1.4. Extension of measures

So far. Content on $\mathcal{J} = \{[a,b[\subseteq \mathbb{R} \mid a \le b\}$, $\mu([a,b[) = b-a$.

$\to$ Unique extension to $R(\mathcal{J})$ = ring generated by $\mathcal{J}$. (Prop. 1.17)

$\to$ this is a premeasure (Thm 1.21) if μ is σ-additive

Question
- Exist extension to Borel algebra on $\mathbb{R}$ (= σ-algebra gen. by $\mathcal{J}$) ?
- Uniqueness ?

H Halbring, μ σ-add Inhalt auf H $\leadsto$ Eindeutige Fortsetzung auf von H erzeugten Ring

Idea. R = ring over X, μ premeasure on R.

$\leadsto$ $\exists$ extension μ^* to $\mathbb{P}X$, where μ^* is an outer measure

$\leadsto$ $\exists$ σ-algebra A^* s.t. $R \subseteq A^*$ and (X, A^*, μ^*) is a measure space.

$\leadsto$ unique, if X is σ-finite

$\leadsto$ there exists a unique extension of the Lebesgue-premeasure to $\mathcal{L}$ = Borel σ-algebra generated by $\mathcal{J}$ (take $\mu_*|_{\mathcal{L}}$).

Definition 1.23. X set, $\mu : \mathbb{P}X \to \mathbb{R} \cup \{\infty\}$ is an outer measure on X

$$:\Longleftrightarrow \begin{cases} \mu(\emptyset) = 0 \\ \mu(A) \le \mu(B) \text{ if } A \subseteq B \\ \mu\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu(A_j) \quad \sigma\text{-subadditivity.} \end{cases}$$

A set $M \subseteq X$ is called μ^*-measurable if and only if

$$\forall A \subseteq X \qquad \mu^*(A) = \mu^*(M \cap A) + \mu^*(A \setminus M).$$

Observation. By subadditivity always $\mu^*(M) \le \mu^*(M \cap A) + \mu^*(M \setminus A)$

$\leadsto$ M μ^*-mb $\Leftrightarrow \left(\mu^*(M) \ge \mu^*(M \cap A) + \mu^*(M \setminus A), \quad A \subseteq X\right)$.

Theorem 1.24. (Carathéodory) $\mu^* : \mathbb{P}X \to \mathbb{R} \cup \{\infty\}$ outer measure, $\mathcal{M} := \{\text{measurable sets}\}$. Then:

i) $\mathcal{M}$ σ-algebra over X

ii) $\bar{\mu} = \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$.

iii) $M \in \mathbb{P}X$, $\mu^*(M) = 0 \to M \in \mathcal{M}$.

Proof

i) ① Empty set & complements:

$\emptyset$ is μ^*-measurable because $\forall A \subseteq X$: $\mu^*(A) = \mu^*(A \setminus \emptyset) + \overbrace{\mu^*(A \cap \emptyset)}^{=\mu^*(\emptyset) = 0, \text{ by Def.}}$

Now assume $M \in \mathcal{M}$.

$\Rightarrow \forall A \subseteq X \quad \mu^*(A) = \mu^*(A \cap M) + \mu^*(A \setminus M) = \mu^*(A \setminus (X \setminus M)) + \mu^*(A \cap (X \setminus M))$

$\Rightarrow X \setminus M \in \mathcal{M}$.

② $M, N \in \mathcal{M} \Rightarrow M \cup N \in \mathcal{M}$.

For every $A \subseteq X$:

$\mu^*(A) = \mu^*(A \cap M) + \mu^*(A \setminus M)$

$\overset{N\ \mu^*\text{-mb}}{=} \mu^*(A \cap M) + \mu^*((A \setminus M) \cap N) + \mu^*((A \setminus M) \setminus N)$

$[(A \cap M) \cup \{(A \setminus M) \cap N\} = (A \cap M) \cup \{(A \cap N) \setminus M\} = (A \cap M) \cup (A \cap N)]$

$\overset{\text{subadditivity}}{\ge} \mu^*((A \cap M) \cup (A \cap N)) + \mu^*((A \setminus M) \setminus N)$

$= \mu^*(A \cap (M \cup N)) + \mu^*(A \setminus (M \cup N))$

$\leadsto \mu^*(A) \ge \mu^*(A \cap (M \cup N)) + \mu^*(A \setminus (M \cup N))$.

By the observation also "$\le$" $\leadsto$ $M \cup N \in \mathcal{M}$.

③ $\mathcal{M}$ is an algebra

... because $\emptyset \in \mathcal{M}$, $A \in \mathcal{M} \Rightarrow X \setminus A \in \mathcal{M}$ (by ①)

and $A, B \in \mathcal{M} \Rightarrow A \cup B \in \mathcal{M}$ (by ②).

④ $\mathcal{M}$ is a σ-algebra.

$(M_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$. To show $\bigcup_{j=1}^{\infty} M_j =: N \in \mathcal{M}$.

Without restriction: the M_j are pairwise disjoint.

(otherwise define $M_1' = M_1$, $M_2' = M_2 \setminus M_1', \dots, M_j' = M_j \setminus \left(\bigcup_{k=1}^{j-1} M_k\right)$

$\rightsquigarrow$ all $M_j' \in R$ because R is a ring, and

$\forall n$: $N_n := \bigcup_{j=1}^{n} M_j = \bigcup_{j=1}^{n} M_j'$.)

$\Rightarrow \forall A \in X$: $\quad = \bigcup_{j=1}^{n} M_n$

$\mu^*(A) = \mu^*(A \cap N_n) + \mu^*(A \setminus N_n)$

$\geq \mu^*(A\cap N_n) + \mu^*(A \setminus N) \quad \leftarrow (N_n \subseteq N \Rightarrow A \setminus N_n \supseteq A \setminus N$

$\overset{(*)}{=} \sum_{j=1}^{n} \mu^*(A \cap M_j) + \mu^*(A\setminus N) \quad \Rightarrow \mu^*(A \setminus N_n) \geq \mu^*(A \setminus N)$ by mon. of μ^*)

$(*)\ \mu^*(A \cap (M_1 \cup M_2)) =$
$= \mu^*(A \cap (M_1 \cup M_2) \cap M_1)$
$+ \mu^*([A \cap (M_1 \cup M_2)] \setminus M_1)$
$= \mu^*(A \cap M_1) + \mu^*(A \cap M_2)$

$\rightsquigarrow$ by Induction:
$\mu^*(A \cap \bigcup_{j=1}^{n} M_j) = \sum_{j=1}^{n} \mu^*(A \cap M_j)$

Since this is true for all $n \in \mathbb{N}$:

(A) $\quad \mu^*(A) \geq \sum_{j=1}^{\infty} \mu^*(A \cap M_j) + \mu^*(A \setminus N)$

σ-sub-additivity of μ^* $\longrightarrow\ \geq \mu^*(A \cap N) + \mu^*(A\setminus N)$

$\mu^*(A) \leq \mu^*(A\cap N) + \mu^*(A \setminus N)$

by ~~monotoni~~ sub-additivity

$\Rightarrow \mu^*(A) = \mu^*(A \cap N) + \mu^*(A \setminus N)$

$\Rightarrow N = \bigcup_{j=1}^{\infty} M_j \in \mathcal{M}$

□

ii) $\bar{\mu}$ is a measure on $\mathcal{M}$:

To show: $(M_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$, pairwise disjoint.
Then: $\bar{\mu}\left(\bigcup_{j=1}^{\infty} M_j\right) = \sum_{j=1}^{\infty} \bar{\mu}(M_j)$.

Take $A = \bigcup_{j=1}^{\infty} M_j$ in **(A)**. Then:

$$\sum_{j=1}^{\infty} \mu^*(M_j) = \sum_{j=1}^{\infty} \mu^*(A \cap M_j) + \overbrace{\mu^*(A \setminus A)}^{=0} \overset{(A)}{\leq} \mu^*(A) \underbrace{\leq}_{\sigma\text{-subadd. of } \mu^*} \sum_{j=1}^{\infty} \mu^*(M_j)$$

$\Rightarrow \sum_{j=1}^{\infty} \mu^*(M_j) = \mu^*\left(\bigcup_{j=1}^{\infty} M_j\right)$.

iii) Let $M \in \mathcal{P}X$ with $\mu^*(M) = 0$.

$\Rightarrow \forall A \subseteq X \quad \mu^*(A) = \mu^*(A) + \overbrace{\mu^*(A \cap M)}^{=0 \text{ because of mon. of } \mu^* \text{ and } M \supseteq A\cap M} \underset{\text{mon. of } \mu^*}{\geq} \mu^*(A \setminus M) + \mu^*(A \cap M)$

$\geq \mu^*(A)$ (subadd).

$\Rightarrow \mu^*(A) = \mu^*(A \cap M) + \mu^*(A \setminus M)$

$\Rightarrow M \in \mathcal{M}$.

□

Theorem 1.25 (Extension of a content to an outer measure)

H semiring over a set X, $\mu: H \to \mathbb{R} \cup \{\infty\}$ ~~content~~ premeasure on H.

Define $\mu^*: \mathcal{P}X \to \mathbb{R} \cup \{\infty\}$,

$$\mu^*(A) := \inf\left\{ \sum_{j=1}^{\infty} \mu(A_j) \;\middle|\; A_j \in H, (j \in \mathbb{N}), A \subseteq \bigcup_{j=1}^{\infty} A_j \right\} \quad (*)$$

Then. i) μ^* is an outer measure and $H \subseteq \mathcal{M} :=$ set of all μ^*-mb. sets

ii) $\mu^*|_H = \mu$.

Observation. If μ is only a content, and not a premeasure, then $\exists A \in H$ s.t. $\mu^*(A) < \mu(A)$.

Proof. μ not a premeasure $\rightsquigarrow \exists (A_j)_j \subseteq H$, pairwise disjoint, with $A := \bigcup_{j=1}^{\infty} A_j \in H$ and $\mu(A) \neq \sum_{j=1}^{\infty} \mu(A_j)$.

Always: $\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j)$ (Thm 1.18, iv) ($\mu(A) \geq \mu(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j)$ true for all n)

$\Rightarrow \mu(A) > \sum_{j=1}^{\infty} \mu(A_j) \geq \mu^*(A)$
Def of μ^*.

□

(*) If $A \notin \bigcup_{B \in R} B \rightsquigarrow \mu^*(A) = \inf \emptyset = \infty$

Proof. By Prop. 1.17 we can assume that H is a ring.

i) First we show: $\boxed{\mu^* \text{ is an outer measure}}$.

Clearly: $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B \subseteq X$.

σ-subadditivity: Let $(A_j)_j \subseteq \mathcal{P}X$. To show: $\mu^*(\bigcup_{j=1}^\infty A_j) \leq \sum_{j=1}^\infty \mu^*(A_j)$

If at least one $\mu^*(A_j) = \infty$, then both sides are $= \infty$. ✓

Now assume all $\mu^*(A_j) < \infty$.

Choose $N_{jk} \subseteq X$ s.t. $\forall j \in \mathbb{N}$ $\bigcup_{k=1}^\infty N_{jk} \supseteq A_j$ and

$$\mu^*(A_j) \geq \sum_{k=1}^\infty \mu(N_{jk}) - \varepsilon/2^j \quad \text{(by Def of } \mu^*) \qquad (*)$$

$$\Rightarrow \bigcup_{j=1}^\infty A_j \subseteq \bigcup_{j=1}^\infty \bigcup_{k=1}^\infty \underbrace{N_{jk}}_{\in H} \quad \text{countable union!}$$

$$\Rightarrow \mu^*(\bigcup_{j=1}^\infty A_j) \leq \sum_{j,k \in \mathbb{N}} \mu(N_{jk}) \overset{(*)}{\leq} \sum_{j=1}^\infty (\mu^*(A_j) + \varepsilon/2^j)$$

$$= \sum_{j=1}^\infty \mu^*(A_j) + \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we have $\mu^*(\bigcup_{j=1}^\infty A_j) \leq \sum_{j=1}^\infty \mu^*(A_j)$.

Now we show: $\boxed{H \subseteq \mathcal{M}}$.

Let $M \in H$. To show: $\forall A \subseteq X$ $\mu^*(A) = \mu^*(A \cap M) + \mu^*(A \setminus M)$

"$\leq$" by subadditivity of μ^*.

"$\geq$" Let $\varepsilon > 0$ and choose $(A_j)_j \subseteq H$ s.t. $A \subseteq \bigcup_{j=1}^\infty A_j$ and $\mu^*(A) + \varepsilon \geq \sum_{j=1}^\infty \mu(A_j)$

$$\Rightarrow \mu^*(A) + \varepsilon \geq \sum_{j=1}^\infty \mu(A_j) \overset{A_j, M \in H,\ \mu \text{ premeasure on } H}{=} \sum_{j=1}^\infty (\mu(A_j \cap M) + \mu(A_j \setminus M))$$

$$= \sum_{j=1}^\infty \mu(A_j \cap M) + \sum_{j=1}^\infty \mu(A_j \setminus M)$$

$$\geq \mu^*(A \cap M) + \mu^*(A \setminus M). \quad \text{by def of } \mu^*.$$

ii) Let $M \in H$. Obviously: $\mu^*(M) \leq \mu(M)$.

For $(A_j)_j \subseteq H$ with $M \subseteq \bigcup_{j=1}^\infty A_j$, ~~do~~ then: $\mu(M) \overset{\mu\ \sigma\text{-subadd. (Prop 1.18 vi)}}{\leq} \sum_{j=1}^\infty \mu(A_j)$

$\Rightarrow \mu^*(M) \geq \inf\{\ldots\} \geq \mu(M)$. □

So far: μ premeasure on a ring R.

$\to \mu^*$ outer measure on $\mathcal{P}X$

$\to \exists$ σ-algebra $\mathcal{M}$ with $R \subseteq \mathcal{M}$ s.t. $\bar{\mu} = \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$

where $\mathcal{M}$ = set of all μ^*-mb. subsets of X.

Since $\sigma(R) = (\sigma\text{-algebra generated by } R) \subseteq \mathcal{M}$ we have an extension of μ to a measure on $\sigma(R)$ (take $\bar{\mu}|_{\sigma(R)}$).

Question. Uniqueness of this extension?

In general: no!

Example: X set, $\neq \emptyset$, $R := \{\emptyset\}$ and $\mu: R \to \mathbb{R} \cup \{\infty\}$ measure on R

$\rightsquigarrow \sigma(R) = \{\emptyset, X\}$. and for every $\alpha \geq 0$:

$\bar{\mu}: \sigma(R) \to \mathbb{R} \cup \{\infty\}$, $\bar{\mu}(\emptyset) := 0$, $\bar{\mu}(X) := \alpha$

is an extension of μ to $\sigma(R)$.

But: Definition 1.26. R ring (or semiring) over X, $\mu: R \to \mathbb{R} \cup \{\infty\}$ measure (or content) on R.

μ is called σ-finite if

$$\exists (A_j)_{j \in \mathbb{N}} \subseteq R \text{ s.t. } R = \bigcup_{j=1}^\infty A_j \text{ and } \forall j\ \mu(A_j) < \infty.$$

Example. $\mathcal{J} = \{[a,b[\mid a \leq b, \in \mathbb{R}\}$ with $\mu([a,b[) := b - a$.

$\rightsquigarrow \mu$ is σ-finite because $\forall n \in \mathbb{N}$ $\mu([-n,n[) = 2n < \infty$ and $\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n,n[$.

Theorem 1.27. (Hahn extension theorem)

R ring over X, μ premeasure on R, σ-finite.

$\Rightarrow$ there ex. exactly one $\bar{\mu}$ measure on $\sigma(R)$ s.t. $\bar{\mu}|_R = \mu$.

($\sigma(R)$ = σ-algebra generated by R)

Proof. Existence: Take $\bar{\mu} = \mu^*|_{\sigma(R)}$ (Thm. 1.24 & 1.25).

Uniqueness:

~~Case: μ σ-finite. Let $m(R)$ = monotone class generated by R. Uniqueness of the extension of μ to $m(R)$: Because: AC~~

Let $\bar{\mu} = \mu^*|_{\sigma(R)}$ and ν a measure on $\sigma(R)$ s.t. $\nu|_R = \mu$.

Let $(A_j)_j \subseteq R$ s.t. $\forall j$ $\mu(A_j) < \infty$, $X = \bigcup_{j=1}^\infty A_j$,

without restriction: $A_j \cap A_k = \emptyset$, $j \neq k$ (otherwise take $A_1' := A_1$,
$A_j' := A_j \setminus (\bigcup_{l=1}^{j-1} A_l)$, $j \geq 2$)

Fix $M \in \sigma(R)$.

$$\Rightarrow \begin{cases} \bar{\mu}(M) = \bar{\mu}(\bigcup_{j=1}^\infty M \cap A_j) = \sum_{j=1}^\infty \bar{\mu}(M \cap A_j) \\ \nu(M) = \nu(\quad '' \quad) = \sum_{j=1}^\infty \nu(M \cap A_j) \end{cases}$$

$\Rightarrow$ suffices to show: $\forall j$ $\bar{\mu}(M \cap A_j) = \nu(M \cap A_j)$

$\Rightarrow$ without restriction: $\exists A \in R$ s.t. $\mu(A) < \infty$ and $M \subseteq A$.

$$\rightsquigarrow \bar{\mu}(M) = \inf\{\sum_{j=1}^\infty \mu(A_j) \mid A_j \in R, j \in \mathbb{N};\ M \subseteq \bigcup_{j=1}^\infty A_j\}$$
$$= \inf\{\underbrace{\sum_{j=1}^\infty \nu(A_j)}_{\geq \nu(M)} \mid \quad '' \quad\} \geq \nu(M) \qquad (*)$$

Apply the same to $A \setminus M \subseteq A$: $(A \setminus M \in \sigma(R))$ $\qquad (**)$

$\rightsquigarrow \bar{\mu}(A \setminus M) \geq \nu(A \setminus M)$

$$\Rightarrow \mu(A) \underset{A \in R}{=} \bar{\mu}((A \cap M) \cup (A \setminus M)) = \bar{\mu}(A \cap M) + \bar{\mu}(A \setminus M)$$
$$\geq \nu(A \cap M) + \nu(A \setminus M) = \nu(A) \underset{A \in R}{=} \mu(A)$$

$\Rightarrow \bar{\mu}(A \cap M) + \bar{\mu}(A \setminus M) = \nu(A \cap M) + \nu(A \setminus M)$

$$\Rightarrow \underbrace{\bar{\mu}(A \cap M) - \nu(A \cap M)}_{\geq 0 \text{ by } (*)} = \underbrace{\nu(A \setminus M) - \mu(A \setminus M)}_{\leq 0 \text{ by } (**)}$$

$\Rightarrow \bar{\mu}(M) = \bar{\mu}(A \cap M) = \nu(A \cap M) = \nu(M)$ $\qquad (M \cap A = M)$. $\square$

More general: **Theorem.**

$\mathfrak{A}$ σ-algebra over X, $\exists\, E \subseteq \mathcal{P}X$, generator of $\mathfrak{A}$, stable under intersection.
Suppose $\exists (E_j)_j \subseteq E$ s.t. $\forall j$ $\mu_1(E_j) = \mu_2(E_j) < \infty$ and $X = \bigcup_{j=1}^\infty E_j$
where μ_1, μ_2 are measures on $\mathfrak{A}$.
Then: $\mu_1 = \mu_2$.

Proof: Bauer, Thm. I.5.4 (uses Dynkin-Systems).

So far: R σ-ring over X, μ σ-finite measure on R

$\sigma(R) :=$ σ-algebra generated by R

$\mathfrak{M} :=$ μ-measurable sets (is a σ-algebra)

$\rightsquigarrow R \subset \sigma(R) \subseteq \mathfrak{M}$.

What sets belong to $\mathfrak{M} \setminus \sigma(R)$?

Definition 1.28. μ measure on a σ-algebra $\mathfrak{A}$ over X.
The measure space $(X, \mathfrak{A}, \mu)$ is called *complete*
$:\Leftrightarrow$ $(M \in \mathfrak{A}, \mu(M) = 0, N \subseteq M \Rightarrow N \in \mathfrak{A})$

Obviously then $\mu(N) = 0$ by isotonicity of μ.

Example. $\mathfrak{M}$ is complete ($\mathfrak{M}$ = μ-mb. sets).

Proof: $M \in \mathfrak{M}$ with $\bar{\mu}(M) = 0$, and $N \subseteq M$.
$\Rightarrow \mu^*(N) \leq \mu^*(M) = \bar{\mu}(M) = 0 \Rightarrow N \in \mathfrak{M}$ by Thm. 1.24.

$\rightsquigarrow$ Every measure can be extended to a complete measure. (take $\mu^*|_{\mathfrak{M}}$)

The "smallest extension" of μ to a complete measure is called its *completion*.

Theorem 1.29. $(X, \mathfrak{A}, \mu)$ measure space, μ σ-finite. $\mathfrak{N}$ = set of μ-zerosets
(that is: $N \in \mathfrak{N} \Rightarrow \exists M \in \mathfrak{A}$ s.t. $N \subseteq M$ and $\mu(M) = 0$).
Then $(X, \mathfrak{M}, \mu^*|_{\mathfrak{M}})$ is the completion of $(X, \mathfrak{A}, \mu)$.

To this end we show:

Theorem 1.30. $(X, \mathfrak{A}, \mu)$ and $\mathfrak{N}$ as above.
Define $\tilde{\mathfrak{A}} := \{A \cup N \mid A \in \mathfrak{A}, N \in \mathfrak{N}\}$,
$\tilde{\mu}: \tilde{\mathfrak{A}} \to \mathbb{R} \cup \{\infty\}$, $\tilde{\mu}(A \cup N) := \mu(A)$.

<u>Thm:</u> i) $\tilde\mu$ is well-defined, $\tilde{\mathcal{A}}$ is a σ-algebra and $(X,\tilde{\mathcal{A}},\tilde\mu)$ is a complete measure space.
$\tilde\mu$ is the unique extension of μ to $\tilde{\mathcal{A}}$.

ii) $(X,\tilde{\mathcal{A}},\tilde\mu)$ is the completion of $(X,\mathcal{A},\mu)$.

<u>Proof.</u>

i) <u>$\tilde\mu$ is well-defined:</u> $A,B\in\mathcal{A}$, $M,N\in\mathcal{N}$ with $A\cup M = B\cup N$.

$\Rightarrow \exists C\in\mathcal{A}$ s.t. $M\subseteq C$ and $\mu(C)=0$.

$\Rightarrow \mu(B) \le \mu(A\cup C) \le \mu(A)+\mu(C) = \mu(A)$.

$B\subseteq A\cup C\in\mathcal{A}$

Analogously: $\mu(A)\le\mu(B)$. $\rightsquigarrow \underbrace{\mu(A)}_{\tilde\mu(A\cup M)} = \underbrace{\mu(B)}_{\tilde\mu(B\cup N)}$.

<u>$\tilde{\mathcal{A}}$ is a σ-algebra and $\tilde\mu$ is a measure on $\tilde{\mathcal{A}}$:</u>

- $X\in\mathcal{A}\subseteq\tilde{\mathcal{A}}$
- Let $(A_j)_j\subseteq\tilde{\mathcal{A}} \Rightarrow \exists (B_j)_j\subseteq\mathcal{A}$, $(N_j)_j\subseteq\mathcal{N}$ and $(C_j)_j\subseteq\mathcal{A}$ s.t. $\forall_j$ $A_j = B_j\cup N_j$, $N_j\subseteq C_j$ and $\mu(C_j)=0$

  $\Rightarrow \bigcup_{j=1}^\infty A_j = \underbrace{\left(\bigcup_{j=1}^\infty B_j\right)}_{\in\mathcal{A}}\cup\underbrace{\left(\bigcup_{j=1}^\infty N_j\right)}_{\in\mathcal{N}} \in\tilde{\mathcal{A}}$
- $A\in\tilde{\mathcal{A}} \Rightarrow \exists B\in\mathcal{A}, N\in\mathcal{N}, C\in\mathcal{A}$ s.t. $A=B\cup N$, $N\subseteq C$ and $\mu(C)=0$.

  $\Rightarrow X\setminus A = X\setminus(B\cup N) = \underbrace{\{X\setminus(B\cup C)\}}_{\in\mathcal{A}}\cup\underbrace{(C\setminus N)}_{\in\mathcal{N}}$.
- $\tilde\mu$ measure: clear (A_j, B_j, C_j, N_j as above, all A_j disjoint $\rightsquigarrow B_j$ pairwise disj.

  $\tilde\mu(\bigcup A_j) = \tilde\mu((\bigcup B_j)\cup(\bigcup C_j))$

  $= \tilde\mu(\bigcup B_j) + \underbrace{\tilde\mu(\overbrace{(\bigcup C_j)\setminus(\bigcup B_j)}^{\subseteq\bigcup C_j})}_{=0} = \sum_{j=1}^\infty \mu(B_j)$

  $= \sum_{j=1}^\infty \mu(B_j\cup C_j) = \sum_{j=1}^\infty \tilde\mu(A_j)$.)

<u>uniqueness of $\tilde\mu$:</u> follows from isotonicity of $\tilde\mu$.

ii) Let $(\tilde{\tilde{\mathcal{A}}},\varrho)$ be a complete extension of $(\mathcal{A},\mu)$.

$\Rightarrow \mathcal{N}\subseteq\tilde{\tilde{\mathcal{A}}} \Rightarrow \tilde{\mathcal{A}}\subseteq\tilde{\tilde{\mathcal{A}}}$ and for all $A\in\mathcal{A}$, $N\in\mathcal{N}$:

$\varrho(A\cup N) \le \varrho(A)+\varrho(N) = \varrho(A) = \mu(A) = \tilde\mu(A\cup N)$

Analogously: $\tilde\mu(A\cup N)\le\varrho(A\cup N)$ $\Rightarrow \tilde\mu = \varrho|_{\tilde{\mathcal{A}}}$. □

<u>Proof of Thm. 1.29:</u>

Only to show: $\mathfrak{M}\subseteq\tilde{\mathcal{A}}$.

Let $B\in\mathfrak{M}$. To show: $B\in\tilde{\mathcal{A}}$.

<u>Case 1</u>. $\mu(B)<\infty$.

$\forall n\in\mathbb{N}$ choose $A_{nk}\in\mathcal{A}$ s.t. $B\subseteq\bigcup_{k=1}^\infty A_{nk}$ and

$\mu^*(B)+\frac{1}{n} \ge \sum_{k=1}^\infty \mu(A_{nk})$ (possible by def. of $\mu^*(\cdot)$).

$\Rightarrow B\subseteq \underbrace{\bigcap_{n=1}^\infty\bigcup_{k=1}^\infty A_{nk}}_{:=A\in\mathcal{A}}$ and $\forall n$: $\mu(B)+1/n \ge \sum_{k=1}^\infty \mu(A_{nk}) \ge \mu(A)\ge\mu(B)$

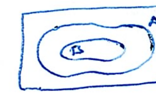
$\Rightarrow \mu(A)=\mu(B)$

Analogously: $\exists C_{nk}\in\mathcal{A}$ st. $A\setminus B\subseteq\bigcap_{n=1}^\infty\bigcup_{k=1}^\infty C_{nk} =: C\in\mathcal{A}$

and $\mu(C) = \mu(A\setminus B) = \mu(A)-\mu(B) = 0$

$\uparrow \mu(B)<\infty$!

$\Rightarrow B = (B\setminus C)\cup(B\cap C) = \underbrace{(A\setminus C)}_{\in\mathcal{A}}\cup\underbrace{(B\cap C)}_{\in\mathcal{N}}\in\tilde{\mathcal{A}}$.



$A\setminus C = \underbrace{[(A\setminus B)\setminus C]}_{=\emptyset}\cup(B\setminus C)$

<u>Case 2</u> $\mu(B)$ arbitrary.

Since X is σ-finite, there exist $(S_j)_j\subseteq\mathcal{A}$ st. $X=\bigcup_{j=1}^\infty S_j$ and $\mu(S_j)<\infty$, $j\in\mathbb{N}$.

~~Without restriction: $S_j\cap S_k=\emptyset$, $j\neq k$.~~

$\Rightarrow B = B\cap\bigcup S_j = \bigcup(B\cap S_j)$

Since $\mu(B\cap S_j)\le\mu(S_j)<\infty$, all $B\cap S_j$ have finite measure, and by Case 1, belong to $\tilde{\mathcal{A}}$.

□

<u>Other descriptions of $\mathfrak{M}$:</u>

$\mathfrak{M} = \{A\,\Delta\,N \mid A\in\mathcal{A}, N\in\mathcal{N}\}$

$= \{A\subseteq X \mid \exists B\in\mathcal{A}, M,N\in\mathcal{N}$ st. $A\cup M = B\cup N\}$.

Lebesgue measure.

So far: Lebesgue-content λ over the semiring $\mathcal{J} = \{[a,b[\subseteq \mathbb{R} \mid a \leq b\}$,
with $\lambda([a,b[) = b-a$

$\rightarrow$ unique extension to the σ-algebra generated by $\mathcal{J}$: Lebesgue-Borel-measure

(Recall: Borel-σ-algebra = σ-algebra gen. by the top on $\mathbb{R}$ $\overset{(*)}{=}$ σ-algebra gen. by $\mathcal{J}$

(*) because obviously all open intervals belong to the σ-alg. gen. by $\mathcal{J}$: $(a,b) = \bigcup_{n\in\mathbb{N}} [a+\frac{1}{n}, b)$ if $a,b \in \mathbb{R}$
similar for sets of the type $(-\infty, b)$, (a,∞))

$\rightarrow$ unique extension to $\mathcal{M}$: Lebesgue-measure

We will see: $\mathcal{M} \neq \mathcal{B} := \sigma(\mathcal{J})$.

in general refers to opt approx.

Theorem 1.31. λ Lebesgue measure on $\mathbb{R}$, $\mathcal{M}$ = Lebesgue mb sets, $A \in \mathcal{M}$.

Then: i) "inner regularity": $\forall \varepsilon > 0\ \exists B \subseteq A$, B closed, s.t. $\lambda(A \setminus B) < \varepsilon$.

ii) outer regularity: $\forall \varepsilon > 0\ \exists U \supseteq A$, U open, s.t. $\lambda(U \setminus A) < \varepsilon$.

Proof.
ii) Fix $\varepsilon > 0$. First assume $\lambda(A) < \infty$.

$\Rightarrow \forall n \in \mathbb{N}\ \exists A_n = [a_n, b_n[\in \mathcal{J}$ s.t. $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$.

and $\cancel{\mu(A)}$ $\lambda(A) \geq \sum_{n\in\mathbb{N}} \lambda(A_n) + \varepsilon/2$.

Choose open intervals B_n s.t. $B_n \supseteq A_n$ and $\lambda(B_n) \leq \lambda(A_n) + \varepsilon/2^{n+1}$

$\Rightarrow A \subseteq \bigcup_{n\in\mathbb{N}} B_n$ $\leftarrow$ open!

Note that $\lambda(\bigcup_{n\in\mathbb{N}} B_n) \leq \sum_{n\in\mathbb{N}} \lambda(B_n) \leq \sum_{n\in\mathbb{N}} \lambda(A_n) + \varepsilon/2^n$

$\leq \sum_{n\in\mathbb{N}} \lambda(A_n) + \varepsilon/2 < \lambda(A) + \varepsilon$

$\Rightarrow \lambda(\bigcup_{n\in\mathbb{N}} B_n \setminus A) = \lambda(\bigcup_{n\in\mathbb{N}} B_n) - \lambda(A) < \varepsilon$.

$\lambda(A) < \infty$

If $\lambda(A) = \infty$: Choose $S_j \in \mathcal{M}$ s.t. $\mathbb{R} = \bigcup_{j=1}^{\infty} S_j$ and all $\mu(S_j) < \infty$
($(\mathbb{R}, \mathcal{M}, \lambda)$ is σ-finite!) Without restriction: $S_j \cap S_k = \emptyset$, $j \neq k$.

(Take, e.g. $S_1 = [-1,1[$, $S_j = [-j, j[\setminus [-(j-1), j-1[$, $j \geq 2$)

Set $A_j := A \cap S_j$. For every A_j choose an open set B_j s.t. $\lambda(B_j \setminus A_j) \leq \varepsilon/2^j$

$\Rightarrow \underbrace{\bigcup_j B_j}_{\text{open}, =: B} \supseteq A$ and $\lambda(B \setminus A) \leq \sum_{j=1}^{\infty} \lambda(B_j \setminus A_j) \leq \sum_{j=1}^{\infty} \varepsilon/2^j = \varepsilon$.

[the B_j are not nec. disjoint] □

i) $A \subseteq \mathcal{M}$, $\varepsilon > 0$.

$\Rightarrow \exists B$ open s.t. $B \supseteq X \setminus A$ and $\lambda(B \setminus (X \setminus A)) < \varepsilon$.

$\Rightarrow X \setminus B =: C$ closed, $C \subseteq A$ and $\lambda(A \setminus C) = \lambda(A \cap B)$
$= \lambda(B \setminus (X \setminus A)) < \varepsilon$ □

Proposition 1.31. The Lebesgue measure is translation invariant,
that is: $M \in \mathcal{M}$, $y \in \mathbb{R} \Rightarrow \begin{cases} x+M \in \mathcal{M} \text{ and} \\ \mu(M) = \mu(x+M) \end{cases}$ μ = Lebesgue measure

Proof. Obviously: if $M = [a,b[$, then $x+M \in \mathcal{M}$ and $\mu(M) = \mu(x+M)$.

Then, by def of μ^*, also $\mu^*(M) = \mu^*(x+M)$ for all $M \in \mathcal{M}$.

$\Rightarrow M \in \mathcal{M} \Leftrightarrow x+M \in \mathcal{M}$ and $\mu(M) = \mu(x+M)$. □

Application. Construction of a non-measurable set: $A \subseteq \mathbb{R}$ but $A \notin \mathcal{M}$.

$\rightarrow$ If a measure on $\mathbb{R}$ is translation invariant and assigns 1 to the interval $[0,1]$, then there are always sets in $\mathbb{R}$ that cannot be measured

Observation. $\mathcal{B}$ = Borel-σ-algebra in $[0,1]$, $\mathcal{M}$ = Lebesgue-mb sets, λ = Lebesgue measure
$\Rightarrow \mathcal{B} \subsetneq \mathcal{M}$.

Idea: C = Cantor set $\in \mathcal{M}$ and $\lambda(C) = 0 \Rightarrow$ Every subset of C belongs to $\mathcal{M}$.
C uncountable $\Rightarrow \#(\mathcal{P}C) = 2^{2^{\aleph_0}} \leq \#\mathcal{M} \leq 2^{2^{\aleph_0}} = \#\mathcal{P}([0,1])$
But $\#\mathcal{B} = 2^{\aleph_0} < \#\mathcal{M} \Rightarrow \mathcal{M} \setminus \mathcal{B} \neq \emptyset$.