2. Measurable Functions & Integration

2.1. Measurable Functions

Definition 2.1. $(X, \mathfrak{A})$ measurable space (Maßraum)
$:\iff$ X conjunto, $\neq \emptyset$, $\mathfrak{A}$ σ-algebra sobre X.

Definition 2.2. $(X, \mathfrak{A})$, $(Y, \mathcal{B})$ measurable spaces, $f: X \to Y$ is called measurable (or: $\mathfrak{A}$-$\mathcal{B}$-measurable) if:

$$\forall M \in \mathcal{B} \quad f^{-1}(M) \in \mathfrak{A}.$$

Lemma 2.3. Assume E ($\subseteq \mathcal{P}(Y)$) generates the σ-algebra $\mathcal{B}$.
Then: f $\mathfrak{A}$-$\mathcal{B}$-measurable $\iff$ $\forall M \in E \quad f^{-1}(E) \in \mathfrak{A}$. $\square$

Lemma 2.4. "Composition rule"
$(X, \mathfrak{A}), (Y, \mathcal{B}), (Z, \mathcal{C})$ measurable spaces, $f: X \to Y$, $g: Y \to Z$ mb.
$\implies g \circ f$ mb. $\square$

Recall $(X, \mathcal{T})$ top. space $\rightsquigarrow$ $\mathcal{B} :=$ Borel-σ-algebra $:=$ σ-algebra gen. by $\mathcal{T}$.

$(X, \mathcal{T}), (Y, \mathcal{S})$ top spaces. $f: X \to Y$ is called Borel-measurable if f is $\mathcal{B}(\mathcal{T})$-$\mathcal{B}(\mathcal{S})$-measurable.

Lemma 2.5. X, Y top spaces, $f: X \to Y$ continuous
$\implies f$ Borel-measurable. $\square$

Lemma 2.6. X measurable space, Y, Z top spaces; $f: X \to Y \times Z$, $f(x) = (f_1(x), f_2(x))$
$\implies (f \text{ mb} \iff f_1, f_2 \text{ mb.})$

Proof. "$\Rightarrow$" Note that $f_1 = pr_Y \circ f$ is mb because pr_Y is cont.
"$\Leftarrow$" Base de la topología en $Y \times Z$: $\{A \times B \mid A \subseteq Y \text{ open}, B \subseteq Z \text{ open}\}$
$f^{-1}(A \times B) = f_1^{-1}(A) \cap f_2^{-1}(B) \in \mathfrak{A}$

Now Functions with values in $\mathbb{R}$ or $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ or $\mathbb{C}$

Corollary 2.7. Sums, products, quotients (if they exist) of mb. functions ~~with values in~~ $f: X \to Y$ ($Y = \mathbb{R}$ or $\overline{\mathbb{R}}$ or $\mathbb{C}$) are mb.

Proof. Sum, product, quotient: $Y \times Y \to Y$ is cont. (if exists)
Also: $|\cdot|: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ cont $\implies$ $|f|$ mb if f mb.

Corollary 2.8. (Characterisation of $\overline{\mathbb{R}}$-valued mb functions)
$(X, \mathfrak{A})$ mb. space, $f: X \to \overline{\mathbb{R}}$.
Then:
$$\begin{aligned} f \text{ mb} &\iff \forall a \in \overline{\mathbb{R}} \quad f^{-1}([-\infty, a[) \in \mathfrak{A} \\ &\iff \forall a \in \overline{\mathbb{R}} \quad f^{-1}([-\infty, a]) \in \mathfrak{A} \\ &\iff \forall a \in \overline{\mathbb{R}} \quad f^{-1}([a, \infty]) \in \mathfrak{A} \\ &\iff \forall a \in \overline{\mathbb{R}} \quad f^{-1}(]a, \infty]) \in \mathfrak{A} \quad \text{etc.} \end{aligned}$$

Proof. The families $([-\infty, a[)_{a \in \overline{\mathbb{R}}}$, ... generate the Borel-σ-algebra of $\overline{\mathbb{R}}$.
(topology on $\overline{\mathbb{R}}$: $A \subseteq \overline{\mathbb{R}}$ open $\iff$ $A \cap \mathbb{R}$ open and $\exists a \in \mathbb{R}:]a, \infty] \subseteq A$ if $\infty \in A$, $\exists b \in \mathbb{R}: [-\infty, b[\subseteq A$ if $-\infty \in A$
$\rightsquigarrow \overline{\mathbb{R}}$ is compact and $\mathbb{R} \subseteq \overline{\mathbb{R}}$ dense)

Corollary 2.9.
1) $f: X \to \mathbb{C}$ mb $\iff$ $\mathrm{Re}(f), \mathrm{Im}(f): X \to \mathbb{C}$ mb.
2) $f: X \to \overline{\mathbb{R}}$ mb $\iff$ $f_+, f_-: X \to \overline{\mathbb{R}}$ mb

where: $\begin{cases} f_+(x) := \max\{f(x), 0\} \\ f_-(x) := \max\{-f(x), 0\} \end{cases}$ $\rightsquigarrow$ $f = f_+ - f_-$, $|f| = f_+ + f_-$

Pointwise limits of mb fct's:

Proposition 2.10. $(X, \mathcal{A})$ mb space, $f_n : X \to \overline{\mathbb{R}}$ mb functions.

In the following: $\sup_n f_n$, $\inf_n f_n$ etc.: pointwise defined.

(E.g. $\sup_n f_n : X \to \overline{\mathbb{R}}$, $(\sup_n f_n)(x) := \sup_n (f_n(x))$.)

i) $\sup f_n$, $\inf f_n : X \to \overline{\mathbb{R}}$ mb.

ii) $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n : X \to \overline{\mathbb{R}}$ mb.

iii) $\lim_{n\to\infty} f_n : X \to \overline{\mathbb{R}}$ mb, if the pointwise limit exists.

Proof.

i) Let $f : X \to \overline{\mathbb{R}}$, $f(x) = \sup_n f_n(x)$.

We will show: $\forall a \in \overline{\mathbb{R}}$: $f^{-1}(]a,\infty]) \in \mathcal{A}$.

Fix $a \in \overline{\mathbb{R}}$. Then:

$$x \in f^{-1}(]a,\infty]) \iff f(x) > a \iff \sup_{n} f_n(x) > a$$
$$\iff \exists n \in \mathbb{N} \text{ s.t. } f_n(x) > a$$
$$\iff x \in \bigcup_{n\in\mathbb{N}} f_n^{-1}(]a,\infty]) \in \mathcal{A}.$$

By Cor. 2.8: f is mb.

$\Rightarrow \inf_n f_n = -\sup_n(-f_n) = -\sup_n(-f_n)$ mb. (f mb $\iff -f$ mb)

ii) $\forall x \in X$ $\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\sup_{k\geq n} f_k(x)\right) = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k(x)$

By i): $F_n := \sup_{k\geq n} f_k$ mb $\overset{i)}{\Rightarrow} f = \inf F_n$ mb.

Analogously for liminf.

iii) Consequence of ii).

$\square$

$(X, \mathcal{A}, \mu)$ measure space. A property holds <u>μ-almost everywhere</u> if it is true for all $X \setminus N$ where $N \in \mathcal{N}$

that is, $\exists M \in \mathcal{A}$ ~~with~~ with $N \subseteq M$ and $\mu(M) = 0$ —

Proposition 2.11. $(X, \mathcal{A}, \mu)$ complete measure space, $f, g : X \to \overline{\mathbb{R}}$.

Assume $f = g$ μ-a.e. (i.e. $\exists N \in \mathcal{A}$ with $\mu(N) = 0$ and $f(x) = g(x)$, $x \in X \setminus N$).

Then: f mb $\iff$ g mb.

Proof. Assume f mb. Let $A := \{x \in X \mid f(x) \neq g(x)\}$.

$\Rightarrow \exists N \in \mathcal{A}$ s.t. $A \subseteq N$ and $\mu(N) = 0$ because $f = g$ μ-a.e.

$\Rightarrow A \in \mathcal{A}$ because X is complete.

Now: $\forall M \in \mathcal{L}$: $g^{-1}(M) = \{x \in X \mid g(x) \in M\}$

Lebesgue mb sets

$$= \{x \in X \mid f(x) \in M, x \notin A\} \cup \{x \in X \mid g(x) \in M, x \in A\}$$
$$= \underbrace{(f^{-1}(M) \setminus A)}_{\in \mathcal{A} \text{ because } f^{-1}(M) \text{ and } A \in \mathcal{A}} \cup \underbrace{(g^{-1}(M) \cap A)}_{\subseteq A, \text{ hence } \in \mathcal{A} \text{ (completeness of } X!)}$$

—

⇝ Extension of the definition of a mb fct. to only μ-a.e. defined fct's:

$(X, \mathcal{A}, \mu)$ complete measure space, $D \subseteq X$ st $D \in \mathcal{A}$, $\mu(X \setminus D) = 0$

A function $f : D \to \overline{\mathbb{R}}$ is <u>measurable</u> $:\iff \exists h : X \to \overline{\mathbb{R}}$ mb st. $h|_D = f$

In this case: Every extension $\tilde{h}$ of f to X is mb (because $\tilde{h} = h$ μ-a.e.)

Approximation of measurable functions by simple functions

Definition 2.12. X set, $M \subseteq X$.

characteristic function of M := $\chi_M : X \to \mathbb{R}$, $\chi_M(x) = \begin{cases} 1, & x \in M \\ 0, & x \notin M. \end{cases}$

Observation. χ_M mb $\iff$ M mb

Proof. Follows from Cor. 2.8, because:

$$\forall a \in \overline{\mathbb{R}}: \quad \chi_M^{-1}([a,\infty]) = \begin{cases} \emptyset, & a > 1 \\ M, & a \in (0,1] \\ X, & a \le 0. \end{cases}$$

Definition 2.13. $(X, \mathfrak{A})$ measurable ~~function~~ space,
$g : X \to \mathbb{R}$ (or $\mathbb{C}$) is a simple function,
$:\iff$ g measurable and assumes only finitely many values.

Note. Sometimes other definition of "simple function": $g : X \to \mathbb{C}$ has only finitely many values; not necessarily measurable (e.g. Rudin).

Proposition 2.14. $(X, \mathfrak{A})$ measurable space, $g : X \to \mathbb{R}$ (or $\mathbb{C}$). The following is equivalent:

i) g is a simple function

ii) $\exists \alpha_1, \dots, \alpha_n \in \mathbb{R}$, $M_1, \dots, M_n \in \mathfrak{A}$, pairwise disjoint
s.t. $g = \sum_{j=1}^n \alpha_j \chi_{M_j}$

iii) g mb, $\exists \beta_1, \dots, \beta_m \in \mathbb{R}$, $N_1, \dots, N_m \subseteq X$ s.t.
$g = \sum_{j=1}^m \beta_j \chi_{N_j}$

Proof. i) $\Rightarrow$ ii) Let $g(X) =: \{\alpha_1, \dots, \alpha_n\}$ with $\alpha_j \neq \alpha_k$ $(j \neq k)$.

g mb $\Rightarrow$ $\forall j = 1, \dots, n$: $M_j := g^{-1}(\{\alpha_j\}) \in \mathfrak{A}$

Obviously: $M_j \cap M_k = \emptyset$ $(j \neq k)$ and $X = \dot\bigcup_{j=1}^n M_j$.

$x \in X$ $\Rightarrow$ $\exists!\, k \in \{1, \dots, n\}$ s.t. $x \in M_k$

$\Rightarrow$ $g(x) = \alpha_k = \alpha_k \cdot \chi_{M_k}(x) = \sum_{j=1}^n \alpha_j \chi_{M_j}(x)$.

ii) $\Rightarrow$ iii) ✓

iii) $\Rightarrow$ i) g mb by definition and obviously range(g) is finite
(range$(g) \subseteq \{ \sum_{j=1}^m \varepsilon_j \beta_j \mid \varepsilon_j \in \{0,1\} \}$). □

Notation. $E(X, \mathfrak{A}) := \{ g : X \to \mathbb{R} \mid g \text{ simple fct} \}$

$E^+(X, \mathfrak{A}) := \{ g \in E(X, \mathfrak{A}) \mid g(x) \ge 0,\ x \in X \}$.

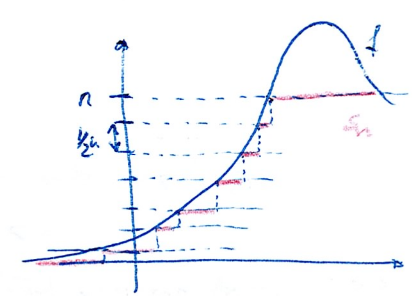
Observation. $E(X, \mathfrak{A})$ is an $\mathbb{R}$-VS.

Theorem 2.15. $(X, \mathfrak{A})$ mb space, $f : X \to \overline{\mathbb{R}}$ mb, $f \ge 0$
Then exists a sequence of simple functions $s_n \in E^+(X, \mathfrak{A})$
s.t. $0 \le s_1 \le s_2 \le \dots \le f$ (pointwise, i.e. $\forall x \in X$ $0 \le s_1(x) \le \dots \le f(x)$)
and $\forall x \in X$ $\lim_{n \to \infty} s_n(x) = f(x)$.

If f is bounded, the s_n can be chosen such that they converge uniformly to f, i.e.

$$\sup_{x \in X} |f(x) - s_n(x)| \longrightarrow 0, \quad n \to \infty.$$

Proof.



$\forall n \in \mathbb{N}$ and $1 \le k \le 2^n n$ define

$A_{n,k} := \{ x \in X \mid \frac{k-1}{2^n} \le f(x) < k/2^n \}$

$B_n := \{ x \in X \mid f(x) \ge n \}$

$\Rightarrow$ all $A_{n,k}, B_n \in \mathfrak{A}$
and $\forall n \in \mathbb{N}$ $X = \dot\bigcup_{k=1}^{2^n n} A_{n,k} \,\dot\cup\, B_n$

$\forall n \in \mathbb{N}$ define

$$s_n := \sum_{k=1}^{2^n n} \frac{k-1}{2^n} \chi_{A_{n,k}} + n \chi_{B_n}.$$

Obviously: $\forall n \in \mathbb{N}$ $s_n \in E^+(X, \mathcal{A})$.

- Monotony. Observe $A_{n,k} = A_{n+1,2k-1} \cup A_{n+1,2k}$ because:

$$x \in A_{n,k} \Leftrightarrow \frac{(2k-1)-1}{2^{n+1}} = \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n} = \frac{2k}{2^{n+1}}$$

Let $x \in X$.

Case 1. $x \in A_{n,k}$.

Case 1.1. $x \in A_{n+1,2k-1}$ $\Rightarrow s_{n+1}(x) = \frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} = s_n(x)$

Case 1.2. $x \in A_{n+1,2k}$ $\Rightarrow s_{n+1}(x) = \frac{2k-1}{2^{n+1}} > \frac{2k-2}{2^{n+1}} = s_n(x)$

Case 2. $x \in B_n$ $\Rightarrow x \in B_{n+1} \cup \bigcup_{k=2^{n+1}n+1}^{(n+1)2^{n+1}} A_{n+1,k}$.

Case 2.1. $x \in A_{n+1,\ell}$ for some $\ell > 2^{n+1}n$

$\Rightarrow s_{n+1}(x) = \ell/2^{n+1} > n = s_n(x)$

Case 2.2. $x \in B_{n+1}$ $\Rightarrow s_{n+1}(x) = n+1 > n = s_n(x)$.

$\rightsquigarrow \forall x \in X$ $s_{n+1}(x) \ge s_n(x)$.

- pointwise converge. Fix $x \in X$.

Case 1. $f(x) = \infty$ $\Rightarrow \forall n$ $x \in B_n(x)$ $\Rightarrow \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} n = \infty$.

Case 2. $f(x) < \infty$.

Choose $\tilde{N}_x \in \mathbb{N}$ s.t. $f(x) < \tilde{N}_x$.

Fix $\varepsilon > 0$ and $N > \tilde{N}$ s.t. $\frac{1}{2^N} < \varepsilon$.

$\Rightarrow \forall n \ge N$ $\exists k \in \{1, \dots, 2^n \cdot n\}$ s.t. $x \in A_{n,k}$.

$\Rightarrow \underbrace{\frac{k-1}{2^n}}_{= s_n(x)} \le f(x) < k/2^n = \frac{k-1}{2^n} + \frac{1}{2^n} < s_n(x) + \varepsilon$.

$\Rightarrow 0 \le f(x) - s_n(x) < \varepsilon;$ $n \ge N$. $(*)$

- uniform convergence if f is bounded.

In this case: $\exists \tilde{N} \in \mathbb{N}$ s.t. $\forall x \in X$ $f(x) < \tilde{N}$

$\Rightarrow$ the N in $(*)$ does not depend on x

$\Rightarrow \forall n \ge N$ $\forall x \in X$ $0 \le f(x) - s_n(x) < \varepsilon$.

$\Rightarrow \forall n \ge N$ $\sup_{x \in X} |f(x) - s_n(x)| < \varepsilon$. $\square$

2.2 Lebesgue integral

$(X, \mathcal{A}, \mu)$ measure space.

Idea: define an integral $\int_X f \, d\mu$ for every measurable functions $f \ge 0$

But: $f \ge 0$ integrable $\Leftrightarrow \int_X f \, d\mu < \infty$.

Then extend the integral for real- and complex valued fct's

Definition 2.16. (Integral of non-neg. simple functions)

$(X, \mathcal{A}, \mu)$ measure space, $g \in E^+(X, \mathcal{A})$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}, \ge 0$, and $M_1, \dots, M_j \in \mathcal{A}$, pairwise disjoint, s.t. $g = \sum_{j=1}^{n} \alpha_j \chi_{M_j}$.

Then: $I(g) := \int_X g \, d\mu := \sum_{j=1}^{n} \alpha_j \mu(M_j)$

with the convention $\alpha_j \mu(M_j) = 0$ if $\alpha_j = 0$, $\mu(M_j) = \infty$.

Observation. $\int_X g \, d\mu$ is well-defined.

Proof. Let g as in the definition, and assume that $\exists \beta_1, \dots, \beta_m \ge 0$, $N_1, \dots, N_m \in \mathcal{A}$, pairwise disjoint with $g(x) = \sum_{j=1}^{m} \beta_j \chi_{N_j}(x)$

To show: $\sum_{j=1}^{n} \alpha_j \mu(M_j) = \sum_{j=1}^{m} \beta_j \mu(N_j)$.

Define $k \in \mathbb{N}$, $\gamma_1, \dots, \gamma_k \in \mathbb{R}$, pairwise distinct, s.t.

$\{\beta_1, \dots, \beta_m\} = \{\gamma_1, \dots, \gamma_k\}$

and for all $j \in \{1, \dots, k\}$ $S_j := \{\ell \in \{1, \dots, m\} \mid \beta_\ell = \gamma_j\}$.

$\Gamma_j := \bigcup_{\ell \in S_j} N_\ell$

$\Rightarrow g = \sum \gamma_j \chi_{\Gamma_j}$ and

$$\sum_{j=1}^{k} \gamma_j \mu(\Gamma_j) = \sum_{j=1}^{k} \gamma_j \sum_{\ell \in S_j} \mu(N_\ell) = \sum_{j=1}^{k} \Big(\sum_{\ell \in S_j} \overset{\beta_\ell}{\overset{\shortparallel}{\gamma_j}} \mu(N_\ell)\Big)$$

$$= \sum_{j=1}^{m} \beta_j \mu(N_j)$$

Analogously: $\sum_{j=1}^{k} \gamma_j \mu(\Gamma_j) = \sum_{j=1}^{n} \alpha_j \mu(M_j)$. $\square$

Proposition 2.17. (Properties of the integral)

$(X, \mathcal{A}, \mu)$ measure space, $g, h \in E^+(X, \mathcal{A})$, $\alpha \geq 0$

$\Rightarrow$ i) $\mathcal{J}(\alpha g) = \alpha \mathcal{J}(g)$

ii) $\mathcal{J}(g+h) = \mathcal{J}(g) + \mathcal{J}(h)$

iii) $g \leq h \Rightarrow \mathcal{J}(g) \leq \mathcal{J}(h)$

Proof. i): ✓

ii) g, h simple $\rightarrow \exists \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathbb{R}$, $N_1, \dots, N_n \in \mathcal{A}$ pairwise disjoint, $M_1, \dots, M_m \in \mathcal{A}$, pairwise disjoint, s.t. $g = \sum_{j=1}^n \alpha_j \chi_{N_j}$, $h = \sum_{j=1}^m \beta_j \chi_{M_j}$ (*)

Without restriction: $n = m$, $M_j = N_j$ $(j = 1, \dots, n)$.

Because: $N_{n+1} := X \setminus \dot{\bigcup}_{j=1}^n N_j$, $M_{m+1} := X \setminus \dot{\bigcup}_{j=1}^m M_j$. Let $C_{jk} := M_j \cap N_k$

$\Rightarrow X = \dot{\bigcup}_{j=1}^{m+1} M_j = \dot{\bigcup}_{j=1}^{m+1} \dot{\bigcup}_{k=1}^{n+1} C_{jk} = \dot{\bigcup}_{j=1}^{n+1} N_j$

Set $\alpha_{jk} := \alpha_j$ $(k = 1, \dots, m+1)$, $\beta_{jk} := \beta_j$ $(k = 1, \dots, n+1)$

$\rightarrow g = \sum_{j=1}^{n+1} \sum_{k=1}^{m+1} \alpha_{jk} \chi_{C_{kj}}$, $h = \sum_{j=1}^{n+1} \sum_{k=1}^{m+1} \beta_{jk} \chi_{C_{jk}}$

$\Rightarrow \mathcal{J}(g+h) = \mathcal{J}\left(\sum_{j=1}^n (\alpha_j + \beta_j) \chi_{N_j}\right) = \sum_{j=1}^n (\alpha_j + \beta_j) \mu(N_j)$

$= \sum_{j=1}^n \alpha_j \mu(N_j) + \sum_{j=1}^n \beta_j \mu(N_j) = \mathcal{J}(g) + \mathcal{J}(h)$.

iii) g, h with representation as in (*)

$\rightarrow \forall j = 1, \dots, n$ $\alpha_j \leq \beta_j \Rightarrow \mathcal{J}(g) = \sum_{j=1}^n \alpha_j \mu(N_j) \leq \sum_{j=1}^n \beta_j \mu(N_j) = \mathcal{J}(h)$

Or: $h - g \in E^+(\mathcal{A}, X)$ by assumption.

$\Rightarrow \mathcal{J}(h) = \mathcal{J}(g + h - g) \overset{ii)}{=} \mathcal{J}(g) + \overbrace{\mathcal{J}(h-g)}^{\geq 0} \geq \mathcal{J}(g)$. ⌋

□

Definition 2.18. (Integral of non-negative functions)

$(X, \mathcal{A}, \mu)$ measure space; $f: X \to [0, \infty]$ measurable.

$$\mathcal{J}(f) := \int_X f \, d\mu := \sup \underbrace{\{ \mathcal{J}(\varphi) \mid \varphi \in E^+(X, \mathcal{A}), 0 \leq \varphi \leq f \}}_{\neq \emptyset}$$

=: integral of f.

Observations.

- $\mathcal{J}(f) \in [0, \infty]$
- $\mathcal{J}(\alpha f) = \alpha \mathcal{J}(f)$, $\alpha > 0$
- $f \leq g \Rightarrow \mathcal{J}(f) \leq \mathcal{J}(g)$

Not so clear: $\mathcal{J}(f+g) = \mathcal{J}(f) + \mathcal{J}(g)$ (will be shown in Cor. 2.22)

Definition 2.19. $(X, \mathcal{A}, \mu)$ measure space

i) $f: X \to \mathbb{R} \cup \{\pm\infty\}$ mb, $f \geq 0$.

f integrable (w.r.t. $(X, \mathcal{A}, \mu)$) $:\Leftrightarrow \int_X f \, d\mu < \infty$.

ii) $f: X \to \mathbb{R} \cup \{\pm\infty\}$ mb, $f_\pm$ as in Cor. 2.9 $\left(\begin{array}{l} f_+(x) := \max\{0, f(x)\} \\ f_-(x) := \max\{0, -f(x)\} \end{array} \right)$

f integrable $:\Leftrightarrow$ $f_\pm$ are integrable

In this case: $\mathcal{J}(f) := \int_X f \, d\mu := \int_X f_+ \, d\mu - \int_X f_- \, d\mu$

iii) $f: X \to \mathbb{C}$ mb.

f integrable $:\Leftrightarrow$ $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ integrable.

In this case: $\mathcal{J}(f) := \int_X f \, d\mu = \int_X \mathrm{Re}\, f \, d\mu + i \int_X \mathrm{Im}\, f \, d\mu$.

$\mathcal{L}_1((X, \mathcal{A}, \mu), \mathbb{K}) := \{ f: X \to \mathbb{K} \mid f \text{ integrable} \}$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{R} \cup \{\pm\infty\}$ or $\mathbb{R}$.

Other notations (when $X, \mathcal{A}, \mu, \mathbb{K}$ are clear):

$\mathcal{L}_1(X, \mathbb{K})$, $\mathcal{L}_1(X)$, $\mathcal{L}_1^{\mathbb{R}}(X)$, ...

Later we will show:

- $\mathcal{L}_1(X, \mathbb{K})$ is a $\mathbb{K}$-VS ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

  and $\|f\|_1 := \int_X |f| \, d\mu$ is a seminorm on $\mathcal{L}_1(X, \mathbb{K})$
- $\int : \mathcal{L}_1(X, \mathbb{K}) \to \mathbb{K}$ is linear.