\cdot) \to f(t_0,\cdot)$ μ-a.e. and $\forall n\in\mathbb{N}$ $|f(t_n,x)| \leq g(x)$, μ-a.e.

By the dominated convergence thm:

$$F(t_n) \to F(t_0)$$

and $\|f(t_n,\cdot) - f(t_0,\cdot)\|_1 \to 0$, $n\to\infty$.

□

Theorem 2.32. (Differentiability)

$(X, \mathcal{A}, \mu)$ measure space, $D \subseteq \mathbb{R}^s$ open, $j \in \{1, \dots, s\}$.

$f: D \to \mathbb{R}$ s.t.

i) $\forall t \in D \quad f(t, \cdot) \in \mathcal{L}_1(X, \mathbb{R})$

ii) for almost all $x \in X \quad f(\cdot, x)$ has jth partial derivative

iii) $\exists \phi \in \mathcal{L}_1(X, \mathbb{R})$ s.t. $\|\frac{\partial f}{\partial t_j}(t, x)\| \le \phi(x)$ for all $t \in D$ and almost all $x \in X$.

Fix $\tau \in D$; let $N := \{x \in X \mid \frac{\partial f}{\partial t_j}(\tau, x)$ does not exist$\}$ and $\tilde{N} \in \mathcal{A}$ with $N \subseteq \tilde{N}$ and $\mu(\tilde{N}) = 0$.
Then:

$$\varphi: X \to \mathbb{R}, \quad \varphi(x) = \begin{cases} \partial f/\partial t_j(\tau, x), & x \in X \setminus \tilde{N} \\ 0, & x \in \tilde{N}. \end{cases}$$

is measurable and the function

$$F: D \to \mathbb{R}, \quad F(t) := \int_X f(t, x)\, d\mu$$

has jth partial derivative $\frac{\partial F}{\partial t_j}(\tau) = \int_{X \setminus \tilde{N}} \frac{\partial f}{\partial t_j}(\tau, x) d\mu = \int_X \varphi(x)\, d\mu$.

Proof.

Let $(\varepsilon_n)_n \subseteq \mathbb{R}$ s.t. $\varepsilon_n \to 0$ and $\tau + \varepsilon_n e_j \in D$ $(n \in \mathbb{N})$ [e_j = jth unit vector]

$\Rightarrow \forall n \in \mathbb{N} \quad \varphi_n(x) := \frac{1}{\varepsilon_n}(f(\tau + \varepsilon_n e_j, x) - f(\tau, x))$ is measurable.

$\Rightarrow \tilde{\varphi} := \limsup_{n \to \infty} \varphi_n$ is measurable.

$\Rightarrow \varphi = \tilde{\varphi} \cdot \chi_{X \setminus \tilde{N}} \quad (x \in X \setminus \tilde{N} \Rightarrow \tilde{\varphi}(x) = \frac{\partial f}{\partial t_j}(\tau, x) = \varphi(x))$

$\Rightarrow \varphi$ measurable.

By the mean value theorem:

$$1/\varepsilon_n \,|f(\tau + \varepsilon_n e_j, x) - f(\tau, x)| \le \sup_{t \in D} \left|\frac{\partial f}{\partial t_j}(t, x)\right| \le \phi(x)$$

μ-a.e.

By the dominated convergence thm:

$$\int_{X \setminus \tilde{N}} \frac{\partial f}{\partial t_j}(\tau, x)\, d\mu = \int_{X \setminus \tilde{N}} \lim_{n \to \infty} \frac{1}{\varepsilon_n}(f(\tau + \varepsilon_n e_j, x) - f(\tau, x))\, d\mu$$

$$\overset{\text{dom. conv.}}{=} \lim_{n \to \infty} \int_{X \setminus \tilde{N}} \frac{1}{\varepsilon_n}(f(\tau + \varepsilon_n e_j, x) - f(\tau, x))\, d\mu$$

$$\overset{\mu(\tilde{N}) = 0}{=} \lim_{n \to \infty} \frac{1}{\varepsilon_n}(F(\tau + \varepsilon_n e_j) - F(\tau)).$$

Since $(\varepsilon_n)_n$ was arbitrary with $\varepsilon_n \to 0$, it follows that the jth partial derivative of F in τ exists, and

$$\frac{\partial F}{\partial t_j}(\tau) = \int \frac{\partial f}{\partial t_j}(\tau, x)\, d\mu.$$

□

Remark. If in the theorem instead of ii) and iii) we assume:

ii') for almost all $x \in X \quad f(\cdot, x)$ is differentiable

iii') $\exists \phi \in \mathcal{L}_1(X)$ s.t. $\|D_t f(t, x)\| \le \phi(x)$ for all $t \in D$ and almost all $x \in X$ (D_t: total derivative w.r.t. t)

Then: F is differentiable and for all $h \in \mathbb{R}^s$:

$$D_t F(t) h = \int_X D_t f(x, t)\, h\, d\mu.$$

Proof. Let $\tilde{N} \in \mathcal{A}$ s.t. $\mu(\tilde{N}) = 0$ and $\tilde{N} \supseteq N := \{x \in X \mid f$ not diff$\}$.

By definition of Differentiability:

$\forall x \in X \setminus \tilde{N}, \ \tau \in D \quad \exists\, \varepsilon(\cdot, x): D \to \mathbb{R}^s$ s.t.

$$\begin{cases} f(t, x) - f(\tau, x) = (D_t f)(\tau, x)(t - \tau) + \varepsilon(t, x)\|t - \tau\| \\ \text{and } \varepsilon(t, x) \xrightarrow[t \to \tau]{} \varepsilon(\tau, x) = 0 \end{cases} \qquad (*)$$

As before $x \mapsto (D_t f)(\tau, x)(t - \tau)$ is mb (linear combination of partial derivatives) and dominated by ϕ, hence it is integrable.

By (*): $\varepsilon(t,\cdot)|t-\tau|$ is integrable.

$$\Rightarrow F(t)-F(\tau) = \int_X f(t,x) - f(\tau,x)\, d\mu$$
$$= \int_X (D_t f)(\tau,x)\,(t-\tau) + \varepsilon(x,t)\,|t-\tau|\, d\mu$$
$$= \int_X (D_t f)(\tau,x)\,(t-\tau)\, d\mu + |t-\tau| \int_X \varepsilon(x,t)\, d\mu$$

Hence it remains to prove $\int_X \varepsilon(x,t)\, d\mu \to 0$, $t\to\tau$.

Since $\varepsilon(x,t)\to 0$, $t\to\tau$, it remains to find an integrable function which dominates $\varepsilon(\cdot,t)$ for all t. The assertion follows then from the dominated convergence theorem.

$$\varepsilon(x,t) = \frac{1}{\|t-\tau\|}\left(f(t,x) - f(\tau,x) - (D_t f)(\tau,x)(t-\tau)\right)$$
$$\overset{\text{MWS}}{=} \frac{1}{\|t-\tau\|}\left(D_t(s(t),x) - (D_t f)(\tau,x)\right)(t-\tau)$$

$$\Rightarrow \|\varepsilon(x,t)\| \le 2\,\phi(x).$$

Since $\phi\in\mathcal{L}_1(X,\mathbb{R})$, the claim is proved. $\square$

Application: Fourier transformation

Let $f\in\mathcal{L}_1(\mathbb{R},\mathbb{C})$.

Since for all $t\in\mathbb{R}$, $x\in\mathbb{R}$, $|e^{ixt}f(x)| = |f(x)|$, $x\mapsto e^{-ixt}f(x) \in \mathcal{L}_1(\mathbb{R},\mathbb{C})$

$\Rightarrow$ The Fourier transform of f is well-defined:

$$\tilde f(t) := \int_{\mathbb{R}} e^{-ixt} f(x)\, dx.$$

Thm. 2.33. $f\in\mathcal{L}_1(\mathbb{R},\mathbb{C})$. Then $\tilde f$ is bounded and continuous.

Proof. $|\tilde f(t)| = \left|\int_{\mathbb{R}} e^{-ixt} f(x)\,dx\right| \le \int_{\mathbb{R}} |e^{-ixt} f(x)|\,dx$
$= \int_{\mathbb{R}} |f(x)|\,dx = \|f\|_1 < \infty$

Since $t\mapsto e^{-ixt}f(x)$ is continuous for all $x\in\mathbb{R}$ and $x\mapsto e^{-ixt}f(x)$ is dominated by $|f|$ for all $t\in\mathbb{R}$

continuity of $\tilde f$ follows from Lebesgue's dominated convergence thm. (Thm. 2.31)

Theorem 2.34.

Assume $f\in\mathcal{L}_1(\mathbb{R},\mathbb{C})$ and $x\mapsto x f(x)\in\mathcal{L}_1(\mathbb{R},\mathbb{C})$.

Then: $\tilde f$ is differentiable and $i\tilde f'(t) = \int_{\mathbb{R}} e^{-ixt} x f(x)\,dx = (xf(x))^\sim(t)$.

Proof. Since $|e^{-ixt} x f(x)| = |x f(x)|$, Lebesgue's dom. conv. thm (Thm 2.32) yields:

$$\int_{\mathbb{R}} e^{-ixt} x f(x)\,dx = \int_{\mathbb{R}} i\frac{d}{dt}\left(e^{-ixt} f(x)\right) dx$$
$$= i\frac{d}{dt}\int_{\mathbb{R}} e^{-ixt} f(x)\,dx = i\frac{d}{dt}\tilde f(t)$$
$\square$

Riemann integral & Lebesgue integral

Recall: $f:[a,b]\to\mathbb{R}$ bounded.

$P := \{x_0, x_1, \dots, x_n\}$ partition of $[a,b]$ with $x_0 = a < x_1 < x_2 < \dots < x_n = b$

$s(f,P) := \sum_{j=1}^n m_j (x_j - x_{j-1})$ with $m_j := \inf\{f(x) \mid x\in(x_{j-1},x_j)\}$

$S(f,P) := \sum_{j=1}^n M_j (x_j - x_{j-1})$ with $M_j := \sup\{f(x)\mid x\in(x_{j-1},x_j)\}$

f is Riemann-integrable

$\Leftrightarrow$ $\sup\{s(f,P)\mid P \text{ part. of } [a,b]\} = \inf\{S(f,P)\mid P \text{ part. of } [a,b]\}$.

$\Leftrightarrow$ $\exists\,(P_n)_n$ sequence of partitions with $P_1\subseteq P_2\subseteq P_3\subseteq\dots$

s.t. $\lim_{n\to\infty} s(f,P_n) = \lim_{n\to\infty} S(f,P_n) =: R\text{-}\int_a^b f(x)\,dx.$

Define functions

$$s_n:[a,b]\to\mathbb{R},\quad s_n(x) = \begin{cases} f(a), & x=a \\ m_{n,j}, & x\in(x_{n,j-1}, x_{n,j}],\ j=1,\dots,n.\end{cases}$$

$$S_n:[a,b]\to\mathbb{R},\quad S_n(x) = \begin{cases} f(a), & x=a \\ M_{n,j}, & x\in(x_{n,j-1}, x_{n,j}],\end{cases}$$

where $P_n := \{a, x_{n,1}, \dots, x_{n,n}\}$ and $m_{n,j}, M_{n,j}$ the corresp. inf{ } and sup{..}.

Then: for almost all $x \in [a,b]$ (at least for $x \in [a,b] \setminus \overbrace{\bigcup_{n \in \mathbb{N}} \{x_0, x_{n,1}, \dots, x_{n,n}\}}^{\text{countable!}}$)

$$s_1 \le s_2 \le \dots \le f \le \dots \le S_2 \le S_1$$

Obviously: all s_n, S_n are simple functions and pointwise almost everywhere.

~~Let s, S: lim~~

Let $s, S : [a,b] \to \mathbb{R}$ s.t. $s(x) = \lim_{n\to\infty} s_n(x)$, $S(x) = \lim_{n\to\infty} S_n(x)$ where the limit exists.

Theorem 2.35. Let $f : [a,b] \to \mathbb{R}$, λ Lebesgue measure on $[a,b]$

i) f R-integrable $\Rightarrow$ f L-integrable and

$$R\text{-}\int_a^b f(x)\,dx = \int_{[a,b]} f \, d\lambda$$

ii) f bdd. Then: f R-integrable $\iff$ f cont. λ-a.e. on $[a,b]$.

Proof. s_n, S_n, s, S as above. Note: s, S measurable.

i) Assume f R-integrable.

$$\Rightarrow \int_{[a,b]} s \, d\lambda \underset{\substack{\uparrow \\ \text{dom. conv} \\ (s_1 \in \mathcal{L}_1([a,b],\mathbb{R}))}}{=} \lim_{n\to\infty} \int_{[a,b]} s_n \, d\lambda \underset{\substack{\uparrow \\ f \text{ R-integrable}}}{=} \lim_{n\to\infty} \int_{[a,b]} S_n \, d\lambda \underset{\substack{\uparrow \\ \text{dom. conv.} \\ (S_1 \in \mathcal{L}_1([a,b],\mathbb{R}))}}{=} \int_{[a,b]} S \, d\lambda$$

$$\Rightarrow 0 = \int_{[a,b]} (S - s)\, d\lambda$$

$$\Rightarrow S = s \quad \lambda\text{-a.e.} \quad (\text{Cor. } 2.22) \qquad (**)$$

Since $\forall n$ $s_n \le f \le S_n$ $\Rightarrow$ $s \le f \le S$ λ-a.e.

With $(**)$: $f = s = S$ λ-a.e.

$\Rightarrow$ f measurable and $\int_{[a,b]} f \, d\lambda = \int_{[a,b]} s \, d\lambda = R\text{-}\int_a^b f(x)\,dx$.

ii) Let $x \in [a,b] \setminus \bigcup_{n \in \mathbb{N}} \{x_0, x_{n,1}, \dots, x_{n,n}\}$

Then: f continuous in x $\iff$ $s(x) = S(x)$

("$\Leftarrow$" $\varepsilon > 0 \Rightarrow \exists N \in \mathbb{N}: \forall n \ge N$ $S_n(x) - \varepsilon \le f(x) \le s_n(x) + \varepsilon$

let j s.t. $x \in (x_{n,j-1}, x_{n,j})$

$\Rightarrow \forall y \in (x_{n,j-1}, x_{n,j})$: $S_n(x) \ge f(y) \ge s_n(y) = s_n(x)$

"$\Rightarrow$" Assume f cont. in x. $\forall n \in \mathbb{N}$ let $j(n) \in \{0, \dots, n\}$ s.t. $x \in (x_{n,j-1}, x_{n,j})$

Assume $s_n(x) \to a \neq f(x)$ ($\Rightarrow a < f(x)$)

$\Rightarrow$ $x_{n_j} - x_{n,j-1} \not\to 0$ (because otherwise $s_n(x) = \inf\{\dots\} \to f(x)$ by cont. of f)

$\Rightarrow \int_a^b S_n(x) - s_n(x)\,dx \ge \int_{x_{n,j-1}}^{x_{nj}} S_n(x) - s_n(x)\,dx$

$\ge (f(x) - a)(x_{n_j} - x_{n_{j-1}}) \not\to 0$, $\Rightarrow$ f not R-integrable ↯]

$\iff \lim \int s_n \, dx = \lim \int S_n \, dx$

$\Rightarrow$ f R-integrabel $\iff$ $s = S$ λ-a.e.

$\iff$ f cont. a.e.

$\square$

Examples

1) $D : [0,1] \to \mathbb{R}$, $D(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap [0,1] \\ 1, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$.

D is not Riemann-integrable, because it is discontinuous everywhere.

~~Howe~~ However: D L-measurable because $= 1$ λ-a.e.

and $\int_{[0,1]} D \, d\lambda = \int_{[0,1]} 1 \, d\lambda = 1$.

2). $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x \in [0,1] \setminus \mathbb{Q} \\ 1/q, & x \in \mathbb{Q} \cap [0,1] \text{ where } x = p/q \text{ with } \operatorname{ggT}(p,q) = 1 \end{cases}$

$\Rightarrow$ f discontinuous only in the countable set $\mathbb{Q} \cap [0,1]$

$\Rightarrow$ f is R-integrable and $f = 0$ λ-a.e.

$\Rightarrow$ $R\text{-}\int_0^1 f(x)\,dx = \int_{[0,1]} f \, d\lambda = \int_{[0,1]} 0 \, d\lambda = 0$.

3) A function that is R-integrable, L-integrable but not Borel-mb:

Let C = Cantor set, $A \subseteq C$ s.t. $A \notin \mathcal{B}$

$f : [0,1] \to \mathbb{R}$, $f = \chi_A$.

$\Rightarrow$ $f \equiv 0$ in $[0,1] \setminus C$ $\Rightarrow$ f R-integrable and $R\text{-}\int_0^1 f \, dx = 0$
(because continuous in $[0,1] \setminus C$ ($\leftarrow$ open set!)

But A not Borel set, $\Rightarrow$ f no Borel-mb.

Remark. Theorem 2.35 is __not__ true for improper Riemann-integrals.

__Example.__ $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{\sin x}{x}$.

Then: $R - \int_0^\infty f(x)dx$ exists, but $f \in \mathcal{L}_1((0,\infty), \mathbb{R})$.

Without proof:

Theorem $J \subseteq \mathbb{R}$ interval, $f: J \to \mathbb{K}$ R-integrable on every cpt. subset of J. Then:

f L-integrable $\Leftrightarrow$ $|f|$ improper R-integrable.

in this case: $\int_J f \, d\lambda = R - \int_J f(x) dx$.

__Proof.__ Elstrodt, Satz 6.3 (S.153). □

2.4. Convergence.

__Recall__ $f_n: X \to \mathbb{K}$ (measurable) where $(X, \mathcal{A}, \mu)$ is a measure space.

Then: $f_n \to f$ __pointwise__, if $f_n(x) \to f(x)$ for all $x \in X$

$f_n \to f$ __μ-a.e.__, if $\exists N \in \mathcal{A}$ s.t. $\mu(N) = 0$ and $f_n(x) \to f(x)$ for all $x \in X \setminus N$.

Proposition 2.36. $f_n, f: X \to \mathbb{K}$ measurable.

i) $f_n \to f$ μ-ae $\Leftrightarrow$ $\forall \varepsilon > 0$ $\mu\left(\bigcap_{n=1}^\infty \bigcup_{k \geq n} \{|f_n - f| \geq \varepsilon\}\right) = 0$

ii) If $\mu(X) < \infty$, then:

$f_n \to f$ μ-ae $\Leftrightarrow$ $\forall \varepsilon > 0$ $\lim_{n\to\infty} \mu\left(\bigcup_{k \geq n} \{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0$

__Proof.__

i) $f_n \to f$ μ-ae $\Leftrightarrow$ $\forall \varepsilon > 0$: $\mu\{x \in X \mid \forall n \geq 1 \; \exists k \geq n$: $|f_n(x) - f(x)| > \varepsilon\} = 0$

ii) If $\mu(X) < \infty$, then $\mu\left(\bigcup_{k \in \mathbb{N}} \{|f_n - f| \geq \varepsilon\}\right) < \infty$

$\Rightarrow \mu\left(\bigcap_{n=1}^\infty \ldots\right) = \lim_{n\to\infty} \mu(\ldots)$ □

__Observation:__ If $\mu(X) = \infty$, then "$\Leftarrow$" is still true.

"$\Rightarrow$" in general does not hold. Example: $f_n := \chi_{[n,\infty)}$

Then: $f_n \to 0$ everywhere, but $\mu\left\{\bigcup_{k \geq n} \{x \in X \mid |f_n(x) - 0| \geq \frac{1}{2}\}\right\} =$

for all $n \in \mathbb{N}$.