Literatur

- Rudin, Real and Complex Analysis
- Widom, Lectures on Measure and Integration
- Bartle, A Modern Theory of Integration
- Stein, Shakarchi: Measure Theory, Integration and Hilbert spaces
- Bauer, Measure and Integration Theory
- Elstrodt, Maß- und Integrationstheorie
- Halmos, Measure Theory.

1. Rings, Algebras, Contents, Measures

1.1. Rings and Algebras

Definition 1.1. X set, $A, B \subseteq X$. Then:

$$A \triangle B := (A\setminus B) \cup (B \setminus A) \quad \text{symmetric difference}$$
$$= (A\cup B) \setminus (A \cap B)$$

Satz 1.2. X set, then: $(\mathcal{P}X, \triangle, \cap)$ is a commutative ring ($\triangle$ as addition, $\cap$ as multiplication) with $\emptyset$ zero and X one.

Proof. Calculate.

Or $K = \{\bar 0, \bar 1\}$ field; $R := \{f: X \to K\}$.

$\rightsquigarrow$ R is a ring with addition and multiplication defined pointwise with $X \to K, x \mapsto \bar 0$ as 0, and $X \to K, x \mapsto \bar 1$ as 1

$\psi: \mathcal{P}X \to R,\ A \mapsto \chi_A$ is a bijection and

$$\begin{cases} \psi(A\cap B) = \chi_{A\cap B} = \chi_A \cdot \chi_B = \psi(A)\cdot\psi(B) \\ \psi(A\triangle B) = \chi_{(A\setminus B)\dot\cup(B\setminus A)} = \chi_{A\setminus B} + \chi_{B\setminus A} = \chi_{A\setminus B} + \overbrace{\chi_{A\cap B} + \chi_{A\cap B}}^{=0} + \chi_{B\setminus A} \end{cases}$$
$$= \chi_A + \chi_B = \psi(A) + \psi(B)$$

$\Rightarrow (\mathcal{P}X, \triangle, \cap)$ is a comm. ring with $0 = \psi^{-1}(x \mapsto \bar 0) = \emptyset$

$1 = \psi^{-1}(x \mapsto \bar 1) = X$

□

Definition 1.3. X set.

i) $R \subseteq \mathcal{P}X$ is a ring over X if it is a subring of $\mathcal{P}X$.

ii) $\mathfrak{A} \subseteq \mathcal{P}X$ is an algebra over X if it ~~is~~ is a ring and $X \in \mathfrak{A}$.

Remarks 1.4.

i) R ring over X. Then: $\emptyset \in R$ because every ring contains the zero element.

$A, B \in R \Rightarrow \begin{cases} A\cup B \in R & \text{because} \quad A\cup B = (A\triangle B)\triangle(A\cap B) \\ A\setminus B \in R & \text{because} \quad A\setminus B = A\triangle(A\cap B) \end{cases}$

By induction: $A_1, \dots, A_n \in R \Rightarrow \bigcup_{j=1}^{n} A_j \in R$.

ii) $\mathfrak{A}$ algebra over X. Then: $A \in \mathfrak{A} \Rightarrow X\setminus A \in \mathfrak{A}$

Theorem 1.5. X set, $R \subseteq \mathcal{P}X$. The following is equivalent:

i) R is a ring over X

ii) $\emptyset \in R$ and $\forall A, B \in R$: $A\triangle B \in R$ and $A \cap B \in R$

iii) $\emptyset \in R$ and $A\triangle B \in R$ and $A\cup B \in R$

iv) $\emptyset \in R$ and $A \cup B \in R$ and $A\setminus B \in R$.

Proof.

i) $\Rightarrow$ ii): ✓ (every ring contains the zero element)

ii) $\Rightarrow$ i) ✓ –"–

i) $\Rightarrow$ iii) ✓ } by the remark

i) $\Rightarrow$ iv) ✓ }

iii) $\Rightarrow$ ii) $\quad A\cap B = (A\cup B)\triangle(A\triangle B)$

iv) $\Rightarrow$ iii) $\quad$ ~~A∩B = A△B = A∪B~~ $A\triangle B = (A\setminus B)\cup(B\setminus A)$

□

Theorem 1.6. X set, $\mathfrak{A} \subseteq \mathcal{P}X$. The following is equivalent:

i) $\mathfrak{A}$ is an algebra

ii) $X \in \mathfrak{A}$ and $\forall A, B \in \mathfrak{A}$: $X\setminus A \in \mathfrak{A}$ and $A\cup B \in \mathfrak{A}$

iii) $X \in \mathfrak{A}$ and $\forall A, B \in \mathfrak{A}$: $X\setminus A \in \mathfrak{A}$ and $A \cap B \in \mathfrak{A}$.

Proof. i) $\Rightarrow$ ii), i) $\Rightarrow$ iii): clear.

ii) $\Rightarrow$ iii) $\quad A\cap B = X\setminus((X\setminus A)\cup(X\setminus B))$

iii) $\Rightarrow$ i) $\quad A\triangle B = (A\cup B)\setminus(A\cap B)$

$$= (A\cup B)\cap(X\setminus(A\cap B))$$
$$= X\setminus((X\setminus A)\cap(X\setminus B)) \cap (X\setminus(A\cap B)).$$

$\Rightarrow \mathfrak{A}$ is a ring (Thm 1.5) and $X \in \mathfrak{A}$

$\Rightarrow \mathfrak{A}$ is an algebra.

□

Examples 1.8.

i) X set, $A \subseteq X$. Then: $\{\emptyset, A\}$ is a ring, (an algebra if and only if $A = X$)

$\{\emptyset, A, X, X\setminus A\}$ is an algebra.

ii) $\mathcal{P}X$ is an algebra

iii) $X \neq \emptyset$, $\quad E := \{A \subseteq X \mid A \text{ finite}\}$ is a ring.

E is an algebra $\Leftrightarrow$ X finite.

iv) $X \neq \emptyset$, $\quad C := \{A \subseteq X \mid A \text{ countable}\}$ is a ring

C is an algebra $\Leftrightarrow$ X countable (Here: "countable" is finite or countably infinite)

Aufgabe v) X countably infinite set.

$\mathfrak{A} := \{A \subseteq X \mid A \text{ or } X\setminus A \text{ finite}\}$ is an algebra.

Definition 1.9 X set.

i) $R \subseteq \mathcal{P}X$ is a σ-ring over X :$\Leftrightarrow$ R is a ring over X and for every sequence $(A_j)_{j\in\mathbb{N}} \subseteq R$: $\bigcup_{j=1}^{\infty} A_j \in R$

ii) $\mathfrak{A} \subseteq \mathcal{P}X$ is a σ-algebra over X :$\Leftrightarrow$ $\mathfrak{A}$ σ-ring over X and $X \in \mathfrak{A}$.

Theorem 1.10. X set, $R \subseteq \mathcal{P}X$, $\mathfrak{A} \subseteq \mathcal{P}X$

i) R σ-ring over X $\Leftrightarrow$ $\left\{\begin{array}{l} \emptyset \in R \\ A,B \in R \Rightarrow A\setminus B \in R \\ (A_n)_n \subseteq R \Rightarrow \bigcup_{j=1}^{\infty} A_j \in R \end{array}\right.$

ii) $\mathfrak{A}$ σ-algebra over X (A)

$\Leftrightarrow$ (B) $\left\{\begin{array}{l} X \in \mathfrak{A} \\ A \in \mathfrak{A} \Rightarrow X\setminus A \in \mathfrak{A} \\ (A_n)_n \subseteq \mathfrak{A} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathfrak{A} \end{array}\right\}$ $\Leftrightarrow$ (C) $\left\{\begin{array}{l} X \in \mathfrak{A} \\ A \in \mathfrak{A} \Rightarrow X\setminus A \in \mathfrak{A} \\ (A_n)_n \subseteq \mathfrak{A} \Rightarrow \bigcap_{j=1}^{\infty} A_j \in \mathfrak{A} \end{array}\right.$

Proof i) "$\Rightarrow$" clear.

"$\Leftarrow$" given $A,B \in R$, let $A_1 := A$, $A_2 := B$, $A_j := \overset{\in R}{\emptyset}$, $j \geq 3$.

$\Rightarrow A \cup B = \bigcup_{j=1}^{\infty} A_j \in R$.

ii) (A) $\Rightarrow$ (B) clear. (B) $\Rightarrow$ (A) by thm. 1.6.

(B) $\Leftrightarrow$ (C) because $\bigcup_{j=1}^{\infty} A_j = X \setminus \left(\bigcap_{j=1}^{\infty} A_j\right)$ and $\bigcap_{j=1}^{\infty} A_j = X \setminus \left(\bigcup_{j=1}^{\infty} A_j\right)$

Examples.

i) Every finite $\left\{\begin{array}{l}\text{ring}\\ \text{algebra}\end{array}\right\}$ is a σ-$\left\{\begin{array}{l}\text{ring}\\ \text{algebra}\end{array}\right.$.

$\mathcal{P}X$ is always a σ-algebra.

ii) X set, $C := \{A \subseteq X \mid A \text{ countable}\}$ is a σ-algebra

iii) X, Y sets, $\mathcal{B}$ σ-$\left\{\begin{array}{l}\text{ring}\\ \text{algebra}\end{array}\right\}$ over Y, $f: X \to Y$.

Then: $f^{-1}(\mathcal{B}) := \{f^{-1}(B) \mid B \in \mathcal{B}\}$ is a σ-$\left\{\begin{array}{l}\text{ring}\\ \text{algebra}\end{array}\right\}$ in X.

iv) special case of iii): $X \subseteq Y$, $f: X \to Y$, $f(x) = x$.

$\Rightarrow f^{-1}(\mathcal{B}) := \{B \cap X \mid B \in \mathcal{B}\} =: \mathcal{B}|X$

$=:$ trace of $\mathcal{B}$ in X.

Definition 1.11: X set, $H \subseteq \mathcal{P}X$ is a semiring over X if

i) $\emptyset \in H$

ii) $A, B \in H \Rightarrow A \cap B \in H$

iii) $A, B \in H \Rightarrow \exists\, C_1, \ldots, C_n \in H$ s.t. $A \setminus B = \bigcup_{j=1}^{n} C_j$

pairwise disjoint.

Examples.

i) $H := \{\emptyset\} \cup \{\{a\} \mid a \in X\}$ semiring over X (ring only if $|X| \in \{0,1\}$)

ii) $\mathcal{J} := \{[a,b[\mid a,b \in \mathbb{R},\ a \le b\}$
$\mathcal{J} := \{]a,b] \mid a,b \in \mathbb{R},\ a \le b\}$ } semirings over $\mathbb{R}$.

Lemma 1.12 H, K semirings over X and Y.

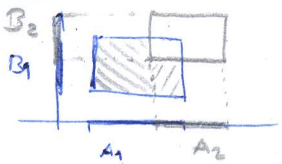
$\Rightarrow$ $H * K := \{A \times B \mid A \in H,\ B \in K\}$ semiring over $X \times Y$.

Proof. i) $\emptyset = \emptyset \times \emptyset \in H * K$

Let $C_1 := A_1 \times B_1$, $C_2 = A_2 \times B_2 \in H * K$.

$\Rightarrow$ ii) $C_1 \cap C_2 = (A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) \in H * K$

iii) $C_1 \setminus C_2 = (A_1 \times B_1) \setminus (A_2 \times B_2) =$
$= [(A_1 \setminus A_2) \times B_1] \,\dot\cup\, [(A_1 \cap A_2) \times (B_1 \setminus B_2)]$
$\in H * K$.



$\Rightarrow$ by def. $H * K$ is a semiring. □

Corollary 1.13. For $a = (a_j)_{j=1}^s \in \mathbb{R}^s$, $b = (b_j)_{j=1}^s \in \mathbb{R}^s$ let
$a \le b \iff \forall j = 1, \dots, s \quad a_j \le b_j$. and $[a,b[:= \bigtimes_{j=1}^s [a_j, b_j[$ etc.

Then $\mathcal{J}^s := \{[a,b[\subseteq \mathbb{R}^s \mid a \le b\}$
$\mathcal{J}^s := \{]a,b] \subseteq \mathbb{R}^s \mid a \le b\}$ } semirings over $\mathbb{R}^s$.

Definition 1.14. $Y \subseteq PX$. Then the smallest ring containing Y is called the ring generated by Y (similar: σ-ring, algebra, σ-algebra). denoted by $R(Y)$

Remark $R(Y) = \bigcap_{R \in \mathcal{R}} R$, $\mathcal{R} = \{R \subseteq PX \mid R \text{ ring over } X \text{ and } Y \subseteq R\}$

obviously: $PX \in \mathcal{R}$.

Theorem 1.14. X set, $H \subseteq PX$ semiring over X. Then the ring generated by H is

$$R(H) = \left\{ \dot\bigcup_{j=1}^n A_j \mid A_j \in H,\ n \in \mathbb{N} \right\} =: R$$

Proof. Obviously: $H \subseteq R$, and if R is a ring, then it is the smallest containing H.
So it suffices to show: R is a ring.
By theorem 1.5 we have to show: $\emptyset \in R$, $A \cup B \in R$, $A \setminus B \in R$ if $A, B \in R$.

i) $\emptyset \in H \subseteq R$

$A, B \in R \Rightarrow \exists \begin{matrix} N_1, \dots, N_n \\ M_1, \dots, M_m \end{matrix}$ } pairwise disjoint, $\in H$ s.t. $A = \dot\bigcup_{j=1}^n N_j$, $B = \dot\bigcup_{j=1}^m M_j$.

ii) We show $A \setminus B \in R$ by induction on m:

m = 1: $A \setminus B = \left(\dot\bigcup_{j=1}^n N_j\right) \setminus M_1 = \dot\bigcup_{j=1}^n \underbrace{N_j}_{\in H} \setminus \underbrace{M_1}_{\in H} = \dot\bigcup_{j=1}^n \dot\bigcup_{k=1}^{n_j} B_{jk}$

for some $B_{jk} \in H$ $\Rightarrow$ $A \setminus B \in H$.

m ⇝ m+1: $A \setminus B = A \setminus \dot\bigcup_{j=1}^{m+1} M_j = \underbrace{\underbrace{\left(A \setminus \dot\bigcup_{j=1}^m M_j\right)}_{\in R \text{ by ind. hyp because } \dot\bigcup_{j=1}^m M_j \in R} \setminus M_{m+1}}_{\in R \text{ by ind. hyp}}$

iii) To show $A \cup B \in R$:

Case 1 $A \cap B = \emptyset \Rightarrow A \cup B = \dot\bigcup_{j=1}^n N_j \,\dot\cup\, \dot\bigcup_{j=1}^m M_j \in R$.

Case 2 $A \cap B \neq \emptyset \Rightarrow A \cup B = A \,\dot\cup\, (B \setminus A) \in R$ by case 1.

□

Observation: $(\mathcal{Y}_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{P}X$ family of rings, σ-rings, algebras, σ-algebras, semirings $\Rightarrow \bigcap_{\lambda\in\Lambda} \mathcal{Y}_\lambda$ is a σ-ring, σ-ring, ...

~~It is called the~~

≠ ∅ because it contains $\mathcal{P}X$

$\Rightarrow \forall\, \mathcal{Y} \subseteq \mathcal{P}X$: $\tilde{\mathcal{Y}} := \bigcap \left\{ \mathcal{B} \subseteq \mathcal{P}X \mid \mathcal{B} \text{ is a } \left\{\begin{matrix}\text{ring}\\ \sigma\text{-ring}\\ \vdots\end{matrix}\right\} \right\}$

is a ring, σ-ring, ... It is the smallest ring, σ-ring ... w.r.t. inclusion, that contains $\mathcal{Y}$. It is called the ring, σ-ring, ... generated by $\mathcal{Y}$.

Special case: X top. space, $\mathcal{T}$ = topology on X.

Then: $\mathcal{B} := \sigma(\mathcal{T}) :=$ σ-algebra generated by $\mathcal{T}$ =: Borel sets

Definition. $\mathfrak{M} \subseteq \mathcal{P}X$ is called a monotone class.

:⟺ i) $(A_n)_n \subseteq \mathfrak{M}$, $A_1 \subseteq A_2 \subseteq \dots \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathfrak{M}$ and

ii) $(A_n)_n \subseteq \mathfrak{M}$, $A_1 \supseteq A_2 \supseteq \dots \Rightarrow \bigcap_{j=1}^{\infty} A_j \in \mathfrak{M}$

Remark:

i) Every σ-ring is a monotone class.

ii) Every monotone ring is a ~~monotone class~~ σ-ring

iii) Intersection of mon. classes is a monotone class ⟹ "Mon. class gen. by $\mathcal{Y}$" makes sense.

Theorem. R ring ~~ab~~ over X.

$S :=$ σ-ring generated by R, $\mathfrak{M} :=$ monotone class generated by R

$\Rightarrow S = \mathfrak{M}$.

Proof. $S \supseteq \mathfrak{M}$ is clear. (Use: $A_1 \supseteq A_2 \supseteq \dots \Rightarrow \bigcap_{j=1}^{\infty} A_j = A_1 \setminus \bigcup_{j=1}^{\infty} (A_1 \setminus A_j)$)

To show $\mathfrak{M} \supseteq S$ it suffices to show: $\mathfrak{M}$ is ~~a ring~~ a σ-ring.

Fix $A \in \mathfrak{M}$ and define

$Q(A) := \{ B \subseteq X \mid A\setminus B,\ B\setminus A,\ A\cup B \in \mathfrak{M} \}$ = alle $B \subseteq X$, für die die Ringoperationen "gut" sind.

Obviously: $Q(A)$ is a monotone class, and for all $A, B \in \mathfrak{M}$: $A \in Q(B) \Leftrightarrow B \in Q(A)$.

Note: $A \in R \Rightarrow R \subseteq Q(A)$. $\Rightarrow \mathfrak{M} \subseteq Q(A)$

Now: $A \in R$, $B \in \mathfrak{M} \Rightarrow B \in Q(A) \Rightarrow A \in Q(B)$.

$A \in R$ arbitrary $\Rightarrow R \subseteq Q(B) \Rightarrow \mathfrak{M} \subseteq Q(B)$

$\Rightarrow \forall\, \tilde{A}, \tilde{B} \in \mathfrak{M}$: $\tilde{A} \cup \tilde{B},\ \tilde{A}\setminus\tilde{B} \in \mathfrak{M}$ (by def. of $Q(B)$) (1)

Obviously: $\emptyset \in Q(B)$ (2)

(1),(2) ⟹ $\mathfrak{M}$ is a ring by thm 1.5. □

Corollary. σ-algebra generated by $\mathfrak{A}$ = monotone class generated by $\mathfrak{A}$, if $\mathfrak{A}$ is an ~~g~~ algebra over X. □

Definition. $D \subseteq \mathcal{P}X$ is a Dynkin system over X

:⟺ $\begin{cases} X \in D \\ A \in D \Rightarrow X\setminus A \in D \\ A_j \text{ pairwise disjoint, elements of } D \Rightarrow \dot{\bigcup}_{j=1}^{\infty} A_j \in D. \end{cases}$

Theorem. Let $D \subseteq \mathcal{P}X$. Then:

D Dynkin system ⟺ $\begin{cases} X \in D \\ \cancel{A, B \in D \text{ and } B \subseteq A \Rightarrow A \setminus B \in D}\ \ A, B \in D \text{ mit } A \supseteq B \Rightarrow A\setminus B \in D \\ D \text{ is a monotone class} \end{cases}$

Proof. "⟹" Assume D is a Dynkin system.

Let $A, B \in D \Rightarrow A\setminus B = X \setminus \big((X\setminus A) \,\dot\cup\, B \,\dot\cup\, \emptyset \,\dot\cup \dots \big) \in D$

X

Let $(A_j)_{j\in\mathbb{N}} \in D$ with $A_1 \subseteq A_2 \subseteq \dots$ and define $B_j := A_j \setminus A_{j-1} \in D$ $(A_0 := \emptyset)$

$\Rightarrow \bigcup_{j=1}^{\infty} A_j = \dot{\bigcup}_{j=1}^{\infty} B_j \in D$

If $A_1 \supseteq A_2 \supseteq \dots$ then $\bigcap_{j=1}^{\infty} A_j = A_1 \setminus \left(\bigcup_{j=1}^{\infty} (A_1 \setminus A_j) \right) \in D$.

"$\Leftarrow$" Assume: $X \in D$, $A, B \in D \Rightarrow A \setminus B \in D$ and D is mon. class.

To show: $A_1, A_2, \dots$ pairwise disjoint, $\in D \Rightarrow \dot{\bigcup}_{j=1}^{\infty} A_j \in D$

Note: $A, B \in D$, $A \cap B = \emptyset$ then: $A \cup B = ((X \setminus A) \setminus B)^c \in D$

$\Rightarrow \forall j \; B_j := \dot{\bigcup}_{l=1}^{j} A_l \in D$ and $B_1 \subseteq B_2 \subseteq \dots$

$\Rightarrow \dot{\bigcup}_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} B_j \in D$.

□

Corollary $D \subseteq \mathbb{P}X$ Dynkin system. Then:

D stable under intersection $\iff$ D σ-algebra.

durchschnittsstabil: $A, B \in D \Rightarrow A \cap B \in D$

Proof "$\Leftarrow$" ✓ "$\Rightarrow$" $\left.\begin{array}{l} D \text{ durchschnittsstabil} \Rightarrow D \text{ algebra} \\ \text{by thm: } D \text{ mon. class} \end{array}\right\} \Rightarrow D$ σ-algebra.

Theorem. $Y \subseteq \mathbb{P}X$ durchschnittsstabil. Then: $D = S$,

where D = Dynkin system generated by Y

S = σ-algebra generated by Y.

Proof. $S \supseteq D$ clear. To show $S \subseteq D$, it suffices to show that D is a σ-algebra. By the corollary, it is enough to show that D is durchschnittsstabil.

Fix $A \in D$ and set $Q(A) := \{M \subseteq X \mid A \cap M \in D\}$

$\Rightarrow Q(A)$ is a Dynkin system.

If $A \in Y$, then $Y \subseteq Q(A)$ because Y is durchschnittsstabil.

$\Rightarrow D \subseteq Q(A)$

Now: $A \in Y$, $B \in D \Rightarrow B \in Q(A) \Rightarrow A \in Q(B)$

$\Rightarrow Y \subseteq Q(B) \Rightarrow D \subseteq Q(B) \Rightarrow \forall C \in D: B \cap C \in D$ (Def of $Q(B)$)

Since B was arbitrary, D is durchschnittsstabil.

□

1.2. Contents and Measure

Definition 1.15 H semiring on a set X. $\mu: H \to \mathbb{R} \cup \{\infty\}$ is a content on H

$$:\Longleftrightarrow \begin{cases} \text{i) } \mu(\emptyset) = 0 \\ \text{ii) positivity: } \mu(M) \geq 0, \; M \in H \\ \text{iii) additivity: } M_1, \dots, M_n \in H, \text{ pairwise disjoint, then } \mu\left(\dot{\bigcup}_{j=1}^{n} M_j\right) = \sum_{j=1}^{n} \mu(M_j). \end{cases}$$

$\mu: H \to \mathbb{R} \cup \{\infty\}$ is a premeasure on H (or measure on H)

$:\Longleftrightarrow$ i) $\mu(\emptyset) = 0$

ii) positivity $\mu(M) \geq 0$, $M \in H$

iii) σ-additivity $(M_j)_j \in H$ pairwise disjoint such that $\dot{\bigcup}_{j=1}^{\infty} M_j \in H$, then $\mu\left(\dot{\bigcup}_{j=1}^{\infty} M_j\right) = \sum_{j=1}^{\infty} \mu(M_j)$.

μ is called a measure if H is a σ-algebra.

In this case: (X, H, μ) is called a measure space.

A content, (pre-) measure is called finite if $\forall A \in H \;\; \mu(A) < \infty$. □

Remarks

i) Convention: $\forall a \in \mathbb{R}$ $\begin{cases} a \cdot \infty = \infty \text{ if } a > 0. \\ a + \infty = \infty, \quad \infty + \infty = \infty. \end{cases}$

$\rightsquigarrow \sum_{j=1} \mu(M_j) = \infty$ if at least one $\mu(M_j) = \infty$

ii) order of summation in $\sum_{j=1}^{\infty} \mu(M_j)$ not important, because all terms are ≥ 0.

iii) the conditions i) excludes the possibility "$\mu(M) = \infty$ for all $M \in H$."

If μ is a finite content, then i) follows from ii) and iii) because:

$\mu(\emptyset) = \mu(\emptyset \dot{\cup} \emptyset) = \mu(\emptyset) + \mu(\emptyset)$.

Examples

i) J^p as before, $M := [a, b[\; \in J^s$.

Then $\mu(M) := \prod_{j=1}^{s} (b_j - a_j)$ defines a content on J^s.

ii) H semiring over a set X. Then: $\mu(A) := \begin{cases} \infty, & A \neq \emptyset \\ 0, & A = \emptyset \end{cases}$ defines a measure on H.

iii) Counting measure. X set, $\mu: \mathcal{P}X \to \mathbb{R} \cup \{\infty\}$, $\mu(A) = \begin{cases} |A| & \text{if } A \text{ finite} \\ \infty & \text{else.} \end{cases}$

iv) Dirac measure. X set. Fix $a \in X$ and define

$\mu_a: \mathcal{P}X \to \mathbb{R} \cup \{\infty\}$, $\mu_a(B) := \chi_B(a) = \begin{cases} 1, & a \in B \\ 0, & \text{else} \end{cases}$

$\to (X, \mathcal{P}X, \mu_a)$ is a measure space.

Generalization: $(a_j)_{j=1}^{\infty} \subseteq X$, pairwise distinct, $\alpha_j \geq 0$.

Then: $\mu := \sum_{j=1}^{\infty} \alpha_j \mu_{a_j}$ defines a measure on X.

"μ is concentrated on $\{a_j \mid j \in \mathbb{N}\}$".

Aufgabe

v) X countably infinite, $\mathfrak{A} := \{A \subseteq X \mid A \text{ or } X \setminus A \text{ finite}\}$ is an algebra (but not a σ-algebra)

and $\mu: \mathfrak{A} \to \mathbb{R} \cup \{\infty\}$, $\mu(A) = \begin{cases} 0, & A \text{ finite} \\ 1, & A \text{ not finite} \end{cases}$

is a content on $\mathfrak{A}$, but not a premeasure.

Lemma 1.16. μ content on a semiring H, $A, B \in H$. Then:

$$A \subseteq B \implies \mu(A) \le \mu(B) \quad \text{(isotonicity)}$$

Proof. Choose $C_1, \dots, C_n \in H$ s.t. $B \setminus A = \dot{\bigcup}_{j=1}^n C_j$.

$$\implies \mu(B) = \mu(A \,\dot\cup\, (B \setminus A)) = \cancel{\mu(A) + \mu(\dot\cup\, C_j)}$$
$$= \mu(A \,\dot\cup\, \dot{\bigcup}_{j=1}^n C_j) = \mu(A) + \underbrace{\sum_{j=1}^n \mu(C_j)}_{\ge 0} \ge \mu(A). \qquad \square$$

Proposition 1.17. Extension of a content.

H semiring over X, $\mu: H \to \mathbb{R} \cup \{\infty\}$ content.

$R(H) :=$ ring generated by H. (recall: $R(H) = \{\dot{\bigcup}_{j=1}^n M_j \mid M_j \in H, n \in \mathbb{N}\}$)

For $A := \dot{\bigcup}_{j=1}^n M_j \in R(H)$ (with $M_j \in H$) define

$$\tilde\mu(A) := \sum_{j=1}^n \mu(M_j).$$

Then:
i) $\tilde\mu$ is a content on $R(Y)$
ii) $\tilde\mu$ is a pre-measure on $R(Y)$ ~~if μ is σ-additive~~ if μ is σ-additive
iii) $\tilde\mu|_H = \mu$
iv) $\tilde\mu$ uniquely determined by i) and iii)
(that is: μ' content on $R(Y)$ with $\mu'|_H = \mu$, then $\mu' = \tilde\mu$).

Proof

First we show that $\tilde\mu$ is well-defined.

Let $M_1, \dots, M_n, M_1', \dots, M_m' \in H$ such that $\dot{\bigcup}_{j=1}^n M_j = \dot{\bigcup}_{k=1}^m M_k' =: A \in R(H)$

$$\implies \forall j = 1, \dots, n: \quad M_j = \dot{\bigcup}_{k=1}^m \underbrace{(M_j \cap M_k')}_{\in H} \implies \mu(M_j) = \sum_{k=1}^m \mu(M_j \cap M_k')$$

$$\implies \sum_{j=1}^n \mu(M_j) = \sum_{j=1}^n \sum_{k=1}^m \mu(M_j \cap M_k') = \dots = \sum_{k=1}^m \mu(M_k'). \qquad \square$$

i) Obviously: $\tilde\mu(\emptyset) = 0$ and $\forall M \in R(H)$ $\mu(M) \ge 0$.

Let $A_1, \dots, A_n \in R(H)$, pairwise disjoint, and choose $B_{jk} \in H$ s.t. $\forall j = 1, \dots, n$ $A_j = \dot{\bigcup}_{k=1}^{n_j} B_{jk}$.

$$\implies \tilde\mu(\dot{\bigcup}_{j=1}^n A_j) = \tilde\mu(\dot{\bigcup}_{j=1}^n \dot{\bigcup}_{k=1}^{n_j} B_{jk}) = \sum_{j=1}^n \sum_{k=1}^{n_j} \mu(B_{jk})$$
$$= \sum_{j=1}^n \tilde\mu(A_j). \qquad \square$$

ii) We have to show: $(A_j)_{j=1}^\infty \subseteq R(H)$, pairwise disjoint with $\dot{\bigcup}_{j=1}^\infty A_j =: A \in R(H)$, then $\tilde\mu(A) = \sum_{j=1}^\infty \tilde\mu(A_j)$.

Choose $B_{jk}, C_j \in H$, s.t. $\forall j \in \mathbb{N}$ $A_j = \dot{\bigcup}_{k=1}^{n_j} B_{jk}$, and $A = \dot{\bigcup}_{k=1}^n C_k$

$$\implies A_j = A_j \cap A = \dot{\bigcup}_{m=1}^n A_j \cap C_m = \dot{\bigcup}_{m=1}^n \dot{\bigcup}_{k=1}^{n_j} \underbrace{B_{jk} \cap C_m}_{\in H}$$

$$\implies \tilde\mu(A_j) = \sum_{m=1}^n \sum_{k=1}^{n_j} \mu(B_{jk} \cap C_m) = \sum_{m=1}^n \tilde\mu(A_j \cap C_m) \qquad (*)$$

$$C_m = C_m \cap A = C_m \cap (\dot{\bigcup}_{j=1}^\infty A_j) = \dot{\bigcup}_{j=1}^\infty C_m \cap A_j = \dot{\bigcup}_{j=1}^\infty \dot{\bigcup}_{k=1}^{n_j} C_m \cap B_{jk}$$

$$\implies \tilde\mu(C_m) = \sum_{j=1}^\infty \sum_{k=1}^{n_j} \mu(C_m \cap B_{jk}) = \sum_{j=1}^\infty \tilde\mu(C_m \cap A_j) \qquad (**)$$

$$\implies \tilde\mu(A) = \sum_{k=1}^n \mu(C_k) = \sum_{k=1}^n \sum_{j=1}^\infty \tilde\mu(C_k \cap A_j)$$
$$= \sum_{j=1}^\infty \sum_{k=1}^n \tilde\mu(C_k \cap A_j) = \sum_{j=1}^\infty \tilde\mu(A_j). \qquad \square$$

iii) Obvious.

iv) Let $\mu': R(H) \to \mathbb{R} \cup \{\infty\}$ content on $R(H)$ with $\mu'|_H = \mu = \tilde\mu|_H$.

For $A \in R(H)$ choose disjoint $C_1, \dots, C_n \in$ ~~$R(H)$~~ H with $A = \dot{\bigcup}_{j=1}^n C_j$

$$\implies \mu'(A) = \mu'(\dot{\bigcup}_{j=1}^n C_j) = \sum_{j=1}^n \mu'(C_j) = \sum_{j=1}^n \mu(C_j)$$
$$= \sum_{j=1}^n \tilde\mu(C_j) = \tilde\mu(\dot\cup\, C_j) = \tilde\mu(A). \qquad \square$$

Theorem 1.18 (Properties of contents)

$R \subseteq \mathcal{P}X$ ring, $A, B, A_1, A_2, \ldots \in R$; $\mu : R \to \mathbb{R} \cup \{\infty\}$ content.

Then:

i) $B \subseteq A$, $\mu(B) < \infty \Rightarrow \mu(A \setminus B) = \mu(A) - \mu(B)$

ii) $\cancel{\mu(A\cup B) =}\ \mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B)$

iii) Subadditivity: $\mu\left(\bigcup_{j=1}^{n} A_j\right) \le \sum_{j=1}^{n} \mu(A_j)$.

iv) If all A_j are pairwise disjoint and $\dot\bigcup_{j=1}^{\infty} A_j \subseteq B$, then:
$\sum_{j=1}^{\infty} \mu(A_j) \le \mu(B)$

v) If all A_j are pairwise disjoint and $\dot\bigcup_{j=1}^{\infty} A_j \in R$, then
$\sum_{j=1}^{n} \mu(A_j) \le \mu\left(\dot\bigcup_{j=1}^{\infty} A_j\right)$

vi) μ premeasure and $B \subseteq \bigcup_{j=1}^{\infty} A_j$, then
$\mu(B) \le \sum_{j=1}^{\infty} \mu(A_j)$ σ-subadditivity

Proof. i) $\mu(A) = \mu(B \,\dot\cup\, (A \setminus B)) = \mu(B) + \mu(A \setminus B)$

$\mu(B) < \infty \Rightarrow$ Beh.

ii) $\mu(A) + \mu(B) \underset{\uparrow}{=} \mu(A) + \mu(B \setminus A) + \mu(B \cap A) \underset{\uparrow}{=} \mu(A \cup B) + \mu(A \cap B)$.

$B = B \,\dot\cup\, (B \setminus A)$ $\qquad$ $A \cup B = A \,\dot\cup\, (B \setminus A)$

~~which shows the claim if $\mu(A\cap B) < \infty$. If $\mu(A \cap B) = \infty$, then also $\mu(A) = \infty$ and~~

iii) $\forall n$: $\bigcup_{j=1}^{n} A_j = \dot\bigcup_{j=1}^{n} \left(A_j \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)\right)$

$\Rightarrow \mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu\left(A_j \setminus \bigcup_{k=1}^{n-1} A_k\right) \underset{\uparrow}{\le} \sum_{j=1}^{n} \mu(A_j)$

Monotonie von μ.

iv) $\forall n \in \mathbb{N}$ $\dot\bigcup_{j=1}^{n} A_j \subseteq B \Rightarrow \mu(B) \ge \mu\left(\dot\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j)$

Since this holds for all $n \in \mathbb{N}$, it also holds in the limit (Note: $\dot\bigcup_{j\in\mathbb{N}} A_j$ does not necessarily belong to R!)

v) is a special case of iv)

vi) $B = B \cap \left(\bigcup_{j=1}^{\infty} A_j\right) = B \cap \left(\dot\bigcup_{j=1}^{\infty} \left(A_j \setminus \bigcup_{k=1}^{j-1} A_k\right)\right)$ $\qquad$ works also for semirings.

$= \dot\bigcup_{j=1}^{\infty} \underbrace{B \cap \left(A_j \setminus \bigcup_{k=1}^{j-1} A_k\right)}_{\in R}$

μ σ-additive $\Rightarrow \mu(B) = \mu\left(\dot\bigcup_{j=1}^{\infty} B \cap \left(A_j \setminus \bigcup_{k=1}^{j-1} A_k\right)\right)$

$= \sum_{j=1}^{\infty} \mu\Big(\underbrace{B \cap \left(A_j \setminus \bigcup_{k=1}^{j-1} A_k\right)}_{\subseteq A_j}\Big) \le \sum_{j=1}^{\infty} \mu(A_j)$ $\square$

Theorem 1.19. (Charakterisierung der σ-Additivität)

R ring over a ~~finite~~ set X, $\mu : R \to \mathbb{R} \cup \{+\infty\}$ content.

Consider:

(A) μ is a premeasure

(B) $\forall (A_n)_n \subseteq R$ with $A_1 \subseteq A_2 \subseteq \ldots$ and $A := \bigcup_{j=1}^{\infty} A_j \in R$:
$\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(C) $\forall (A_n)_n \subseteq R$ with $A_1 \supseteq A_2 \supseteq \ldots$ and $A := \bigcap_{j=1}^{\infty} A_j \in R$ and $\mu(A_1) < \infty$:
$\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(D) $\forall (A_n)_n \subseteq R$ with $A_1 \supseteq A_2 \supseteq \ldots$ and $\bigcap_{j=1}^{\infty} A_j = \emptyset$ and $\mu(A_1) < \infty$:
$\lim_{n\to\infty} \mu(A_n) = 0$.

Then: (A) $\Leftrightarrow$ (B) $\Rightarrow$ (C) $\Leftrightarrow$ (D). If μ is finite then also (B) $\Leftrightarrow$ (C)

Proof. (A) $\Rightarrow$ (B): Let $M_0 := \emptyset$ and $\forall j \in \mathbb{N}$ $N_j := M_j \setminus M_{j-1}$.

$\Rightarrow \forall j \in \mathbb{N}$ $M_j = \dot\bigcup_{k=1}^{j} N_k$ and $M = \dot\bigcup_{j=1}^{\infty} N_j$.

$\Rightarrow \mu(M) = \sum_{j=1}^{\infty} \mu(N_j)$ $\qquad$ (μ is a premeasure)

Case 1: $\exists j \in \mathbb{N}$ s.t. $\mu(M_j) = \infty \Rightarrow \mu(M) \overset{\text{Monotonie von } \mu}{=} \infty = \lim_{j\to\infty} \mu(M_j)$

Case 2: $\forall j \in \mathbb{N}$ $\mu(M_j) \neq \infty$

$$\Rightarrow \mu(M) = \sum_{j=1}^{\infty} \mu(N_j) = \sum_{j=1}^{\infty} \mu(M_j) - \mu(M_{j-1}) = \lim_{j\to\infty} \mu(M_j).$$

(B) $\Rightarrow$ (A) Only to show: μ is σ-additive.

Let $A_1, A_2, \dots \in R$, pairwise disjoint with $A := \bigcup_{j=1}^{\infty} A_j \in R$.

Let $B_j := \bigcup_{k=1}^{j} A_k$. Then: $B_1 \subseteq B_2 \subseteq \dots$, $\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} A_j$ and all $B_j \in R$.

$$\Rightarrow \mu(A) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) = \sum_{j=1}^{\infty} \mu(A_j)$$

(B) $\Rightarrow$ (C) By assumption: $\mu(A) < \infty$ and $\forall n \in \mathbb{N}$ $\mu(A_n) < \infty$

Since $A_1 \supseteq A_2 \supseteq \dots \Rightarrow A_1 \setminus A_1 \subseteq A_1 \setminus A_2 \subseteq A_1 \setminus A_3 \subseteq \dots$

All $A_1 \setminus A_j \in R$ and $\bigcup_{j=1}^{\infty} A_1 \setminus A_j = A_1 \setminus \bigcap_{j=1}^{\infty} A_j = A_1 \setminus A$

By (B): $\mu(A_1 \setminus A) \underset{A, A_j \subseteq A_1 \text{ and } \mu(A) < \infty}{=} \mu(A_1) - \mu(A)$ (1)

and $\mu(A_1 \setminus A) = \lim_{n\to\infty} \mu(A_1 \setminus A_n) = \mu(A_1) - \lim_{n\to\infty} A_n$ (2)

(1) = (2) $\Rightarrow$ Beh.

(C) $\Rightarrow$ (D) ✓

(D) $\Rightarrow$ (C) Let $B_n := A_n \setminus A$. $\Rightarrow$ all $B_n \in R$, $B_1 \supseteq B_2 \supseteq \dots$, $\bigcap_{j=1}^{\infty} B_j = \emptyset$

Since $B_1 \subseteq A_1 \Rightarrow \mu(B_1) < \infty$.

$$\Rightarrow \mu(A_n) - \mu(A) = \mu(B_n) \to 0, \quad n \to \infty$$

$$\Rightarrow \mu(A_n) \to \mu(A), \quad n \to \infty.$$

Now assume μ is finite.

(C) $\Rightarrow$ (B) We show (D) $\Rightarrow$ (B). $A, (A_n)_n \subseteq R$ as in (B).

Let $B_n := A \setminus A_n \Rightarrow$ all $B_n \in R$, $B_1 \supseteq B_2 \supseteq \dots$, $\bigcap B_j = \emptyset$.

1.3. Content & Premeasure on $\mathbb{R}$

Goal: Find all finite contents and premeasures on $\mathcal{J} = \{[a,b[\mid a \leq b\} \subseteq \mathbb{R}$.

$\to$ All measures on the ring generated by $\mathcal{J}$.

Theorem 1.20.

i) For $F: \mathbb{R} \to \mathbb{R}$ increasing ($x \leq y \Rightarrow F(x) \leq F(y)$ (not nec. strictly!)) let $\mu_F: \mathcal{J} \to \mathbb{R}$, $\mu_F([a,b[) := F(b) - F(a)$.

Then: μ_F is a finite content.

F, G increasing. Then $\mu_F = \mu_G \Leftrightarrow F - G = \text{const.}$

ii) $\mu: \mathcal{J} \to \mathbb{R}$ finite content. Then: the fct.

$$F: \mathbb{R} \to \mathbb{R}, \quad F(x) := \begin{cases} \mu([0,x[), & x \geq 0 \\ -\mu([x,0[), & x < 0 \end{cases}$$

is increasing and $\mu = \mu_F$.

μ_F is called the Stieltjes content corresponding to F.

Proof

i) $\mu_F(\emptyset) = 0$ and $\mu([a,b[) \geq 0$, $a \leq b$, is clear.

Now we show finite additivity: Let $a \leq b$, $a_k \leq b_k$, s.t.

$[a,b[= \bigcup_{k=1}^{n} [a_k, b_k[$. Without restriction: $a = a_1 \leq b_1 = a_2 \leq b_2 \dots \leq b_n = b$

$$\Rightarrow \sum_{k=1}^{n} \mu([a_k,b_k[) = \sum_{k=1}^{n} F(b_k) - F(a_k) \underset{b_k = a_{k-1}}{=} F(b) - F(a)$$

$$= \mu([a,b[).$$

Let $F, G: \mathbb{R} \to \mathbb{R}$ increasing. Then:

$\mu_F = \mu_G \Leftrightarrow \forall a \leq b \in \mathbb{R}$ $\mu_F([a,b[) = \mu_G([a,b[)$

$\Leftrightarrow \forall a \leq b \in \mathbb{R}$ $F(b) - F(a) = G(b) - G(a)$

$\Leftrightarrow \forall a \leq b \in \mathbb{R}$ $F(b) - G(b) = F(a) - G(a)$

$\Leftrightarrow F - G = \text{const.}$

$[a,0[\, = [b,a[\, \dot\cup\, [a,0[$

(iv) F increasing: ✓ To prove $\mu = \mu_F$ let $a \leq b \in \mathbb{R}$.

Case 1 $\quad a \leq b \leq 0 \implies F(b) - F(a) = -\mu([b,0[) + \mu([a,0[) = \mu([a,b[)$

Case 2 $\quad a \leq 0 \leq b \implies F(b) - F(a) = \mu([0,b[) + \mu([a,0[) = \mu([a,b[)$

Case 3 $\quad 0 \leq a \leq b \implies F(b) - F(a) = \mu([0,b[) - \mu([0,a[) = \mu([a,b[)$. □

Theorem 1.21. (Premeasures on $\mathcal{J}$)

$F: \mathbb{R} \to \mathbb{R}$ increasing, μ_F as in theorem 1.20.

Then: μ_F premeasure $\iff$ F left-continuous.

Proof. "$\Rightarrow$" Assume μ_F premeasure on $\mathcal{J}$ and fix $a \in \mathbb{R}$.

~~Let $(b_n)_n \subseteq (a,\infty)$ s.t. $b_n \searrow a$~~

Let $(a_n)_n \subseteq (-\infty, b]$ s.t. $a_n \nearrow b$.

$\implies F(b) - F(a_n) = \mu_F([a_n, b[) \longrightarrow \mu_F\left(\bigcap_{n\in\mathbb{N}} [a_n, b[\right) = \mu(\emptyset) = 0$

$\implies F(a_n) \nearrow F(b) \qquad \implies F$ left-cont.

"$\Leftarrow$" Assume F left-continuous.

Let $a \leq b \in \mathbb{R}$, $a_k \leq b_k$ st $\quad [a,b[\, = \dot\bigcup_{k=1}^{\infty} [a_k, b_k[$

To show: $\mu([a,b[) = \sum_{k=1}^{\infty} \mu([a_k, b_k[)$.

- $\mu([a,b[) \geq \sum \mu([a_k, b_k[)$ by theorem 1.18 (iv)
- "$\leq$". Let $\varepsilon > 0$.

$\implies \exists \beta \in [a,b[$ s.t. $F(b) \leq F(\beta) + \varepsilon$.

$\forall k \in \mathbb{N}\ \exists \alpha_k \in \mathbb{R}$ s.t. $F(\alpha_k) \geq F(a_k) - \varepsilon/2^k$

$\implies$ ~~$[a,b[\, \subseteq$~~ $[a,\beta] \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k[\, \subseteq \bigcup_{k=1}^{\infty}]\alpha_k, b_k[$

Since $[a,\beta]$ cpt and all $]\alpha_k, b_k[$ open, $\exists\, K \in \mathbb{N}$ s.t. $\quad [a,\beta] \subseteq \dot\bigcup_{k=1}^{K}]\alpha_k, b_k[\implies [a,\beta[\, \subseteq \dot\bigcup_{k=1}^{K} [\alpha_k, b_k[$

$$\implies \mu([a,\beta[) = \sum_{k=1}^{K} \mu([\alpha_k, b_k[) = \sum_{k=1}^{K} F(b_k) - F(\alpha_k) \leq \sum_{k=1}^{K} F(b_k) - F(a_k) + \varepsilon/2^k$$

~~$\implies \mu([a,\beta[) \leq F(b) - F(a) + \varepsilon/2 = \mu([a,b[) + \varepsilon/2$.~~

$$\implies \mu([a,b[) = \mu([a,\beta[) + \underbrace{\mu([\beta,b[)}_{<\varepsilon}$$

$$< \sum_{k=1}^{K} \mu([\alpha_k, b_k[) + \sum_{k=1}^{K} \varepsilon/2^k + \varepsilon$$

$$\leq \sum_{k=1}^{\infty} \mu([a_k, b_k[) + 2\varepsilon.$$

ε arbitrary $\implies \mu([a,b[) \leq \sum_{k=1}^{\infty} \mu([a_k, b_k[)$. □

μ_F is called the Stieltjes-Lebesgue-(pre)measure on $\mathcal{J}$.

Special case: $F: \mathbb{R} \to \mathbb{R}$, $F(x) = x$.

$\rightsquigarrow \mu_F =: \lambda =:$ Lebesgue premeasure on $\mathcal{J}$

Note. $F: \mathbb{R} \to \mathbb{R}$ increasing, then for every $x_0 \in \mathbb{R}$ the limit $\lim_{x \nearrow x_0} F(x)$ ex. and the fct $\phi: \mathbb{R} \to \mathbb{R}$, $\phi(x) = \lim_{y \nearrow x} F(y)$ is increasing and left-continuous.

Example.

$F: \mathbb{R} \to \mathbb{R}$, $F(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$ is not left-cont. in 0 and

μ_F is not σ-additive, because

~~$\mu([0,1[)$~~ $\mu([-1,0[) = F(0) - F(-1) = 2$

but ~~$\sum_{k=1}^{\infty} \mu([-1,-\frac{1}{k}[) = F(-\frac{1}{k}) - F(-1) = 0$~~

$$\sum_{k=1}^{\infty} \mu\left(\left[-\tfrac{1}{k}, -\tfrac{1}{k+1}\right[\right) = \sum_{k=1}^{\infty} F\left(-\tfrac{1}{k+1}\right) - F\left(-\tfrac{1}{k}\right) = 0$$

despite $[-1,0[\, = \dot\bigcup_{k=1}^{\infty} \left[-\tfrac{1}{k}, -\tfrac{1}{k+1}\right[$ □

Decomposition of finite premeasure on $\mathcal{J}$:

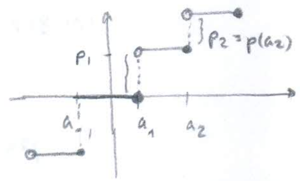
Proposition 1.22. $F: \mathbb{R} \to \mathbb{R}$ increasing, left-cont.

$\Rightarrow$ $\exists$ $H: \mathbb{R} \to \mathbb{R}$ cont, $G: \mathbb{R} \to \mathbb{R}$ left-cont jump function s.t. $F = G + H$.

G, H are unique up to additive constants, and $\mu_F = \mu_G + \mu_H$.

G is called a jump fct, if $\exists A \subseteq \mathbb{R}$ countable subset and $p: A \to \mathbb{R}(0,\infty)$ s.t. $\forall n \in \mathbb{N}$ $\sum_{y \in A \cap [-n,n]} p(y) < \infty$ and $\exists \alpha \in \mathbb{R}$ s.t.

$$(*) \qquad G(x) = \begin{cases} \alpha + \sum_{y \in A \cap [0,x[} p(y), & x > 0 \\ \alpha - \sum_{y \in A \cap [x,0[} p(y), & x \le 0 \end{cases}$$



Proof. $A := \{x \in \mathbb{R} \mid F \text{ discont in } x\}$.

$\Rightarrow$ A countable (F discont $\Rightarrow$ F jumps because F mon. $\Rightarrow$ jumps over an $q \in \mathbb{Q}$)

For $y \in A$ define $p(y) := \lim_{h \searrow 0} F(y+h) - F(y-h)$

For all $n \in \mathbb{N}$ $\sum_{A \cap [-n,n]} p(y) \le F(n) - F(-n) < \infty$.

With this A and p define G as in $(*)$

$\Rightarrow$ G is a jump fct. and $H := F - G$ is continuous and increasing.

$\square$

Finite Premeasures on $\mathcal{J}^s$:

Thm. $\lambda^s: \mathcal{J}^s \to \mathbb{R}$, $\lambda^s([a,b[) = \prod_{j=1}^{s} (b_j - a_j)$ is a premeasure, called the Lebesgue premeasure.

This is a special case of the following thm:

Thm.

- $\mu: \mathcal{J}^s \to \mathbb{R}$ finite premeasure on $\mathcal{J}^s$ $\Rightarrow$ $F(x) := \left(\prod_{j=1}^{s} \operatorname{sgn} x_j\right) \mu([x^-, x^+[)$ is increasing & left cont.
- $F: \mathbb{R}^p \to \mathbb{R}$ increasing and left cont $\Rightarrow$ $\mu_F: \mathcal{J}^s \to \mathbb{R}$ is a finite premeasure.

For definition of "increasing" and the proof of the thm: see

Bauer, Measure and Integration Theory, I. §4,
Elstrodt, Maß- und Integrationstheorie, II. §3.