Linear Algebra

Analysis Series

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Chigüiro Collection

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Contents

Chapter 1

Introduction

This chapter serves as an introduction to the main themes of linear algebra, namely the problem of solving systems of linear equations for several unknowns. We are not only interested in an efficient way to find their solutions, but we also wish to understand how the solutions can possibly look like and what we can say about their structure. For the latter, it will be crucial to find a geometric interpretation of systems of linear equations. In this chapter we will use the "solve and insert"-strategy for solving linear systems. A systematic and efficient formalism will be given in Chapter 3.

Everything we discuss in this chapter will appear again lateron, so you may read it quickly or even skip (parts of) it.

A *linear system* is a set of equations for a number of unknowns which have to be satisfied simultaneously and where the unknowns appear only linearly. If the number of equations is m and the number of unknowns is n, then we call it an $m \times n$ *linear system.* Typically the unknowns are called x, y, z or $x_1, x_2 \dots, x_n$. The following is an example of a linear system of 3 equations for 5 unknowns:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3, \quad 2x_1 + 3x_2 - 5x_3 + x_4 = 1, \quad 3x_1 - 8x_5 = 0.$$

An example of a non-linear system is

$$x_1 x_2 + x_3 + x_4 + x_5 = 3, \quad 2x_1 + 3x_2 - 5x_3 + x_4 = 1, \quad 3x_1 - 8x_5 = 0$$

because in the first equation we have a product of two of the unknowns. Also expressions like x^2, $\sqrt[3]{x}$, xyz, x/y or $\sin x$ make a system non-linear.

Now let us briefly discuss the simplest non-trivial case: A system consisting of one linear equation for one unknown x. Its most general form is

$$ax = b \tag{1.1}$$

where a and b are given constants and we want to find all $x \in \mathbb{R}$ which satisfy (1.1). Clearly, the solution to this problem depends on the coefficients a and b. We have to distinguish several cases.

Case 1. $a \neq 0$. In this case, there is only *one solution*, namely $x = b/a$.

Case 2. $a = 0$, $b \neq 0$. In this case, there is *no solution* because whatever value we choose for x, the left hand side ax will always be zero and therefore cannot be equal to b.
Case 3. $a = 0$, $b = 0$. In this case, there are *infinitely many solutions*. In fact, every $x \in \mathbb{R}$ solves the equation.

So we see that already in this simple case we have three very different types of solution of the system (1.1): no solution, exactly one solution or infinitely many solutions.

Now let us look at a system of one linear equation for two unknowns x, y. Its most general form is

$$ax + by = c. \tag{1.1'}$$

Here, a, b, c are given constants and we want to find all pairs x, y so that the equation is satisfied. For example, if $a = b = 0$ and $c \neq 0$, then the system has no solution, whereas if for example $a \neq 0$, then there are infinitely many solutions because no matter how we choose y, we can always satisfy the system by taking $x = \frac{1}{a}(c - y)$.

Question 1.1

Is it possible that the system has exactly one solution?
(Come back to this question again after you have studied Chapter 3.)

The general form of a system of two linear equations for one unknown is

$$a_1 x = b_1, \quad a_2 x = b_2$$

and that of a system of two linear equations for two unknowns is

$$a_{11}x + a_{12}y = c_1, \quad a_{21}x + a_{22}y = c_2$$

where a_1, a_2, b_1, b_2, respectively $a_{11}, a_{12}, a_{21}, a_{22}, c_1, c_2$ are constants and x, respectively x, y are the unknowns.

Question 1.2

Can you find find examples for the coefficients such that the systems have

(i) no solution,
(ii) exactly one solution,
(iii) exactly two solutions,
(iv) infinitely many solutions?

Can you maybe even give a general rule for when which behaviour occurs?
(Come back to this question again after you have studied Chapter 3.)

Before we discuss general linear systems, we will discuss in this introductory chapter the special case of a system of two linear equations with two unknowns. Although this is a very special type of system, it exhibits many porperties of general linear systems and they appear very often in problems.

1.1 Examples of systems of linear equations; coefficient matrices

Let us start with a few examples of systems of linear equations.

Example 1.1. Assume that a car dealership sells motorcycles and cars. Altogether they have 25 vehicles in their shop with a total of 80 wheels. How many motorcycles and cars are in the shop?

Solution. First, we give names to the quantities we want to calculate. So let M = number of motorcyles, C = number of cars in the dealership. If we write the information given in the exercise in formulas, we obtain

$$\begin{array}{lrl} ① & M + \ \ C = 25, & \text{(total number of vehicles)} \\ ② & 2M + 4C = 80, & \text{(total number of wheels)} \end{array}$$

since we assume that every motorcycle has 2 wheels and every car has 4 wheels. Equation ① tells us that $M = 25 - C$. If we insert this into equation ②, we find

$$80 = 2(25 - C) + 4C = 50 - 2C + 4C = 50 + 2C \quad \Longrightarrow \quad 2C = 30 \quad \Longrightarrow \quad C = 15.$$

This implies that $M = 25 - C = 25 - 15 = 10$. Note that in our calculations and arguments, all the implication arrows go "from left to right", so what we can conclude at this instance is that *the system has only one possible candidate for a solution and this candidate is* $M = 10, C = 15$. We have *not* (yet) shown that it really *is* a solution. However, inserting these numbers in the original equation we see easily that our candidate is indeed a solution.

So the answer is: There are 10 motorcycles and 15 cars (and there is no other possibility). ◇

Let us put one more equation into the system.

Example 1.2. Assume that a car dealership sells motorcycles and cars. Altogether they have 28 vehicles in their shop with a total of 80 wheels. Moreover, the shop arranges them in 7 distinct areas of the shop so that in each area there are either 3 cars or 5 motorcycles. How many motorcycles and cars are in the shop?

Solution. Again, let M = number of motorcyles, C = number of cars. The information of the exercise leads to the following system of equations:

$$\begin{array}{lrl} ① & M + \ \ \ C = 25, & \text{(total number of vehicles)} \\ ② & 2M + \ 4C = 80, & \text{(total number of wheels)} \\ ③ & M/5 + C/3 = \ \ 7. & \text{(total number of areas)} \end{array}$$

As in the previous exercise, we obtain from ① and ② that $M = 10, C = 15$. Clearly, this also satisfies equation ③. So again the answer is: There are 10 motorcycles and 15 cars (and there is no other possibility). ◇

Example 1.3. Assume that a car dealership sells motorcycles and cars. Altogether they have 25 vehicles in their shop with a total of 80 wheels. Moreover, the shop arranges them in 5 distinct areas of the shop so that in each area there are either 3 cars or 5 motorcycles. How many motorcycles and cars are in the shop?

Solution. Again, let M = number of motorcycles, C = number of cars. The information of the exercise gives the following equations:

$$\begin{aligned} &① \quad & M + C &= 25, \qquad &&\text{(total number of vehicles)}\\ &② \quad & 2M + 4C &= 80, \qquad &&\text{(total number of wheels)}\\ &③ \quad & M/5 + C/3 &= 5. \qquad &&\text{(total number of areas)} \end{aligned}$$

As in the previous exercise, we obtain that $M = 10$, $C = 15$ using only equations ① and ②. However, this does not satisfy equation ③; so there is no way to choose M and C such that all three equations are satisfied simultaneously. Therefore, a shop as in this example does not exist. ◇

Example 1.4. Assume that a zoo has birds and cats. The total count of legs of the animals is 60. Feeding a bird takes 5 minutes, feeding a cat takes 10 minutes. The total time to feed the animals is 150 minutes. How many birds and cats are in the zoo?

Solution. Let B = number of birds, C = number of cats in the zoo. The information of the exercise gives the following equations:

$$\begin{aligned} &① \quad & 2B + 4C &= 60, \qquad &&\text{(total number of legs)}\\ &② \quad & 5B + 10C &= 150, \qquad &&\text{(total time for feeding)} \end{aligned}$$

The first equation gives $B = 30 - 2C$. Inserting this into the second equation, gives

$$150 = 5(30 - 2C) + 10C = 150 - 10C + 10C = 150$$

which is always true, independently of the choice of B and C. Indeed, for instance $B = 10, C = 10$ or $B = 14, C = 8$, or $B = 0, C = 15$ are solutions. We conclude that the information given in the exercise it no sufficient to calculate the number of animals in the zoo. ◇

Remark. The reason for this is that both equations ① and ② are basically the same equation. If we divide the first one by 2 and the second one by 5, then we end up in both cases with the equation $B + 2C = 30$, so both equations contain exactly the same information.

Algebraically, the linear system has infinitely many solutions. But our variables represent animals and the only come in nonnegativ integer quantities, so we have the 16 different solutions $B = 30 - C$ where $C \in \{0, 1, \dots, 15\}$. Since our variables represnet

We give a few more examples.

Example 1.5. Find a polynomial P of degree at most 3 with

$$P(0) = 1, \quad P(1) = 7, \quad P'(0) = 3, \quad P'(2) = 23. \tag{1.2}$$

Solution. A polynomial of degree at most 3 is known if we know its 4 coefficients. In this exercise, the unknowns are the coefficients of the polynomial P. If we write $P(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$, then we have to find $\alpha, \beta, \gamma, \delta$ such that (1.2) is satisfied. Note that $P'(x) = 3\alpha x^2 + 2\beta x + \gamma$. Hence (1.2) is equivalent to the following system of equations:

$$\left.\begin{aligned} P(0) &= 1, \\ P(1) &= 7, \\ P'(0) &= 3, \\ P'(2) &= 23. \end{aligned}\right\} \qquad \iff \qquad \begin{cases} (1) & \delta = 1, \\ (2) & \alpha + \beta + \gamma + \delta = 7, \\ (3) & \gamma = 3, \\ (4) & 24\alpha + 8\beta + 2\gamma + \delta = 23. \end{cases}$$

Clearly, $\delta = 1$ and $\gamma = 3$. If we insert this in the remaining equations, we obtain a system of two equations for the two unknowns α, β:

$$\begin{aligned} &(2') & \alpha + \beta &= 3, \\ &(4') & 24\alpha + 8\beta &= 16. \end{aligned}$$

From (2') we obtain $\beta = 4 - \alpha$. If we insert this into (4'), we get that $16 = 24\alpha + 8(4 - \alpha) = 16\alpha + 32$, that is, $\alpha = (32 - 16)/16 = 1$. So the only possible solution is

$$\alpha = 1, \quad \beta = 2, \quad \gamma = 3, \quad \delta = 1.$$

It is easy to verify that the polynomial $P(x) = x^3 + 2x^2 + 3x + 1$ has all the desired properties. $\diamond$

Example 1.6. A pole is 5 metres long and shall be coated with varnish. There are two types of varnish available: The blue one adds 3 g per 50 cm to the pole, the red one adds 6 g per meter to the pole. Is it possible to coat the pole in a combination of the varnishes so that the total weight added is

(a) 35 g? (b) 30 g?

Solution. (a) We denote by b the length of the pole which will be covered in blue and r the length of the pole which will be covered in red. Then we obtain the system of equations

$$\begin{aligned} &(1) & b + r &= 5 & &\text{(total length)} \\ &(2) & 6b + 6r &= 35 & &\text{(total weight)} \end{aligned}$$

The first equation gives $r = 5 - b$. Inserting into the second equation yields $35 = 6b + 6(5 - b) = 30$ which is a contradiction. This shows that there is no solution.

(b) As in (a), we obtain the system of equations

$$\begin{aligned} &(1) & b + r &= 5 & &\text{(total length)} \\ &(2) & 6b + 6r &= 30 & &\text{(total weight)} \end{aligned}$$

Again, the first equation gives $r = 5 - b$. Inserting into the second equation yields $30 = 6b + 6(5 - b) = 30$ which is always true, independently of how we choose b and r as long as (1) is satisfied. This means that in order to solve the system of equations, it is sufficient to solve only the first equation since then the second one is automatically satisfied. So we have infinitely many solutions. Any pair b, r such that $b + r = 5$ gives a solution. So for any b that we choose, we only have to set $r = 5 - b$ and we have a solution of the problem. Of course, we could also fix r and then choose $b = 5 - r$ to obtain a solution.

For example, we could choose $b = 1$, then $r = 4$, or $b = 0.00001$, then $r = 4.99999$, or $r = -2$ then $b = 7$. Clearly, the last example does not make sense for the problem at hand, but it still does satisfy our system of equations. ◇

Example 1.7. When octan reacts with oxigen, the result is carbon dioxide and water. Find the equation for this reaction

Solution. The chemical formulas for the substances are C_8H_{18}, O_2, CO_2 and H_2O. Hence the reaction equation is

$$a\,\mathrm{C_8H_{18}} + b\,\mathrm{O_2} \longrightarrow c\,\mathrm{CO_2} + d\,\mathrm{H_2O}$$

with unkonwn integers a, b, c, d. Clearly the solution will not be unique since if we have one set of numbers a, b, c, d which works and we multiply all of then by the same number, then we obtain another solution. Let us write down the system of equations. To this end we note that the number of atoms of each element has to be equal on both sides of the equation. We obtain:

$$\begin{array}{lrcll} 1 & 8a &=& c & \text{(carbon)} \\ 2 & 18a &=& 2d & \text{(hydrogen)} \\ 3 & 2b &=& 2c + d & \text{(oxygen)} \end{array}$$

or, if we put all the variables on the left hand side,

$$\begin{array}{lrrrrl} 1 & 8a & & -\ c & & = 0, \\ 2 & 18a & & & -\ 2d & = 0, \\ 4 & & 2b & -\ 2c & -\ d & = 0. \end{array}$$

Let us express all the unknowns in terms of a: 1 and 2 show that $c = 8a$ and $d = 9a$. Inserting this in 3 we obtain $0 = 2b - 2 \cdot 8a - 9a = 2b - 25a$, hence $b = \frac{25}{2}a$. If we want all coefficients to be integer, we can choose $a = 2,\ b = 25,\ c = 16,\ d = 18$ and the reaction equation becomes

$$2\,\mathrm{C_8H_{18}} + 25\,\mathrm{O_2} \longrightarrow 16\,\mathrm{CO_2} + 18\,\mathrm{H_2O}\,. \qquad ◇$$

All the examples we discussed in this section are so-called systems of linear equations. Let us give a precise definition of what we mean by this.

Definition 1.8 (Linear system). An $m \times n$ *system of linear equations* (or simply a *linear system*) is a system of m linear equations for n unknowns of the form

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &=& b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &=& b_2 \\ \vdots \qquad\qquad \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &=& b_m \end{array} \tag{1.3}$$

The unknowns are $x_1, \dots, x_n$ while the numbers a_{ij} and b_i ($i = 1, \dots, m,\ j = 1, \dots, n$) are given. The numbers a_{ij} are called the *coefficients of the linear system* and the numbers $b_1, \dots, b_n$ are called the *right side of the linear system*.

A *solution* of the system (1.3) is a tuple $(x_1, \dots, x_n)$ such that all m equations of (1.3) are satisfied simultaneously. The system (1.3) is called *consistent* if it has at least one solution. It is called *inconsistent* if it has no solution.

In the special case when all b_i are equal to 0, the system is called a *homogeneous system*; otherwise it is called *inhomogeneous*.

Definition 1.9 (Coefficient matrix). The *coefficient matrix* A of the system is the collection of all coefficients a_{ij} in an array as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \tag{1.4}$$

The numbers a_{ij} are called the *entries* or *components* of the matrix A.
The *augmented coefficient matrix* A of the system is the collection of all coefficients a_{ij} and the right hand side; it is denoted by

$$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{array}\right). \tag{1.5}$$

The coefficient matrix is nothing else than the collection of the coefficients a_{ij} ordered in some sort of table or rectangle such that the place of the coefficient a_{ij} is in the ith row of the jth column. The augmented coefficient matrix contains additionally the constants from the right hand side.

Important observation. There is a one-to-one correspondence between linear systems and augmented coefficient matrices: Given a linear system, it is easy to write down its augmented coefficient matrix and vice versa.

Let us write down the coefficient matrices of our examples.

Example 1.1: This is a 2×2 system with coefficients $a_{11} = 1$, $a_{11} = 1$, $a_{21} = 2$, $a_{22} = 4$ and right hand side $b_1 = 60, b_2 = 200$. The system has a unique solution. The coefficient matrix and the augmented coefficient matrix are

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, \qquad (A|b) = \left(\begin{array}{cc|c} 1 & 1 & 60 \\ 2 & 4 & 200 \end{array}\right).$$

Example 1.2: This is a 3×2 system with coefficients $a_{11} = 1$, $a_{11} = 1$, $a_{21} = 2$, $a_{22} = 4$, $a_{31} = 2$, $a_{32} = 3$, and right hand side $b_1 = 60, b_2 = 200$, $b_3 = 140$. The system has a unique solution. The coefficient matrix and the augmented coefficient matrix are

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 2 & 3 \end{pmatrix}, \qquad (A|b) = \left(\begin{array}{cc|c} 1 & 1 & 60 \\ 2 & 4 & 200 \\ 2 & 3 & 140 \end{array}\right),$$

Example 1.3: This is a 3×2 system with coefficients $a_{11} = 1$, $a_{11} = 1$, $a_{21} = 2$, $a_{22} = 4$, $a_{31} = 2$, $a_{32} = 3$, and right hand side $b_1 = 60, b_2 = 200$, $b_3 = 100$. The system has no solution. The coefficient matrix is the same as in Example 1.2, the augmented coefficient matrix is

$$(A|b) = \left(\begin{array}{cc|c} 1 & 1 & 60 \\ 2 & 4 & 200 \\ 2 & 3 & 100 \end{array}\right),$$

Example 1.5: This is a 4×4 system with coefficients $a_{11} = 0$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = 1$, $a_{21} = 1$, $a_{22} = 1$, $a_{23} = 1$, $a_{24} = 1$, $a_{31} = 0$, $a_{32} = 0$, $a_{33} = 1$, $a_{34} = 0$, $a_{41} = 24$, $a_{42} = 8$, $a_{43} = 2$, $a_{44} = 1$, and right hand side $b_1 = 1, b_2 = 7$, $b_3 = 3$, $b_4 = 23$. The system has a unique solution. The coefficient matrix and the augmented coefficient matrix are

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 24 & 8 & 2 & 1 \end{pmatrix}, \quad (A|b) = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 & 3 \\ 24 & 8 & 2 & 1 & 23 \end{array}\right).$$

Example 1.7: This is a 3×4 homogeneous system with coefficients $a_{11} = 8$, $a_{12} = 0$, $a_{13} = -1$, $a_{14} = 0$, $a_{21} = 18$, $a_{22} = 0$, $a_{23} = 0$, $a_{24} = -2$, $a_{31} = 0$, $a_{32} = 2$, $a_{33} = -2$, $a_{34} = -1$, and right hand side $b_1 = 1, b_2 = 7$, $b_3 = 3$, $b_4 = 23$. The system has a unique solution. The coefficient matrix and the augmented coefficient matrix are

$$A = \begin{pmatrix} [r]8 & 0 & -1 & 0 \\ 18 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{pmatrix}, \quad (A|b) = \left(\begin{array}{cccc|c} 8 & 0 & -1 & 0 & 0 \\ 18 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right).$$

We saw that Examples 1.1, 1.2, 1.5, 1.6 (a) have unique solutions. In Examples 1.6 (b) and 1.7 the solution is not unique; they even have infinitely many solutions! Examples 1.3 and 1.6(a) do not admit solutions. So given an $m \times n$ system of linear equations, two important questions arise naturally:

- **Existence**: Does the system have a solution?
- **Uniqueness**: If the system has a solution, is it unique?

More generally, we would like to be able to say something about the structure of solutions of linear systems. For example, is it possible that there is only one solution? That there are exactly two solutions? That there are infinite solutions? That there is is no solution? Can we give criteria for existence and/or uniqueness of solutions?
Can we give criteria for existence of infinitely many solutions? Is there an efficient way to calculate all the solutions of a given linear system?

(Spoiler alert: *A system of linear equations has either no or exactly one or infinitely many solutions. It is not possible that it has, e.g., exactly* 7 *solutions.* This will be discussed in detail in Chapter 3.)

Before answering these questions for general $m \times n$ systems in Chapter 3, we will have a closer look at the special case of 2×2 systems in the next section.

You should now have understood

- what a linear system is,
- what a coefficient matrix and an augmented coefficient matrix are,
- their relation with linear systems,
- that a linear system can have different types of solutions,
- ...

You should now be able to

- pass easily from a linear $m \times n$ system to its (augmented) coefficient matrix and back,
- solve linear systems by the "solve and substitute"-method,
- ...

1.2 Linear 2×2 systems

Let us come back to the equation from Example 1.1. For convenience, we write now x instead of B and y instead of C. Recall that the system of equations that we are interested in solving is

$$\begin{array}{ll} (1) & x + y = 60, \\ (2) & 2x + 4y = 200. \end{array} \tag{1.6}$$

We want to give a **geometric** meaning to this system of equations. To this end we think of pairs x, y as points (x, y) in the plane. Let us forget about the equation (2) for a moment and concentrate only on (1). Clearly, it has infinitely many solutions. If we choose an arbitrary x, we can always find y such that (1) satisfied (just take $y = 60 - x$). Similarly, if we choose any y, then we only have to take $x = 60 - y$ and we obtain a solution of (1).

Where in the xy-plane lie *all* solutions of (1)? Clearly, (1) is equivalent to $y = 60 - x$ which we easily identify as the equation of the line L_1 in the xy-plane which passes through $(0, 60)$ and has slope -1. In summary, a pair (x, y) is a solution of (1) if and only if it lies on the line L_1, see Figure 1.1.

If we apply the same reasoning to (2), we find that a pair (x, y) satisfies (2) if and only if (x, y) lies on the line L_2 in the xy-plane given by $y = \frac{1}{4}(200 - 2x)$ (this is the line in the xy-plane passing through $(0, 50)$ with slope $-\frac{1}{2}$).

Now it is clear that a pair (x, y) satisfies both (1) and (2) if and only if it lies on both lines L_1 and L_2. So finding the solution of our system (1.6) is the same as finding the intersection of the two lines L_1 and L_2. From elementary geometry we know that there are exactly three possibilities for their intersection:

(i) L_1 and L_2 are not parallel. Then they intersect in exactly one point.

(ii) L_1 and L_2 are parallel and not equal. Then they do not intersect.

(iii) L_1 and L_2 are parallel and equal. Then $L_1 = L_2$ and they intersect in infinitely many points (they intersect in every point of $L_1 = L_2$).

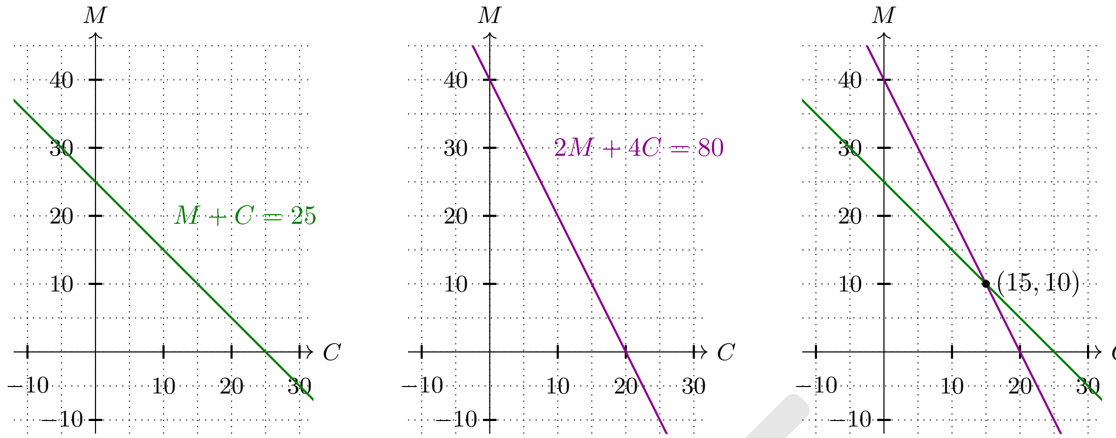


FIGURE 1.1: Graphs of the lines L_1, L_2 which represent the equations from the system (1.6) (see also Example 1.1). Their intersection represents the unique solution of the system.

In our example we know that the slope of L_1 is -1 and that the slope of L_2 is $-\frac{1}{2}$, so they are not parallel and therefore intersect in exactly one point. Consequently, the system (1.6) has exactly one solution.

If we look again at Example 1.6, we see that in Case (a) we have to determine the intersection of the lines

$$L_1 : y = 5 - x, \qquad L_2 : y = \frac{35}{6} - x.$$

Both lines have slope -1 so they are parallel. Since the constant terms in both lines are not equal, they intersect nowhere, showing that the system of equations has no solution, see Figure 1.2.
In Case (b), the two lines that we have to intersect are

$$G_1 : y = 5 - x, \qquad G_2 : y = 5 - x.$$

We see that $G_1 = G_2$, so every point on G_1 (or G_2) is solution of the system and therefore we have infinite solutions, see Figure 1.2.

Important observation. If a linear 2×2 system has a unique solution or not, has nothing to do with the right hand side of the system because this only depends on whether the two lines are parallel or not, and this in turn depends only on the coefficients on the left hand side.

Now let us consider the general case.

One linear equation with two unknowns

The general form of one linear equation with two unknowns is

$$\alpha x + \beta y = \gamma. \tag{1.7}$$

For the set of solutions, there are three possibilities:

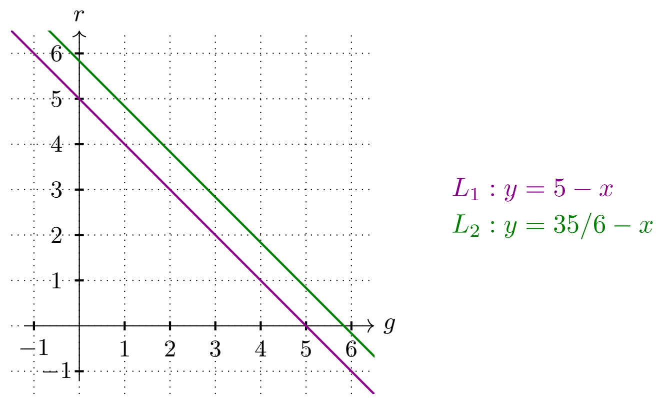


FIGURE 1.2: Example 1.6. Graphs of L_1, L_2.

(i) *The set of solutions forms a line.* This happens if at least one of the coefficients α or β is different from 0. If $\beta \neq 0$, then set of all solutions is equal to the line $L : y = -\frac{\alpha}{\beta}\, x + \frac{\gamma}{\beta}$ which is a line with slope $-\frac{\alpha}{\gamma}$. If $\beta = 0$ and $\alpha \neq 0$, then the set of solutions of (1.7) is a line parallel to the y-axis passing through $(0, \frac{\gamma}{\alpha})$.

(ii) *The set of solutions is all of the plane.* This happens if $\alpha = \beta = \gamma = 0$. In this case, clearly every pair (x, y) is a solution of (1.7).

(iii) *There is no solution.* This happens if $\alpha = \beta = 0$ and $\gamma \neq 0$. In this case, no pair (x, y) is a solution of (1.7) since the left hand side is always 0.

In the first two cases, (1.7) has infinitely many solutions, in the last case it has no solution.

Two linear equations with two unknowns

The general form of one linear equation with two unknowns is

$$\begin{aligned} &① \quad Ax + By = U \\ &② \quad Cx + Dy = V. \end{aligned} \tag{1.8}$$

We are using the letters A, B, C, D instead of $a_{11}, a_{12}, a_{21}, a_{22}$ in order to make the calculations more readable. If we interprete the system of equations as intersection of two geometrical objects, in our case lines, we already know the there are exactly three possible types of solutions:

(i) *A point* if ① and ② describe two non-parallel lines.

(ii) *A line* if ① and ② describe the same line; or if one of the equations is a plane and the other one is a line.

(iii) *A plane* if both equations describe a plane.

(iv) *The empty set* if the two equations describe parallel but different lines; or if one of the equations has no solution.

In case (i), the system has exactly one solution, in cases (ii) and (iii) the system has infinitely many solutions and in case (iv) the system has no solution.

In summary, we have the following very important observation.

Remark 1.10. The system (1.8) has either exactly one solution or infinitely many solutions or no solution.

It is not possible to have for instance exactly 7 solutions.

Question 1.3

What is the geometric interpretation of

(i) a system of 3 linear equations for 2 unknowns?

(ii) a system of 2 linear equations for 3 unknowns?

What can be said about the structure of its solutions?

Algebraic proof of Remark 1.10. Now we want to prove the Remark 1.10 algebraically and we want to find a criterion on a, b, c, d which allows us to decide easily how many solutions there are. Let us look at the different cases.

$\boxed{\text{Case 1. } B \neq 0.}$ In this case we can solve ① for y and obtain $y = \frac{1}{B}(U - Ax)$. Inserting ② we find $Cx + \frac{D}{B}(U - Ax) = V$. If we put all terms with x on one side and all other terms on the other side, we obtain

$$②' \quad (AD - BC)x = DU - BV.$$

(i) If $\boxed{AD - BC \neq 0}$ then there is at most one solution, namely $x = \frac{DU-BV}{AD-BC}$ and consequently $y = \frac{1}{B}(U - Ax) = \frac{AV-CU}{AD-BC}$. Inserting these expressions for x and y in our system of equations, we see that they indeed solve the system (1.8), so that we have exactly one solution.

(ii) If $\boxed{AD - BC = 0}$ then equation ②' reduces to $0 = DU - BV$. This equation has either no solution (if $DU - BV \neq 0$) or it is true for every possible choice of x and y (if $DU - BV = 0$). Since ① has infinitely many solutions, it follows that the system (1.8) has either no solution or infinitely many solutions.

$\boxed{\text{Case 2. } D \neq 0.}$ This case is analogous to Case 1. In this case we can solve ② for y and obtain $y = \frac{1}{D}(V - Cx)$. Hence ① becomes $Ax + \frac{B}{D}(V - Cx) = U$. If we put all terms with x on one side and all other terms on the other side, we obtain

$$①' \quad (AD - BC)x = DU - BV$$

We have the same subcases as before:

(i) If $\boxed{AD - BC \neq 0}$ then there is exactly one solution, namely $x = \frac{DU-BV}{AD-BC}$ and consequently $y = \frac{1}{B}(U - Ax) = \frac{AV-CU}{AD-BC}$.

(ii) If $\boxed{AD - BC = 0}$ then equation ①' reduces to $0 = DU - BV$. This equation has either no solution (if $DU - BV \neq 0$) or holds for every x and y (if $DU - BV = 0$). Since ② has infinitely many solutions, it follows that the system (1.8) has either no solution or infinitely many solutions.

Case 3. $B = 0$ and $D = 0$. Observe that in this case $\boxed{AD - BC = 0}$. In this case the system (1.8) reduces to

$$Ax = U, \quad Cx = V. \tag{1.9}$$

We see that the system no longer depends on y. So, if the system (1.9) has at least one solution, then we automatically have infinitely many solutions since we may choose y freely. If the system (1.9) has no solution, then the original system (1.8) cannot have a solution either.

Note that there are no other possible cases for the coefficients. □

In summary, we proved the following theorem.

Theorem 1.11. *Let us consider the linear system*

$$\begin{array}{ll} ① & Ax + By = U \\ ② & Cx + Dy = V. \end{array} \tag{1.10}$$

(i) *The system* (1.10) *has exactly one solution if and only if* $\boxed{AD - BC \neq 0}$. *In this case, the solution is*

$$x = \frac{DU - BV}{AD - BC}, \quad y = \frac{AV - CU}{AD - BC}. \tag{1.11}$$

(ii) *The system* (1.10) *has no solution or infinitely many solutions if and only if* $\boxed{AD - BC = 0}$.

Definition 1.12. The number $\boxed{d = AD - BC}$ is called the *determinant* of the system (1.10).

In Chapter 4.1 we will generalise this concept to $n \times n$ systems for $n \geq 3$.

Remark 1.13. Let us see how the determinant connects to our geometric interpretation of the system of equations. Assume that $B \neq 0$ and $D \neq 0$. Then we can solve ① and ② for y to obtain equations for a pair of lines

$$L_1: \quad y = -\frac{A}{B}x + \frac{1}{B}U, \qquad L_2: \quad y = -\frac{C}{D}x + \frac{1}{D}V.$$

The two lines intersect in exactly one point if and only if they have different slopes, i.e., if $-\frac{A}{B} \neq -\frac{C}{D}$. After multiplication by $-BD$ we see that this is the same as $AD \neq BC$, or in other words, $AD - BC \neq 0$.
On the other hand, the lines are parallel (hence they are either equal or they have no intersection) if $-\frac{A}{B} \neq -\frac{C}{D}$. This is the case if and only if $AD = BC$, or in other words, if $AD - BC = 0$.

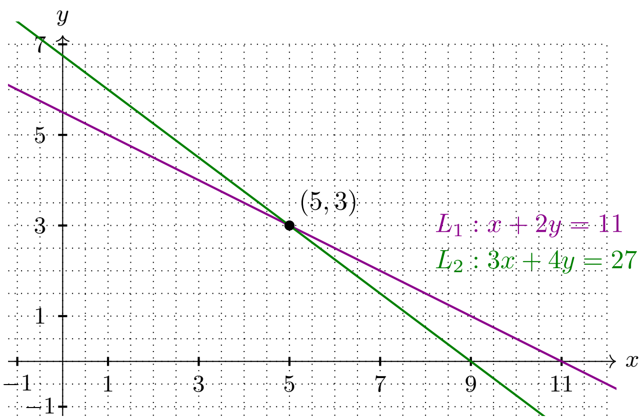


FIGURE 1.3: Example 1.14(a). Graphs of L_1, L_2 and their intersection $(5, 3)$.

Question 1.4

Consider the cases when $B = 0$ or $D = 0$ and make the connection between Theorem 1.11 and the geometric interpretation of the system of equations.

Let us consider some more examples.

Examples 1.14. (a)

$$\begin{array}{rl} ① & x + 2y = 11 \\ ② & 3x + 4y = 27. \end{array}$$

Clearly, the determinant is $d = 4 - 6 = -2 \neq 0$. So the system has *exactly one solution.*

We can check this easily: The first equation gives $x = 11 - 2y$. Inserting this into the second equations leads to

$$3(11 - 2y) + 4y = 27 \quad \Longrightarrow \quad -2y = -6 \quad \Longrightarrow \quad y = 3 \quad \Longrightarrow \quad x = 11 - 2 \cdot 3 = 5.$$

So the solution is $x = 5, y = 3$. (If we did not have Theorem 1.11, we would have to check that this is not only a candidate for a solution, but indeed is one.)

Check that the formula (1.11) is satisfied.

(b)

$$\begin{array}{rl} ① & x + 2y = 1 \\ ② & 2x + 4y = 5. \end{array}$$

Here, the determinant is $d = 4 - 4 = 0$, so we expect *either no solution or infinitely many solutions*. The first equations gives $x = 1 - 2y$. Inserting into the second equations gives $2(1 - 2y) + 4y = 5$. We see that the terms with y cancel and we obtain $2 = 5$ which is a contradiction. Therefore, the system of equations has *no solution.*

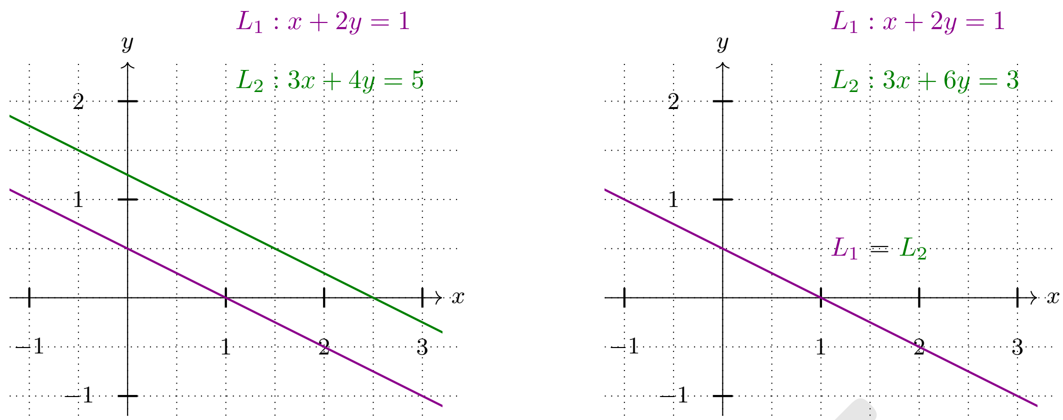


FIGURE 1.4: Picture on the left: The lines L_1, L_2 from Example 1.14(b) are parallel and do not intersect. Therefore the linear system has no solution.
Picture on the right: The lines L_1, L_2 from Example 1.14(c) are equal. Therefore the linear system has infinitely many solutions.

(c)
$$\begin{aligned} &① \quad x + 2y = 1 \\ &② \quad 3x + 6y = 3. \end{aligned}$$

The determinant is $d = 6 - 6 = 0$, so again we expect *either no solution or infinitely many solutions*. The first equations gives $x = 1 - 2y$. Inserting into the second equations gives $3(1 - 2y) + 6y = 3$. We see that the terms with y cancel and we obtain $3 = 3$ which is true. Therefore, the system of equations has *infinitely many solutions* given by $x = 1 - 2y$.

Remark. This was somewhat clear since we can obtain the second equation from the first one by multiplying both sides by 3 which shows that both equations carry the same information and we loose nothing if we simply forget about one of them.

Exercise 1.15. Find all $k \in \mathbb{R}$ such that the system

$$\begin{aligned} &① \quad kx + (15/2 - k)y = 1 \\ &② \quad 4x + \qquad\quad 2ky = 3 \end{aligned}$$

has exactly one solution.

Solution. We only need to calculate the determinant and find all k such that it is different from zero. So let us start by calculating

$$d = k \cdot 2k - (15/2 - k) \cdot 4 = 2k^2 + 4k - 30 = 2(k^2 + 2k - 15) = 2[(k + 1)^2 - 16].$$

Hence there are exactly two values for k where $d = 0$, namely $k = -1 \pm 4$, that is $k_1 = 3$, $k_2 = -5$. For all other k, we have that $d \neq 0$.
So the answer is: The system has exactly one solution if and only if $k \in \mathbb{R} \setminus \{-5, 3\}$. $\diamond$

Remark 1.16. (a) Note that the answer does not depend on the right hand side of the system of the equation. Only the coefficients on the left hand side determine if there is exactly one solution or not.

(b) If we wanted to, we could also calculate the solution x, y in the case $k \in \mathbb{R} \setminus \{-5, 3\}$. We could do it by hand or use (1.11). Either way, we find

$$x = \frac{1}{d}[2k - 3(15/2 - k)] = \frac{5k - 45/2}{2k^2 + 4k - 30}, \qquad y = \frac{1}{d}[6k - 4] = \frac{6k - 4}{2k^2 + 4k - 30}.$$

Note that the denominators are equal to d and they are equal to 0 exactly for the "forbidden" values of $k = -5$ or $k = 3$.

(c) What happens if $k = -5$ or $k = 3$? In both cases, $d = 0$, so we will either have no solution or infinitely many solutions.

If $k = -5$, then the system becomes $-5x + 25/2y = 1, \quad 4x - 10y = 3$.
Multiplying the first equation by $-4/5$ and not changing the second equation, we obtain

$$4x - 10y = -\frac{4}{5}, \qquad 4x - 10y = 3$$

which clearly cannot be satisfied simultaneously.

If $k = 3$, then the system becomes $3x - 9/2y = 1, \quad 4x + 6y = 3$.
Multiplying the first equation by $4/3$ and not changing the second equation, we obtain

$$4x - 6y = \frac{4}{3}, \qquad 4x - 6y = 3$$

which clearly cannot be satisfied simultaneously.

In conclusion, if $k = -5$ or $k = 3$, then the linear system has no solution.

You should have understood

- the geometric interpretation of a linear $m \times 2$ system and how it helps to understand the qualitative structure of solutions,
- how the determinant helps to decide whether a linear 2×2 system has a unique solution or not,
- that it depends only on the coefficients of the system if its solution is unique; it does *not* depend on the right side of the equation (the actual values of the solutions of course do depend on the right side of the equation),
- ...

You should now be able to

- pass easily from a linear $m \times 2$ system to its geometric interpretation and back,
- calculate the determinant of a linear 2×2 system,

- determine if a linear 2×2 system has a unique, no or infinitely many solutions and calculate them,
- give criteria for existence/uniqueness of solutions,
- ...

1.3 Summary

A *linear system* is a system of equations

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ \cdots + a_{1n}x_n &= \ b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \ \cdots + a_{2n}x_n &= \ b_2 \\ \vdots \qquad\qquad\qquad \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $x_1, \dots, x_n$ are the unknowns and the numbers a_{ij} and b_i ($i = 1, \dots, m$, $j = 1, \dots, n$) are given. The numbers a_{ij} are called the *coefficients of the linear system* and the numbers $b_1, \dots, b_n$ are called the *right side of the linear system*.

In the special case when all b_i are equal to 0, the system is called a *homogeneous*; otherwise it is called *inhomogeneous*.

The *coefficient matrix* A and the *augmented coefficient matrix* $(A|b)$ of the system is are

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad (A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{array}\right).$$

The general form of linear 2×2 system is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{1.12}$$

and its *determinant* is

$$d = a_{11}a_{22} - a_{21}a_{12}.$$

The determinant tells us if the system (1.12) has a unique solution:

- If $d \neq 0$, then (1.12) has a unique solution.
- If $d = 0$, then (1.12) has either no or infinitely many solutions (it depends on b_1 and b_2 which case prevails).

Observe that d does **not** depend on the right hand side of the linear system.

1.4 Exercises

1. Para los siguientes sistemas, si es posible,

   (i) escribe la matriz de coeficientes y la matriz aumentada,

   (ii) calcule el determinante y concluya sobre existencia y unicidad de soluciones,

   (iii) encuentre todas las soluciones,

   (iv) haga un dibujo.

   (a) $-3x + 2y = 18, \quad x + 2y = 2.$

   (b) $2x + 8y = 6, \quad 3x + 12y = 2.$

   (c) $2x - 4y = 6, \quad -x + 2y = -1.$

   (d) $3x - 2y = -1, \quad x + 3y = 18, \quad 2x - 5y = -8.$

   (e) $x - y = 5, \quad -3x + 2y = 3, \quad 2x + 3y = 14.$

2. Encuentre todas las soluciones de los siguientes sistemas y visualice las ecuaciones y las soluciones en el plano.

   (a) $3x + 5y = 7, \quad -9x - 15y = 10,$

   (b) $2x + 5y = 10, \quad x + 2y + 3 = 0,$

   (c) $2x + y = 4, \quad 3x - 2y = -1, \quad 5x + 3y = 7,$

   (d) $x + 5y = 3, \quad -3x + 2y = 8, \quad 2x + 3y = -1.$

3. (a) Encuentre todos los números k tal que el siguiente sistema de ecuaciones tiene exactamente una solución y calcule esta solución. ¿Qué pasa para los otros k?

$$kx + 5y = 0, \qquad 3x + (2 + k)y = 0.$$

   (b) Haga lo mismo para el sistema

$$kx + 5y = 5, \qquad 3x + (2 + k)y = -3.$$

4. (a) Encuentre todos los números k tal que e. siguiente sistema de ecuaciones tiene exactamente una solución y calcule esta solución. ¿Qué pasa para los otros k?

$$kx + 2y = 0, \qquad 2x - (3 + k)y = 0.$$

   (b) Haga lo mismo para el sistema

$$kx + 2y = 6, \qquad 2x - (3 + k)y = -3.$$

5. (a) Encuentre un polinomio P de grado 3 con

$$P(1) = 2, \quad P(-1) = 6, \quad P'(1) = 8, \quad P(0) + 4P'(0) = 0.$$

(b) ¿Existe un polinomio de grado 2 que satisface lo de arriba? De ser así, ¿cuántos hay? Justifique su respuesta.

(c) ¿Existe un polinomio de grado 4 que satisface lo de arriba De ser así, ¿cuántos hay? Justifique su respuesta.

6. En una bodega hay soluciones de un cierto químico con concentraciones de 1% y de 13%. ¿Cuántos mililitros de cada una de las soluciónes disponibles se requieren para obtener 500 ml de una solucón de este químico con contentración de 5%?

7. Se puede mostrar que toda solución de la ecuación diferencial $y'' + y = \mathrm{e}^x$ es de la forma

$$y(x) = A\cos x + B\sin x + \frac{1}{2}\,\mathrm{e}^x \tag{1.13}$$

con constantes A y B.

(a) Demuestre que (1.13) soluciona la ecuación diferencial.

(b) Encuentre una solución que satisfaga las condiciones de frontera $y(0) = 0$, $y(1) = 0$.

(c) Encuentre una solución que satisfaga las condiciones iniciales $y(0) = 0$, $y'(0) = 1$.

8. Considere la ecuación

$$3x + 4y = 5. \tag{1.14}$$

(a) ¿Existe otra ecuación lineal tal que la solución del sistema de (1.14) y la nueva ecuación es $(3, -1)$? Encuentre tal ecuación o diga por qué no existe.

(b) ¿Existen otras dos ecuaciones lineales tal que la solución del sistema de (1.14) y las nuevas ecuaciones es $(3, -1)$? Encuentre tales ecuaciones o diga por qué no existen.

(c) ¿Existe otra ecuación lineal tal que la solución del sistema de (1.14) y la nueva ecuación es $(2, -3)$? Encuentre tal ecuación o diga por qué no existe.

(d) ¿Existen otras dos ecuaciones lineales tal que la solución del sistema de (1.14) y las nuevas ecuaciones es $(2, -3)$? Encuentre tales ecuaciones o diga por qué no existen.

(e) Encuentre otra ecuación lineal tal que el sistema de (1.14) y la nueva ecuación no tenga solución.

(f) Encuentre otra ecuación lineal tal que el sistema de (1.14) y la nueva ecuación tenga infinitas soluciones.

Chapter 2

$\mathbb{R}^2$ and $\mathbb{R}^3$

In this chapter we will introduce the vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$. We will define algebraic operations in these spaces and interpret them geometrically. Then we will add some additional structure to these spaces, namely an inner product. It allows to assign a norm (length) to a vector and talk about the angle between two vectors; in particular, it gives us the concept of orthogonality. In Section 2.3 we will define orthogonal projections in $\mathbb{R}^2$ and we will give a formula for the orthogonal projection of a vector onto another. This formula is easily generalised to projections onto a vector in $\mathbb{R}^n$ with $n \geq 3$. Section 2.5 is dedicated to the special and very important case $\mathbb{R}^3$ since this is the space that physicists use in classical mechanics to describe our world. In the last two sections we study lines and planes in $\mathbb{R}^n$ and in $\mathbb{R}^3$. We will see how we can describe them in formulas and we will learn how to calculate their intersections. This naturally leads to the question on how to solve linear systems efficiently which will be addressed in the next chapter.

2.1 Vectors in $\mathbb{R}^2$

Recall that the xy-plane is the set of all pairs (x, y) with $x, y \in \mathbb{R}$. We will denote it by $\mathbb{R}^2$.
Maybe you already encountered vectors in a physics lecture. For instance velocities and forces are described by vectors. The velocity of a particle says how fast it is and in which direction the particle moves. Usually, the velocity is represented by an arrow which points in the direction in which the particle moves and whose length is proportional to the magnitude of the velocity.
Similarly, a force has strength and a direction so it is represented by an arrow which points in the direction in which it acts and with length proportional to its strength.
Observe that it is not important where in the space $\mathbb{R}^2$ or $\mathbb{R}^3$ we put the arrow. As long it points in the same direction and has the same length, it is considered the same vector. We call two arrows *equivalent* if they have the same direction and the same length. A *vector* is the set of all arrows which are equivalent to a given arrow. Each specific arrow in this set is called a *representation* of the vector. A special representation is the arrow that starts in the origin $(0, 0)$. Vectors are usually denoted by a small letter with an arrow on top, for example $\vec{v}$.

Given two points P, Q in the xy-plane, we write $\overrightarrow{PQ}$ for the vector which is represented by the arrow that starts in P and ends in Q. For example, let $P(1,1)$ and $Q(3,4)$ be points in the xy-plane. Then the arrow from P to Q is $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

We can identify a point $P(p_1, p_2)$ in the xy-plane with the vector starting in $(0,0)$ and ending in P. We denote this vector by $\overrightarrow{OP}$ or $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ or sometimes by $(p_1,\, p_2)^t$ in order to save space (the subscript ^{t} stands for "transposed"). p_1 is called the *x-coordinate* or the *x-component* of $\vec{v}$ and p_2 is called the *y-coordinate* or the *y-component* of $\vec{v}$.

On the other hand, every vector $\begin{pmatrix} a \\ b \end{pmatrix}$ describes a unique point in the xy-plane, namely the tip of the arrow which represents the given vector and starts in the origin. Clearly its coordinates are (a, b). Therefore we can identify the set of all vectors in $\mathbb{R}^2$ with $\mathbb{R}^2$ itself.

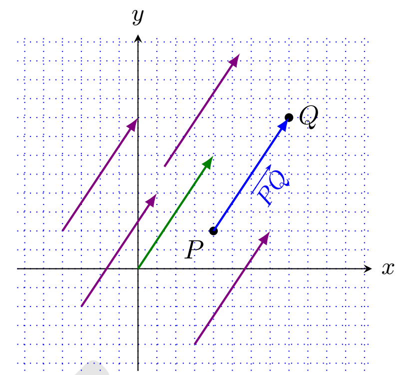


FIGURE 2.1: The vector $\vec{v}$ and several of its representations. The green arrow is the special representation whose initial point is in the origin.

Observe that the slope of the arrow $\vec{v} = (a, b)$ is $\frac{b}{a}$ if $a \neq 0$. If $a = 0$, then the vector is parallel to the y-axis.

For example, the vector $\vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, can be represented as an arrow whose initial point is in the origin and its tip is at the point $(2, 5)$. If we put its initial point anywhere else, then we find the tip by moving 2 units to the right (parallel to the x-axis) and 5 units up (parallel to the y-axis).

A very special vector is the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Is is usually denoted by $\vec{0}$.

We call numbers in $\mathbb{R}$ *scalars* in order to distinguish them from vectors.

Algebra with vectors

If we think of a force and we double its strength then the corresponding vector should be twice as long. If we *multiply* the force by 5, then the length of the corresponding vector should be 5 times as long, that is, if for instance a force $\vec{F} = (3, 4)$ is given, then $5\vec{F}$ should be $(5\cdot 3,\ 5\cdot 4) = (15, 20)$.

In general, if a vector $\vec{v} = (a, b)$ and a scalar c are given, then $c\vec{v} = (ca, cb)$. Note that the resulting vector is always parallel to the original one. If $c > 0$, then the resulting vector points in the same direction as the original one, if $c < 0$, then it points in the opposite direction, see Figure 2.2.

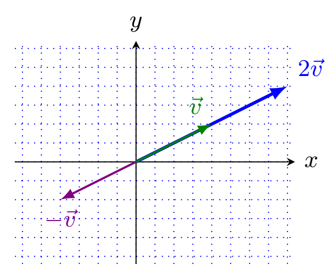


FIGURE 2.2: Multiplication of a vector by a scalar.

Given two points $P(p_1, p_2)$, $Q(q_1, q_2)$ in the xy-plane. Convince yourself that $\overrightarrow{PQ} = -\overrightarrow{QP}$.

How should we *sum* two vectors? Again, let us think of forces. Assume we have two forces $\vec{F}_1$ and $\vec{F}_2$ both acting on the same particle. Then we get the resulting force if we draw the arrow representing $\vec{F}_1$ and attach to its end point the initial point of the arrow representing $\vec{F}_2$. The total force is then represented by the arrow starting in the initial point of $\vec{F}_1$ and ending in the tip of $\vec{F}_2$.

Convince yourself that we obtain the same result if we start with $\vec{F}_2$ and put the initial point of $\vec{F}_1$ at the tip of $\vec{F}_2$.

We could also think of the sum of velocities. For example, if the have a train with velocity $\vec{v}_t$ and on the train a passenger is moving with relative velocity $\vec{v}_p$, then the total velocity is the vector sum of the two.

Now assume that $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} p \\ q \end{pmatrix}$. Algebraically, we obtain the components of their sum by summing the components: $\vec{v} + \vec{w} = \begin{pmatrix} a+p \\ b+q \end{pmatrix}$, see Figure 2.3.

When you sum vector, you should always think of triangles (or polygons if you sum more than two vectors).

Given two points $P(p_1, p_2)$, $Q(q_1, q_2)$ in the xy-plane. Convince yourself that $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ and consequently $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$.
How could you write $\overrightarrow{QP}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$? What is its relation with $\overrightarrow{PQ}$?

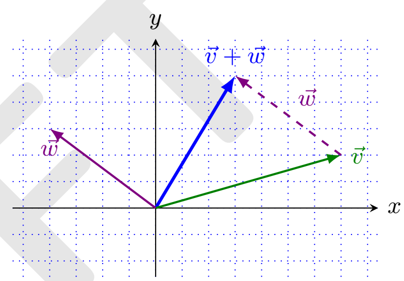


FIGURE 2.3: Sum of two vectors.

Our discussion of how the product of a vector and a scalar and how the sum of two vectors should be, leads us to the following formal definition.

Definition 2.1. Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{w} = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^2$, $c \in \mathbb{R}$. Then:

Vector sum:	$\vec{v} + \vec{w} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a+p \\ b+q \end{pmatrix},$
Product with a scalar:	$c\vec{v} = c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}.$

It is easy to see that the vector sum satisfies what one expects from a sum: $\vec{(u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ (associativity) and $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ (commutativity). Moreover, we have the distributivity laws $(a+b)\vec{v} = a\vec{v} + b\vec{v}$ and $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$. Let verify for example associativity. To this end, let

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then

$$\begin{aligned}
(\vec{u}+\vec{v})+\vec{w} &= \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} (u_1+v_1)+w_1 \\ (u_2+v_2)+w_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1+(v_1+w_1) \\ u_2+(v_2+w_2) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} (v_1+w_1) \\ (v_2+w_2) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right] \\
&= \vec{u} + (\vec{v}+\vec{w}).
\end{aligned}$$

In the same fashion, verify commutativity and distributivity of the vector sum.

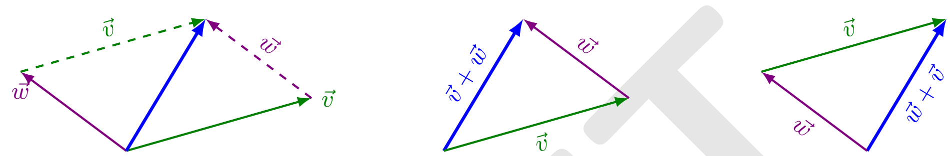


FIGURE 2.4: The picture illustrates the commutativity of the vector sum.

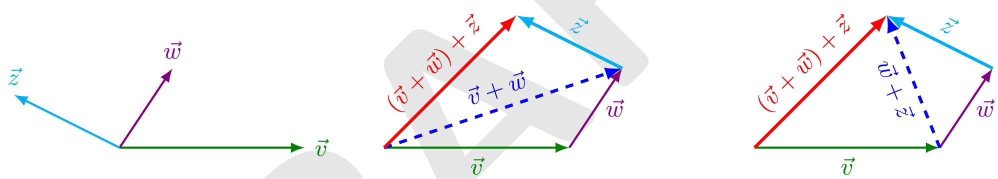


FIGURE 2.5: The picture illustrates associativity of the vector sum.

Draw pictures that illustrate the distributiviy laws.

We can take these properties and define an *abstract vector space*. We shall call a set of things, called *vectors*, with a "well-behaved" sum of its elements and a "well-behaved" product of its elements with scalars a *vector space*. The precise definition is the following.

Vector Space Axioms. Let V be a set together with two operations

$$\begin{aligned}
\text{vector sum} \quad & + : V \times V \to V, \quad (v,w) \mapsto v+w, \\
\text{product of a scalar and a vector} \quad & \cdot : \mathbb{K} \times V \to V, (\lambda, v) \mapsto \lambda \cdot v.
\end{aligned}$$

Note that we will usually write λv instead of $\lambda \cdot v$. Then V is called an $\mathbb{R}$-vector space and its elements are called *vectors* if the following holds:

(a) **Associativity:** $(u+v)+w = u+(v+w)$ for every $u,v,w \in V$.

(b) **Commutativity:** $v+w = w+v$ for every $u,v \in V$.

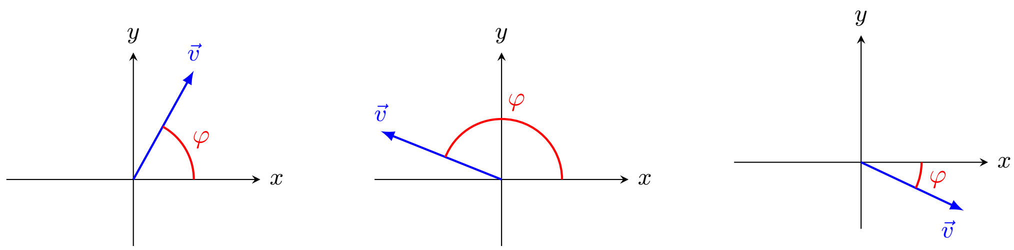


FIGURE 2.6: Angle of a vector with the x-axis.

(c) **Identity element of addition:** There exists an element $\mathbb{O} \in V$, called the *additive identity* such that for every $v \in V$, we have $\mathbb{O} + v = v + \mathbb{O} = v$.

(d) **Inverse element:** For all $v \in V$, we have an inverse element v' such that $v + v' = \mathbb{O}$.

(e) **Identity element of multiplication by scalar:** For every $v \in V$, we have that $1v = v$.

(f) **Compatibility:** For every $v \in V$ and $\lambda, \mu \in \mathbb{R}$, we have that $(\lambda\mu)v = \lambda(\mu v)$.

(g) **Distributivity laws:** For all $v, w \in V$ and $\lambda, \mu \in \mathbb{R}$, we have

$$(\lambda + \mu)v = \lambda v + \mu v \quad \text{and} \quad \lambda(v + w) = \lambda v + \lambda w.$$

These axioms are fundamental for linear algebra and we will come back to them in Chapter 5.1.

Check that $\mathbb{R}^2$ is a vector space, that its additive identity is $\mathbb{O} = \vec{0}$ and that for every vector $\vec{v} \in \mathbb{R}^2$, its additive inverse is $-\vec{v}$.

It is important to note that there are vector spaces that do not look like $\mathbb{R}^2$ and that we cannot always write vectors as columns. For instance, the set of all polynomials form a vector space (we can add them, the sum is additive and commutative; the additive identity is the zero polynomial and for every polynomial p, its additive inverse is the polynomial $-p$; we can multiply polynomials with scalars and obtain another polynomial, etc.). The vectors in this case are polynomials and it does not make sense to speak about its "components" or "coordinates". (We will however learn how to represent certain subspaces of the space of polynomials as subspaces of some $\mathbb{R}^n$ in Chapter 6.3.)

After this brief excursion about abstract vector spaces, let us return to $\mathbb{R}^2$. We know that it can be identified with the xy-plane. This means that $\mathbb{R}^2$ has more structure than only being a vector space. For example, we can measure angles and lengths. Observe that these concepts do *not* appear in the definition of a vector space. They are something in addition to the the vector space properties. Let us now look at some more geometric properties of vectors in $\mathbb{R}^2$. Clearly a vector is known if we know its length and its angle with the x-axis. From the Pythagoras theorem it is clear that the length of a vector $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $\sqrt{a^2 + b^2}$.

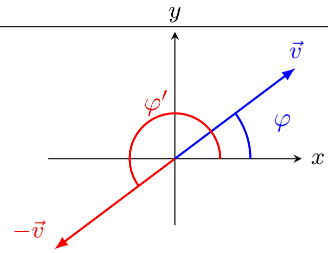


FIGURE 2.7: The angle of $\vec v$ and $-\vec v$ with the x-axis. Clearly, $\varphi' = \varphi + \pi$.

Definition 2.2 (Norm of a vector in $\mathbb{R}^2$). The *length* of $\vec v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ is denoted by $\|\vec v\|$. It is given by

$$\|\vec v\| = \sqrt{a^2 + b^2}\ .$$

Other names for the length of $\vec v$ are *magnitude of $\vec v$* or *norm of $\vec v$*.

As already mentioned earlier, the slope of vector $\vec v$ is $\frac{b}{a}$ if $a \neq 0$. If φ is the angle of the vector $\vec v$ with the x-axis then $\tan\varphi = \frac{b}{a}$ if $a \neq 0$. If $a = 0$, then $\varphi = 0$ or $\varphi = \pi$. Recall that the range of arctan is $(-\pi/2,\ \pi/2)$, so we cannot simply take arctan of the fraction $\frac{a}{b}$ in order to obtain φ. Observe that $\arctan\frac{b}{a} = \arctan\frac{-b}{-a}$, however the vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} -a \\ -b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$ are parallel but point in opposite directions, so they do *not* have the same angle with the x-axis. In fact, their angles differ by π, see Figure 2.7. From elementary geometry, we find

$$\tan\varphi = \frac{b}{a} \text{ if } \ a \neq 0 \qquad \text{and} \qquad \varphi = \begin{cases} \arctan\frac{b}{a} & \text{if } a > 0, \\ \pi - \arctan\frac{b}{a} & \text{if } a < 0, \\ \pi/2 & \text{if } a = 0,\ b > 0, \\ -\pi/2 & \text{if } a = 0,\ b < 0. \end{cases}$$

Note that this formula gives angles with values in $[-\pi/2,\ 3\pi/2)$.

Remark 2.3. In order to obtain angles with values in $(-\pi,\ \pi]$, we could use the formula

$$\varphi = \begin{cases} \arccos\frac{a}{\sqrt{a^2+b^2}} & \text{if } b > 0, \\ -\arccos\frac{a}{\sqrt{a^2+b^2}} & \text{if } b < 0, \\ \pi & \text{if } a < 0,\ b = 0. \end{cases}$$

Proposition 2.4 (Properties of the norm). *Let $\lambda \in \mathbb{R}$ and $\vec v$, $\vec w \in \mathbb{R}^2$. Then the following is true:*

(i) $\|\vec v\| = 0$ *if and only if* $\vec v = \vec 0$.

(ii) $\|\lambda\vec v\| = |\lambda|\|\vec v\|$,

(iii) $\|\vec v + \vec w\| \leq \|\vec v\| + \|\vec w\|$ *(triangle inequality)*,

Proof. Let $\vec v = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec w = \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

(i) Since $\|\vec{v}\| = \sqrt{a^2+b^2}$ it follows that $\|\vec{v}\| = 0$ if and only if $a = 0$ and $b = 0$. This is the case if and only if $\vec{v} = \vec{0}$.

(ii) $\|\lambda\vec{v}\| = \left\|\lambda\begin{pmatrix} a \\ b \end{pmatrix}\right\| = \left\|\begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}\right\| = \sqrt{(\lambda a)^2 + (\lambda b)^2} = \sqrt{\lambda^2(a^2+b^2)} = |\lambda|\sqrt{a^2+b^2} = |\lambda|\|\vec{v}\|.$

(iii) We postpone the proof of the triangle inequality to Corollary 2.20 when we already have the cosine theorem at our disposal. □

Geometrically, the triangel inequality says that in the plane the shortest way to get from one point to the other it a straight line. Figure 2.8 shows that it is shorter to go directly from the origin of the blue vector to its tip than taking a detour along $\vec{v}$ and $\vec{w}$. In other words, $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\|+\|\vec{w}\|$.

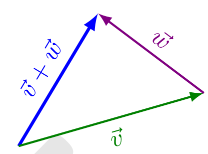


FIGURE 2.8: Triangle inequality.

Definition 2.5. A vector $\vec{v} \in \mathbb{R}^2$ is called a *unit vector* if $\|\vec{v}\| = 1$.

Note that every vector $\vec{v} \neq \vec{0}$ defines a unit vector pointing in the same direction as itself by $\|\vec{v}\|^{-1}\vec{v}$.

Remark 2.6. (i) The tip of every unit vector lies on the unit circle, and, conversely, every vector whose initial point is the origin and whose tip lies on the unit circle is a unit vector.

(ii) Every unit vector is of the form $\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$ where φ is its angle with the positive x-axis.

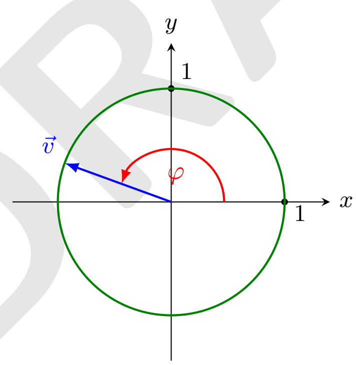


FIGURE 2.9: Unit vectors.

Finally, we define two very special unit vectors:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Clearly, $\vec{e}_1$ is parallel to the x-axis, $\vec{e}_2$ is parallel to the y-axis and $\|\vec{e}_1\| = \|\vec{e}_2\| = 1$.

Remark 2.7. Every vector $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ can be written as

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = a\vec{e}_1 + b\vec{e}_2.$$

Remark 2.8. Another notation for $\vec{e}_1$ and $\vec{e}_2$ is $\hat{\imath}$ and $\hat{\jmath}$.

You should have understood

- the concept of an abstract vector space and vectors,
- the vector space $\mathbb{R}^2$ and how to calculate with vectors in $\mathbb{R}^2$,
- the difference between a point $P(a, b)$ in $\mathbb{R}^2$ and a vector $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$,
- geometric concepts (angles, length of a vector),
- ...

You should now be able to

- perform algebraic operations in the vector space $\mathbb{R}^2$ and visualise them in the plane,
- calculate lengths and angles,
- calculate unit vectors, scale vectors,
- perform simple abstract proofs (e.g., prove that $\mathbb{R}^2$ is a vector space).
- ...

2.2 Inner product in $\mathbb{R}^2$

In this section we will explore further geometric properties of $\mathbb{R}^2$ and we will introduce the so-called inner product. Many of these properties carry over almost literally to $\mathbb{R}^3$ and more generally, to $\mathbb{R}^n$. Let us start with a definition.

Definition 2.9 (Inner product). Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$. The *inner product* of $\vec{v}$ and $\vec{w}$ is

$$\langle \vec{v}\,, \vec{w} \rangle := v_1 w_1 + v_2 w_2.$$

The inner product is also called *scalar product* or *dot product* and it can also be denoted by $\vec{v} \cdot \vec{w}$.

We usually prefer the notation $\langle \vec{v}\,, \vec{w} \rangle$ since this notation is used frequently in physics and extends naturally to abstract vector spaces with an inner product. Moreover, the the notation with the dot seems to suggest that the dot product behaves like a usual product, whereas in reality it does not, see Remark 2.12.

Before we give properties of the inner product and explore what it is good for, we first calculate a few examples to familiarise ourselves with it.

Examples 2.10.

(i) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \end{pmatrix} \right\rangle = 2 \cdot (-1) + 3 \cdot 5 = -2 + 15 = 13.$

(ii) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\rangle = 2^2 + 3^2 = 4 + 9 = 13.$ Observe that this is equal to $\left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\|^2$.

(iii) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 2, \quad \left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 3.$

(iv) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\rangle = 0.$

Proposition 2.11 (Properties of the inner product). *Let $\vec{u}$, $vecv, \vec{w} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Then the following holds.*

(i) $\langle \vec{v}\,, \vec{v} \rangle = \|\vec{v}\|^2$. *In dot notation:* $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

(ii) $\langle \vec{u}\,, \vec{v} \rangle = \langle \vec{v}\,, \vec{u} \rangle$. *In dot notation:* $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

(iii) $\langle \vec{u}\,, \vec{v} + \vec{w} \rangle = \langle \vec{u}\,, \vec{v} \rangle + \langle \vec{u}\,, \vec{w} \rangle$. *In dot notation:* $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

(iv) $\langle \lambda\vec{u}\,, \vec{v} \rangle = \lambda \langle \vec{u}\,, \vec{v} \rangle$. *In dot notation:* $(\lambda\vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v})$.

Proof. Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

(i) $\langle \vec{v}\,, \vec{v} \rangle = v_1^2 + v_2^2 = \|\vec{v}\|^2$.

(ii) $\langle \vec{u}\,, \vec{v} \rangle = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \langle \vec{v}\,, \vec{u} \rangle$.

(iii)
$$\begin{aligned} \langle \vec{u}\,, \vec{v} + \vec{w} \rangle &= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \right\rangle \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) = u_1v_1 + u_2v_2 + u_1w_1 + u_2w_2 \\ &= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle = \langle \vec{u}\,, \vec{v} \rangle + \langle \vec{u}\,, \vec{w} \rangle. \end{aligned}$$

(iv) $\langle \lambda\vec{u}\,, \vec{v} \rangle = \left\langle \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \lambda u_1v_1 + \lambda u_2v_2 = \lambda(u_1v_1 + u_2v_2) = \lambda\langle \vec{u}\,, \vec{v} \rangle.$ $\square$

Remark 2.12. Observe that the proposition shows that the inner product is commutative and distributive, so it has some properties of the "usual product" that we are used to from the product in $\mathbb{R}$ or $\mathbb{C}$, but there are some properties that show that the inner product is **not** a product.

(a) The inner products takes two vectors and gives back a number, so it gives back an object that is **not** of the same type as the two things we put in.

(b) In Example 2.10(iv) we saw that it may happen that $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$ but still $\langle \vec{v}\,, \vec{w} \rangle = 0$ which is impossible for a "decent" product.

(c) Given a vector $\vec{v} \neq 0$ and a number $c \in \mathbb{R}$, there are many solutions of the equation $\langle \vec{v}\,, \vec{x} \rangle = c$ for the vector $\vec{x}$, in stark contrast to the usual product in $\mathbb{R}$ or $\mathbb{C}$. Look for instance at Example 2.10(i) and (ii). Therefore it makes no sense to write something like $\vec{v}^{-1}$.

(d) There is no such thing as a neutral element for scalar multiplication.

Now let us see what the inner product is good for. We will see that the inner product between two vectors is connected to the angle between them and it will help us to define orthogonal projections of one vector onto another. Let us start with a definition.

Definition 2.13. Let $\vec{v}, \vec{w}$ be vectors in $\mathbb{R}^2$. The *angle between* $\vec{v}$ *and* $\vec{w}$ is the smallest nonnegative angle between them, see Figure 2.10. It is denoted by $\sphericalangle(\vec{v}, \vec{w})$.

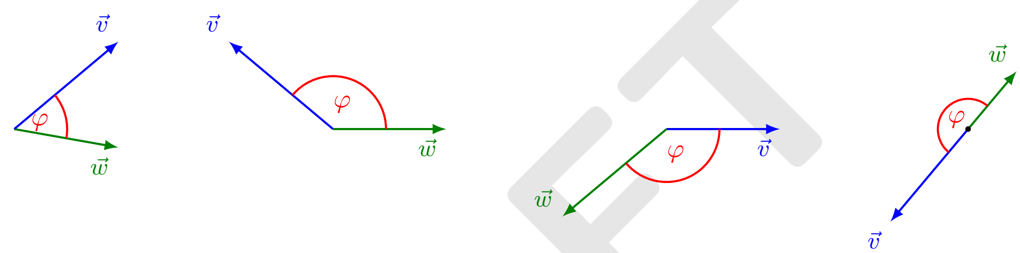


FIGURE 2.10: Angle between two vectors.

The following properties of the angle are easy to see.

Proposition 2.14. (i) $\sphericalangle(\vec{v}, \vec{w}) \in [0, \pi]$ *and* $\sphericalangle(\vec{v}, \vec{w}) = \sphericalangle(\vec{w}, \vec{v})$.

(ii) *If* $\lambda > 0$, *then* $\sphericalangle(\lambda\vec{v}, \vec{w}) = \sphericalangle(\vec{v}, \vec{w})$.

(iii) *If* $\lambda < 0$, *then* $\sphericalangle(\lambda\vec{v}, \vec{w}) = \pi - \sphericalangle(\vec{v}, \vec{w})$.

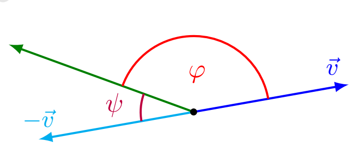


FIGURE 2.11: Angle between the vector $\vec{w}$ and the vectors $\vec{v}$ and $-\vec{v}$. $\varphi = \sphericalangle(\vec{w}, \vec{v})$, $\psi = \sphericalangle(\vec{w}, -\vec{v}) = \pi - \sphericalangle(\vec{w}, \vec{v}) = \pi - \varphi$.

Definition 2.15. (a) Two non-zero vectors $\vec{v}$ and $\vec{w}$ are called *parallel* if $\sphericalangle(\vec{v}, \vec{w}) = 0$ or π. In this case we use the notation $\vec{v} \parallel \vec{w}$. In addition, the vector $\vec{0}$ is parallel to every vector.

(b) Two non-zero vectors $\vec{v}$ and $\vec{w}$ are called *orthogonal* (or *perpendicular*) if $\sphericalangle(\vec{v}, \vec{w}) = \pi/2$. In this case we use the notation $\vec{v} \perp \vec{w}$. In addition, the vector $\vec{0}$ is perpendicular to every vector.

The following properties should be intuitively clear from geometry. A formal proof of (ii) and (iii) can be given easily after Corollary 2.20. The proof of (i) will be given after Remark 2.24.

Proposition 2.16. *Let $\vec{v}, \vec{w}$ be vectors in $\mathbb{R}^2$. Then:*

(i) *If $\vec{v} \parallel \vec{w}$ and $\vec{w} \neq \vec{0}$, then there exists $\lambda \in \mathbb{R}$ such that $\vec{v} = \lambda\vec{w}$.*

(ii) *If $\vec{v} \parallel \vec{w}$ and $\lambda, \mu \in \mathbb{R}$, then also $\lambda\vec{v} \parallel \mu\vec{w}$.*

(iii) *If $\vec{v} \perp \vec{w}$ and $\lambda, \mu \in \mathbb{R}$, then also $\lambda\vec{v} \perp \mu\vec{w}$.*

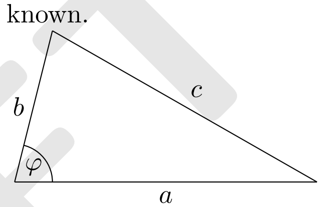
Remark 2.17. (i) Observe that ((i)) is wrong if we do not assume that $\vec{v} \neq \vec{0}$ because if $\vec{v} = \vec{0}$, then it is parallel to every vector $\vec{w}$ in $\mathbb{R}^2$, but there is no $\lambda \in \mathbb{R}$ such that $\lambda\vec{v}$ could ever become different from $\vec{0}$.

(ii) Observe that the reverse direction in ((ii)) is true only if $\lambda \neq 0$ and $\mu \neq 0$.

Without proof, we state the following theorem which should be known.

Theorem 2.18 (Cosine Theorem). *Let a, b, c be the sides or a triangle and let φ be the angle between the sides a and b. Then*

$$c^2 = a^2 + b^2 - 2ab\cos\varphi. \tag{2.1}$$



Theorem 2.19. *Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ and let $\varphi = \sphericalangle(\vec{v}, \vec{w})$. Then*

$$\boxed{\langle \vec{v}\,, \vec{w} \rangle = \|\vec{v}\|\|\vec{w}\| \cos\varphi.}$$

Proof.

The vectors $\vec{v}$ and $\vec{w}$ define a triangle in $\mathbb{R}^2$, see Figure 2.12. Now we apply the cosine theorem with $a = \|\vec{v}\|$, $b = \|\vec{w}\|$, $c = \|\vec{v} - w\|$. We obtain

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\varphi. \tag{2.2}$$

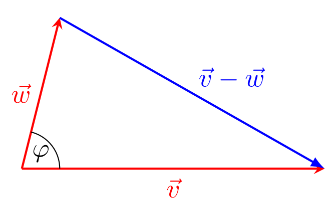


FIGURE 2.12: Triangle given by $\vec{v}$ and $\vec{w}$.

On the other hand,

$$\begin{aligned} \|\vec{v} - \vec{w}\|^2 &= \langle \vec{v} - \vec{w}\,, \vec{v} - \vec{w} \rangle = \langle \vec{v}\,, \vec{v} \rangle - \langle \vec{v}\,, \vec{w} \rangle - \langle \vec{w}\,, \vec{v} \rangle + \langle \vec{w}\,, \vec{w} \rangle = \langle \vec{v}\,, \vec{v} \rangle - 2\langle \vec{v}\,, \vec{w} \rangle + \langle \vec{w}\,, \vec{w} \rangle \\ &= \|\vec{v}\|^2 - 2\langle \vec{v}\,, \vec{w} \rangle + \|\vec{w}\|^2. \end{aligned} \tag{2.3}$$

Comparison of (2.2) and (2.3) shows that

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\varphi = \|\vec{v}\|^2 - 2\langle \vec{v}\,, \vec{w} \rangle + \|\vec{w}\|^2,$$

which gives the claimed formula. □

A very important consequence of this theorem is that we can now determine if two vectors are parallel or perpendicular to each other by simply calculating their inner product as can be seen from the following corollary.

Corollary 2.20. *Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ and $\varphi = \sphericalangle(\vec{v}, \vec{w})$. Then:*

(i) $\vec{v} \parallel \vec{w} \quad \iff \quad \|\vec{v}\|\,\|\vec{w}\| = |\langle \vec{v}\,, \vec{w} \rangle|$.

(ii) $\vec{v} \perp \vec{w} \quad \iff \quad \langle \vec{v}\,, \vec{w} \rangle = 0,$

(iii) *Cauchy-Schwarz inequality:* $|\langle \vec{v}\,, \vec{w} \rangle| \leq \|\vec{v}\|\,\|\vec{w}\|$.

(iv) *Triangle inequality:*

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|. \tag{2.4}$$

Proof. The claims are clear if one of the vectors is equal to $\vec{0}$ since the zero vector is parallel and orthogonal to every vector in $\mathbb{R}^2$. So let us assume now that $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$.

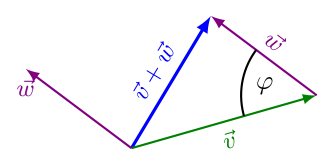
(i) From Theorem 2.19 we have that $|\langle \vec{v}\,, \vec{w} \rangle| = \|\vec{v}\|\,\|\vec{w}\|$ if and only if $\cos\varphi = 1$. This is the case if and only if $\varphi = 0$ or π, that is, if and only if $\vec{v}$ and $\vec{w}$ are parallel.

(ii) From Theorem 2.19 we have that $|\langle \vec{v}\,, \vec{w} \rangle| = 0$ if and only if $\cos\varphi = 0$. This is the case if and only if $\varphi = \pi/2$, that is, if and only if $\vec{v}$ and $\vec{w}$ are perpendicular.

(iii) By Theorem 2.19 we have that $|\langle \vec{v}\,, \vec{w} \rangle| = \|\vec{v}\|\,\|\vec{w}\| \cos\varphi \leq \|\vec{v}\|\,\|\vec{w}\|$ since $0 \leq \cos\varphi \leq 1$ for $\varphi \in [0, \pi]$.

(iv) Consider the triangle whose sides are $\vec{v}$, $\vec{w}$ and $\vec{v} + \vec{w}$ and let φ be the angle opposite to the side $\vec{v}+\vec{w}$ (hence $\varphi = \pi - \sphericalangle(\vec{v}, \vec{w})$). The cosine theorem gives

$$\begin{aligned} \|\vec{v} + \vec{w}\|^2 &= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\,\vec{w}\| \cos\varphi \\ &\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\,\vec{w}\| \\ &= (\|\vec{v}\| + \|\vec{w}\|)^2. \end{aligned}$$



Taking the square root on both sides gives us the desired inequality. □

Question 2.1

When does equality hold in the triangle inequality (2.4)? Draw a picture and prove your claim using the calculations in the proof of (iv).

Exercise. Prove (ii) and (iii) of Proposition 2.16 using Corollary 2.20.

Exercise. (i) Prove Corollary 2.20 (iii) without the cosine theorem.

Hint. Start with the inequality $0 \leq \Big\| \|\vec{w}\|\vec{v} - \|\vec{v}\|\vec{w} \Big\|^2$ and expand the right hand side similar as in the proof of Proposition 8.6. You will find that $0 \leq 2\|\vec{w}\|^2\|\vec{v}\|^2 - 2(\langle \vec{v}\,, \vec{w} \rangle)^2$.

(ii) Prove Corollary 2.20 (iv) without the cosine theorem.

Hint. Cf. the proof of the triangle inequalit in $\mathbb{C}^n$ (Proposition 8.6).

We give a proof of (iii) and (iii) in Proposition 8.6 without the use of the cosine theorem which works also in the complex case.

Example 2.21. Theorem 2.19 allows us to calculate the angle of a given vector with the x-axis easily (see Figure 2.13):

$$\cos\varphi_x = \frac{\langle \vec{v}\,, \vec{\mathrm{e}}_1\rangle}{\|\vec{v}\|\|\vec{e}_1\|}, \qquad \cos\varphi_y = \frac{\langle \vec{v}\,, \vec{\mathrm{e}}_2\rangle}{\|\vec{v}\|\|\vec{e}_2\|}.$$

If we now use that $\|\vec{e}_1\| = \|\vec{e}_2\| = 1$ and that $\langle \vec{v}\,, \vec{e}_1\rangle = v_1$ and $\langle \vec{v}\,, \vec{\mathrm{e}}_2\rangle = v_2$, then we can simplify the expressions to

$$\cos\varphi_x = \frac{v_1}{\|\vec{v}\|}, \qquad \cos\varphi_y = \frac{v_2}{\|\vec{v}\|}.$$

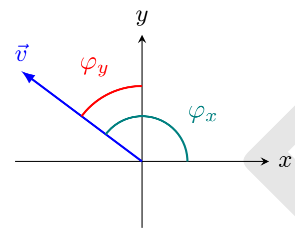


FIGURE 2.13: Angle of $\vec{v}$ with the axes.

You should have understood

- the concepts of being parallel and of being perpendicular,
- the relation of the inner product with the length of a vector and the angle between two vectors,
- that the inner product is commutative and associative, but that it is not a product,
- ...

You should now be able to

- calculate the inner product of two vectors,
- use the inner product to calculate angles between vectors
- use the inner product to determine if two vectors are parallel, perpendicular or neither,
- ...

2.3 Orthogonal Projections in $\mathbb{R}^2$

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ and $\vec{w} \neq \vec{0}$. Geometrically, we have an intuition of what the orthogonal projection of $\vec{v}$ onto $\vec{w}$ should be and that we should be able to construct it as described in the following procedure: We move $\vec{v}$ such that its initial point coincides with that of $\vec{w}$. Then we extend $\vec{w}$ to a line and construct a line that passes through the tip of $\vec{v}$ and is perpendicular to $\vec{w}$. The vector from the initial point to the intersection of the two lines should then be the *orthogonal projection of* $\vec{v}$ *onto* $\vec{w}$. see Figure 2.14

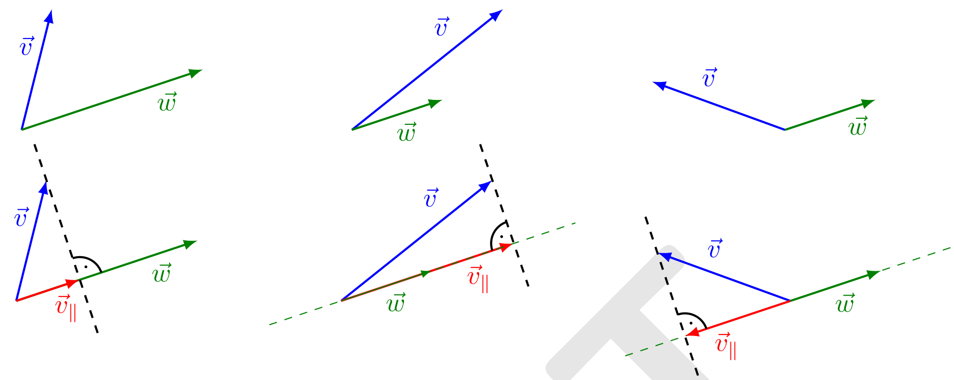


FIGURE 2.14: Some examples for the orthogonal projection of $\vec{v}$ onto $\vec{w}$ in $\mathbb{R}^2$.

We decompose the vector $\vec{v}$ in a part parallel to $\vec{w}$ and a part perpendicular to $\vec{w}$ so that their sum gives us back $\vec{v}$. The parallel part is the orthogonal projection of $\vec{v}$ onto $\vec{w}$.
In the following theorem we give the precise meaning of the orthogonal projection, we show that a decomposition as described above always exists and we even a formula for orthogonal projection. A more general version of this theorem is Theorem 7.33.

Theorem 2.22 (Orthogonal projection). *Let* $\vec{v}$ *and* $\vec{w}$ *be vectors in* $\mathbb{R}^2$ *and* $\vec{w} \neq \vec{0}$. *Then there exist uniquely determined vectors* $\vec{v}_{\|}$ *and* $\vec{v}_{\perp}$ *(see Figure 2.15) such that*

$$\vec{v}_{\|} \parallel \vec{w}, \quad \vec{v}_{\perp} \perp \vec{w} \qquad \text{and} \qquad \vec{v} = \vec{v}_{\|} + \vec{v}_{\perp}. \tag{2.5}$$

The vector $\vec{v}_{\|}$ *is called the* orthogonal projection of $\vec{v}$ onto $\vec{w}$ *and it is given by*

$$\vec{v}_{\|} = \frac{\langle \vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2}\, \vec{w}. \tag{2.6}$$

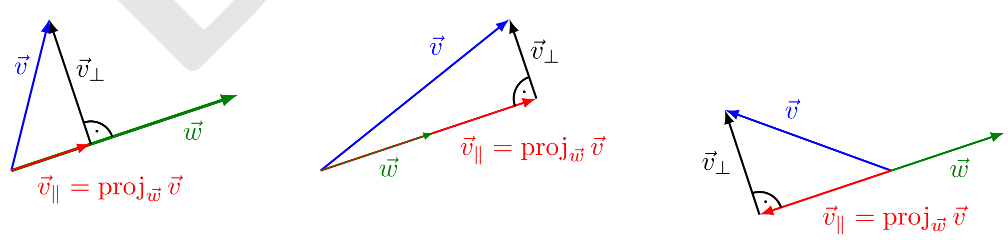


FIGURE 2.15: Decomposition of $\vec{v}$ into $\vec{v} = \vec{v}_{\|} + \vec{v}_{\perp}$ with $\vec{v}_{\|} \parallel \vec{w}$ and $\vec{v}_{\perp} \perp \vec{w}$. Note that by definition $\vec{v}_{\|} = \operatorname{proj}_{\vec{w}} \vec{v}$.

Proof. Assume we have vectors $\vec{v}_\|$ and $\vec{v}_\perp$ satisfying (2.5). Since $\vec{v}_\|$ and $\vec{w}$ are parallel by definition and since $\vec{w} \neq \vec{0}$, there exists $\lambda \in \mathbb{R}$ such that $\vec{v}_\| = \lambda\vec{w}$, so in order to find $\vec{v}_\|$ it is sufficient to determine λ. For this, we notice that $\vec{v} = \lambda\vec{w} + \vec{v}_\perp$ by (2.5). Taking the inner product on both sides with $\vec{w}$ leads to

$$\langle \vec{v}\,, \vec{w}\rangle = \langle \lambda\vec{w} + \vec{v}_\perp\,, \vec{w}\rangle = \langle \lambda\vec{w}\,, \vec{w}\rangle + \underbrace{\langle \vec{v}_\perp\,, \vec{w}\rangle}_{= 0 \text{ since } \vec{v}_\perp \perp \vec{w}} = \langle \lambda\vec{w}\,, \vec{w}\rangle = \lambda\langle \vec{w}\,, \vec{w}\rangle = \lambda\|\vec{w}\|^2$$

$$\implies \qquad \lambda = \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}.$$

So if a sum representation of $\vec{v}$ as in (2.5) exists, then the only possibility is

$$\vec{v}_\| = \lambda\vec{w} = \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w} \qquad \text{and} \qquad \vec{v}_\perp = \vec{v} - \vec{v}_\| = \vec{v} - \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w}.$$

This already proves uniqueness of the vectors $\vec{v}_\|$ and $\vec{v}_\perp$. It remains to show that they indeed have the properties that we want. Clearly, by construction $\vec{v}_\|$ is parallel to $\vec{w}$ and $\vec{v} = \vec{v}_\| + \vec{v}_\perp$ since we defined $\vec{v}_\perp = \vec{v} - \vec{v}_\|$. It remains to verify that $\vec{v}_\perp$ is orthogonal to $\vec{w}$. This follows from

$$\langle \vec{v}_\perp\,, \vec{w}\rangle = \left\langle \vec{v} - \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w}\,, \vec{w}\right\rangle = \langle \vec{v}\,, \vec{w}\rangle - \left\langle \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w}\,, \vec{w}\right\rangle = \langle \vec{v}\,, \vec{w}\rangle - \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\langle \vec{w}\,, \vec{w}\rangle = 0$$

where in the last step we used that $\langle \vec{w}\,, \vec{w}\rangle = \|\vec{w}\|^2$. □

Notation 2.23. Instead of $\vec{v}_\|$ we often write $\operatorname{proj}_{\vec{w}} \vec{v}$, in particular when we want to emphasise onto which vector we are projecting.

Remark 2.24.

(i) $\operatorname{proj}_{\vec{w}} \vec{v}$ depends only on the direction of $\vec{w}$. It does **not** depend on its length.

(ii) For every $c \in \mathbb{R}$, we have that $\operatorname{proj}_{\vec{w}}(c\vec{v}) = c \operatorname{proj}_{\vec{w}} \vec{v}$.

(iii) As special cases of the above, we find $\operatorname{proj}_{\vec{w}}(-\vec{v}) = -\operatorname{proj}_{\vec{w}} \vec{v}$ and $\operatorname{proj}_{-\vec{w}} \vec{v} = \operatorname{proj}_{\vec{w}} \vec{v}$.

(iv) $\vec{v} \parallel \vec{w} \quad\implies\quad \operatorname{proj}_{\vec{w}} \vec{v} = \vec{v}$.

(v) $\vec{v} \perp \vec{w} \quad\implies\quad \operatorname{proj}_{\vec{w}} \vec{v} = \vec{0}$.

(vi) $\operatorname{proj}_{\vec{w}} \vec{v}$ is the unique vector in $\mathbb{R}^2$ such that

$$(\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}) \perp \vec{v} \quad \text{and} \quad \operatorname{proj}_{\vec{w}} \vec{v} \parallel \vec{w}.$$

Proof. (i): By our geometric intuition, this should be clear. Let us give a formal proof. Suppose we want to project $\vec{v}$ onto $c\vec{w}$ for some $c \in \mathbb{R} \setminus \{0\}$. Then

$$\operatorname{proj}_{c\vec{w}} \vec{v} = \frac{\langle \vec{v}\,, c\vec{w}\rangle}{\|c\vec{w}\|^2}(c\vec{w}) = \frac{c\langle \vec{v}\,, \vec{w}\rangle}{c^2\|\vec{w}\|^2}(c\vec{w}) = \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w} = \operatorname{proj}_{\vec{w}} \vec{v}.$$

Convince yourself graphically that it does not matter if we project $\vec{v}$ on $\vec{w}$ or on $5\vec{w}$ or on $-\frac{7}{5}\vec{w}$; only the direction of $\vec{w}$ matters, not its length.

(ii): Again, by geometric considerations, this should be clear. The corresponding calculation is

$$\operatorname{proj}_{\vec{w}}(c\vec{v}) = \frac{\langle c\vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w} = \frac{c\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w} = c\operatorname{proj}_{\vec{w}}\vec{v}.$$

(iii) follows directly from (i) and (ii).

(iv), (v) and (vi) follow from the uniqueness of the decomposisition of the vecotr $\vec{v}$ as sum of a vector parallel and a vector perpendicular to $\vec{w}$. □

Now the proof of Proposition 2.16 (i) follows easily.

Proof of Proposition 2.16 (i). We have to show that if $\vec{v} \parallel \vec{w}$ and if $\vec{w} \neq \vec{0}$, then there exists $\lambda \in \mathbb{R}$ such that $\vec{w} = \lambda\vec{v}$. From Remark **??** (iv) it follows that $\vec{v} = \operatorname{proj}_{\vec{w}}\vec{v} = \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w}$, hence the claim follows if we can choose $\lambda = \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}$. □

We end this section with some examples.

Example 2.25. Let $\vec{u} = 2\vec{e}_1 + 3\vec{e}_2$, $\vec{v} = 4\vec{e}_1 - \vec{e}_2$.

(i) $\operatorname{proj}_{\vec{e}_1}\vec{u} = \frac{\langle \vec{u}\,, \vec{e}_1\rangle}{\|\vec{e}_1\|^2}\vec{e}_1 = \frac{2}{1^2}\vec{e}_1 = 2\vec{e}_1$.

(ii) $\operatorname{proj}_{\vec{e}_2}\vec{u} = \frac{\langle \vec{u}\,, \vec{e}_2\rangle}{\|\vec{e}_2\|^2}\vec{e}_2 = \frac{3}{1^2}\vec{e}_2 = 3\vec{e}_2$.

(iii) Similarly, we can calculate $\operatorname{proj}_{\vec{e}_1}\vec{v} = 4\vec{e}_1$, $\operatorname{proj}_{\vec{e}_2}\vec{v} = -\vec{e}_2$.

(iv) $\operatorname{proj}_{\vec{u}}\vec{v} = \frac{\langle \vec{u}\,, \vec{v}\rangle}{\|\vec{u}\|^2}\vec{u} = \dfrac{\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \end{pmatrix} \right\rangle}{\|\vec{u}\|^2}\vec{u} = \frac{8-3}{2^2+3^2}\vec{u} = \frac{5}{13}\vec{u} = \frac{5}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(v) $\operatorname{proj}_{\vec{v}}\vec{u} = \frac{\langle \vec{v}\,, \vec{u}\rangle}{\|\vec{v}\|^2}\vec{v} = \dfrac{\left\langle \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\rangle}{\|\vec{u}\|^2}\vec{u} = \frac{8-3}{4^2+(-1)^2}\vec{v} = \frac{5}{17}\vec{v} = \frac{5}{17}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Example 2.26 (Angle with coordinate axes). Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$. Then $\cos\sphericalangle(\vec{v}, \vec{e}_1) = \frac{a}{\|\vec{v}\|}$, $\cos\sphericalangle(\vec{v}, \vec{e}_2) = \frac{b}{\|\vec{v}\|}$, hence

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \|\vec{v}\| \begin{pmatrix} \cos\sphericalangle(\vec{v}, \vec{e}_1) \\ \cos\sphericalangle(\vec{v}, \vec{e}_2) \end{pmatrix}$$

and

$$\begin{aligned}
\text{projection of } \vec{v} \text{ onto the } x\text{-axis} &= \operatorname{proj}_{\vec{e}_1}\vec{v} = \|\vec{v}\|\,\cos\sphericalangle(\vec{v}, \vec{e}_1)\,\vec{e}_1 = \|\vec{v}\|\,\cos\varphi_x\,\vec{e}_1, \\
\text{projection of } \vec{v} \text{ onto the } y\text{-axis} &= \operatorname{proj}_{\vec{e}_2}\vec{v} = \|\vec{v}\|\,\cos\sphericalangle(\vec{v}, \vec{e}_2)\,\vec{e}_2 = \|\vec{v}\|\,\cos\varphi_y\,\vec{e}_2.
\end{aligned}$$

Question 2.2

Let $\vec{w}$ be a vector in $\mathbb{R}^2 \setminus \{\vec{0}\}$.

(i) Can you describe geometrically all the vectors $\vec{v}$ whose projection onto $\vec{w}$ is equal to $\vec{0}$?

(ii) Can you describe geometrically all the vectors $\vec{v}$ whose projection onto $\vec{w}$ have length 2?

(iii) Can you describe geometrically all the vectors $\vec{v}$ whose projection onto $\vec{w}$ have length $3\|\vec{w}\|$?

You should have understood

- the concept of orthogonal projections in $\mathbb{R}^2$,
- why the orthogonal projection of $\vec{w}$ onto $\vec{w}$ does not depend on the length of $\vec{w}$,
- ...

You should now be able to

- calculate the projection of a given vector onto another vector,
- ...

2.4 Vectors in $\mathbb{R}^n$

In this section we extend our calculations from $\mathbb{R}^2$ to $\mathbb{R}^n$. If $n = 3$, then we obtain $\mathbb{R}^3$ which usually serves as model for our everyday physical world and which you probably already are familiar with from physics lectures. We will discuss $\mathbb{R}^3$ and some of its peculiarities in more detail in the Section 2.5.

First, let us define $\mathbb{R}^n$.

Definition 2.27. For $n \in \mathbb{N}$ we define the set

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}.$$

Again we can think of vectors as arrows. As in $\mathbb{R}^2$, we can identify every point in $\mathbb{R}^n$ with the arrow that starts in the origin of coordinate system and ends in the given point. The set of all arrows with the same length and the same direction is called a vector in $\mathbb{R}^n$. So every point $P(p_1, \dots, p_n)$ defines a vector $\vec{v} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ and vice versa. As before, we sometimes denote vectors as $(p_1, \dots, p_n)^t$ in order to save (vertical) space. The superscript t stands for "transposed".

$\mathbb{R}^n$ becomes a vector space with the operations

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ \vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}, \qquad \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,\ c\vec{v} = \begin{pmatrix} cv_1 \\ \vdots \\ cv_n \end{pmatrix}. \tag{2.7}$$

Exercise. Show that $\mathbb{R}^n$ is a vector space. That is, you have to show that the vector space axioms on page 30 hold.

As in $\mathbb{R}^2$, we can define the norm of a vector, the angle between two vectors and an inner product. Note that the definition of a the angle between two vectors is not different from the one in $\mathbb{R}^2$ since when we are given two vectors, they always lie in a common plane which we can imagine as some sort of rotated $\mathbb{R}^2$. Let us give now the formal definitions.

Definition 2.28 (Inner product; norm of a vector). For vectors $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ the *inner product* (or *scalar product* or *dot product*) is defined as

$$\langle \vec{v} \,, \vec{w} \rangle = \left\langle \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = v_1 w_1 + \cdots + v_n w_n.$$

The *length* of $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ is denoted by $\|\vec{v}\|$ and it is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}\ .$$

Other names for the length of $\vec{v}$ are *magnitude of* $\vec{v}$ or *norm of* $\vec{v}$.

As in $\mathbb{R}^2$, we have the following properties:

(i) *Symmetry of the inner product:* For all vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, we have that $\langle \vec{v} \,, \vec{w} \rangle = \langle \vec{w} \,, \vec{v} \rangle$.

(ii) *Bilinearity of the inner product:* For all vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$, we have that $\langle \vec{u} \,, \vec{v} + c\vec{w} \rangle = \langle \vec{u} \,, \vec{v} \rangle + c\langle \vec{u} \,, \vec{w} \rangle$.

(iii) *Relation of the inner product with the angle between vectors:* Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let $\varphi = \sphericalangle(\vec{v}, \vec{w})$. Then
$$\langle \vec{v} \,, \vec{w} \rangle = \|\vec{v}\| \, \|\vec{w}\| \, \cos\varphi.$$

In particular, we have (cf. Proposition 2.16):

$$\begin{array}{lllllll} \text{(a)} & \vec{v} \parallel \vec{w} & \iff & \sphericalangle(\vec{v}, \vec{w}) \in \{0, \pi\} & \iff & |\langle \vec{v} \,, \vec{w} \rangle| = \|\vec{v}\| \, \|\vec{w}\| \\ \text{(b)} & \vec{v} \perp \vec{w} & \iff & \sphericalangle(\vec{v}, \vec{w}) = \pi/2 & \iff & \langle \vec{v} \,, \vec{w} \rangle = 0. \end{array}$$

Remark 2.29. In abstract inner product spaces, the inner product is actually used to *define* orthogonality.

(iv) *Relation of the inner product with the norm:* For all vectors $\vec{v} \in \mathbb{R}^n$, we have $\|\vec{v}\|^2 = \langle \vec{v}\,, \vec{v}\rangle$.

(v) *Properties of the norm:* For all vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$, we have that $\|c\vec{v}\| = |c|\|\vec{v}\|$ and $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

(vi) *Orthogonal projections of one vector onto another:* For all vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$ the *orthogonal projection of $\vec{v}$ onto $\vec{w}$* is

$$\operatorname{proj}_{\vec{w}} \vec{v} = \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\, \vec{w}. \tag{2.8}$$

As in $\mathbb{R}^2$, we have n “special vectors” which are parallel to the coordinate axes and have norm 1:

$$\vec{\mathrm{e}}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{\mathrm{e}}_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \vec{\mathrm{e}}_n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

In the special case $n = 3$, the vectors $\vec{\mathrm{e}}_1$, $\vec{\mathrm{e}}_2$ and $\vec{\mathrm{e}}_3$ are sometimes denoted by $\hat{\imath}, \hat{\jmath}, \hat{k}$.

For a given vector $\vec{v} \neq \vec{0}$, we can now easily determine its projections onto the n coordinate axes and its angle with the coordinate axes. By (2.8), the projection onto the x_j-axis is

$$\operatorname{proj}_{\vec{e}_j} \vec{v} = v_j \vec{\mathrm{e}}_j.$$

Let φ_j be the angle between $\vec{v}$ and the x_j-axis. Then

$$\varphi_j = \sphericalangle(\vec{v}, \vec{\mathrm{e}}_j) \quad \Longrightarrow \quad \cos\varphi_x = \frac{\langle \vec{v}\,, \vec{\mathrm{e}}_j\rangle}{\|\vec{v}\|\,\|\vec{\mathrm{e}}_j\|} = \frac{v_j}{\|\vec{v}\|}.$$

From this we see that $\vec{v} = \|\vec{v}\| \begin{pmatrix} \cos\varphi_1 \\ \vdots \\ \cos\varphi_n \end{pmatrix}$. Sometimes the notation

$$\hat{v} := \frac{\vec{v}}{\|\vec{v}\|} = \|\vec{v}\| \begin{pmatrix} \cos\varphi_1 \\ \vdots \\ \cos\varphi_n \end{pmatrix}$$

is used. Clearly $\|\hat{v}\| = 1$ because $\|\hat{v}\| = \|\|\vec{v}\|^{-1}\vec{v}\| = \|\vec{v}\|^{-1}\|\vec{v}\| = 1$. Therefore $\hat{v}$ is a unit vector pointing in the same direction as the original vector $\vec{v}$.

You should have understood

- the vector space $\mathbb{R}^n$ and vectors in $\mathbb{R}^n$,
- geometric concepts (angles, length of a vector) in $\mathbb{R}^n$,
- that $\mathbb{R}^2$ from chapter 2.1 is a special case of $\mathbb{R}^n$ from this section,

- ...

You should now be able to

- perform algebraic operations in the vector space $\mathbb{R}^3$ and, in the case $n = 3$, visualise them in space,
- calculate lengths and angles,
- calculate unit vectors, scale vectors,
- perform simple abstract proofs (e.g., prove that $\mathbb{R}^n$ is a vector space).
- ...

2.5 Vectors in $\mathbb{R}^3$ and the cross product

The space $\mathbb{R}^3$ is very important since it is used in mechanics to model the space we live in. On $\mathbb{R}^3$ we can define an additional operation with vectors, the so-called *cross product*. Another name for it its *vector product*. It takes two vectors and gives back two vectors. It does have several properties which makes it look like a product, however we will see that it is **not** a product. Here is the definition.

Definition 2.30 (Cross product). Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$. Their *cross product* (or *vector product* or *wedge product*) is

$$\vec{v} \times \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} := \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}.$$

Another notation for the cross product is $\vec{v} \wedge \vec{w}$.

A way to remember this formula is as follows. Write the first and the second component of the vectors underneath themthat formally you get a column of 5 components. Then make crosses as in the sketch below, starting with the cross consisting of a line from v_2 to w_3 and then from w_2 to v_3. Each line represents a product of the corresponding components; if the line goes from top left to bottom right then it is counted positive, if it goes from top right to bottom left then it is counted negative.

$$\begin{array}{ccccc} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} & \times & \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} & = & \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix} \\ v_1 & & w_1 & & \\ v_2 & & w_2 & & \end{array}$$

The cross product is defined only in $\mathbb{R}^3$!

Before we collect some easy properties of the cross product, let us calculate a few examples.

Examples 2.31. Let $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$.

- $\vec{u} \times \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 2\cdot 7 - 3\cdot 6 \\ 3\cdot 5 - 1\cdot 7 \\ 1\cdot 6 - 2\cdot 5 \end{pmatrix} = \begin{pmatrix} 14-18 \\ 15-7 \\ 6-10 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -4 \end{pmatrix}$,

- $\vec{v} \times \vec{u} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6\cdot 3 - 7\cdot 2 \\ 7\cdot 1 - 3\cdot 5 \\ 5\cdot 2 - 6\cdot 1 \end{pmatrix} = \begin{pmatrix} 18-14 \\ 7-15 \\ 10-6 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}$,

- $\vec{v} \times \vec{e}_1 = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6\cdot 0 - 7\cdot 0 \\ 7\cdot 0 - 7\cdot 1 \\ 5\cdot 0 - 6\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \\ -6 \end{pmatrix}$,

- $\vec{v} \times \vec{v} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \times \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 6\cdot 7 - 7\cdot 6 \\ 7\cdot 5 - 5\cdot 7 \\ 5\cdot 6 - 6\cdot 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Proposition 2.32 (Properties of the cross product). *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and let $c \in \mathbb{R}$. Then:*

(i) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$.

(ii) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

(iii) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$.

(iv) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$.

(v) $\vec{u} \parallel \vec{v} \quad \Longrightarrow \quad \vec{u} \times \vec{v} = \vec{0}$. *In particular,* $\vec{v} \times \vec{v} = \vec{0}$.

(vi) $\langle \vec{u}\,, \vec{v} \times \vec{w} \rangle = \langle \vec{u} \times \vec{v}\,, \vec{w} \rangle$.

(vii) $\langle \vec{u}\,, \vec{u} \times \vec{v} \rangle = 0$ *and* $\langle \vec{v}\,, \vec{u} \times \vec{v} \rangle = 0$, *in particular*

$$\boxed{\vec{v} \perp \vec{v} \times \vec{u}, \quad \vec{u} \perp \vec{v} \times \vec{u}}$$

that means that the vector $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

Proof. The proofs of the formulas (i) to (v) are easy calculations (you should do them!).

(vi) The proof is a long but straightforward calculation:

$$\begin{aligned}
\langle \vec{u}\,, \vec{v}\times\vec{w}\rangle &= \left\langle \begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix}, \begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix} \times \begin{pmatrix} w_1\\ w_2\\ w_3\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix}, \begin{pmatrix} v_2w_3 - v_3w_2\\ v_3w_1 - w_3v_1\\ v_1w_2 - v_2w_1\end{pmatrix} \right\rangle\\
&= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)\\
&= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1\\
&= u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3\\
&= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3\\
&= \langle \vec{u}\times\vec{v}\,, \vec{w}\rangle.
\end{aligned}$$

(vii) It follows from (vi) and (v) that

$$\langle \vec{u}\,, \vec{u}\times\vec{v}\rangle = \langle \vec{u}\times\vec{u}\,, \vec{v}\rangle = \langle \vec{0}\,, \vec{v}\rangle = 0. \qquad \square$$

Note that the cross product is distributive but it is neither commutative nor associative.

Exercise. Prove the formulas in (i) – (v).

Remark. A geometric interpretation of the number $\langle \vec{u}\,, \vec{v}\times\vec{w}\rangle$ from (vi) will be given in Proposition 2.36.

Remark 2.33. The property (vii) explains why the cross product makes sense only in $\mathbb{R}^3$. Given two non-parallel vectors $\vec{v}$ and $\vec{w}$, their cross product is a vector which is orthogonal to both of them and whose length is $\|\vec{v}\|\,\|\vec{w}\|\sin\varphi$ (see Theorem 2.34; $\varphi = \sphericalangle(\vec{v},\vec{w})$) and this should define the result uniquely up to a factor ± 1. This factor has to do with the relative orientation of $\vec{v}$ and $\vec{w}$ to each other. However, if $n \neq 3$, then one of the following holds:

- If we were in $\mathbb{R}^2$, the problem is that "we do not have enough space" because then the only vector orthogonal to $\vec{v}$ and $\vec{w}$ at the same time would be the zero vector $\vec{0}$ and it would not make too much sense to define a product where the result is always $\vec{0}$.
- If we were in some $\mathbb{R}^n$ with $n \geq 4$, the problem is that "we have too much choice". We will see later in Chapter 7.3 that the orthogonal complement of the plane generated by $\vec{v}$ and $\vec{w}$ has dimension $n-2$ and every vector in the orthogonal complement is orthogonal to both $\vec{v}$ and $\vec{w}$. For example, if we take $\vec{v} = (1,0,0,0)^t$ and $\vec{w} = (0,1,0,0)^t$, then every vector of the form $\vec{a} = (0,0,x,y)^t$ is perpendicular to both $\vec{v}$ and $\vec{w}$ and it easy to find infinitely many vectors of this form which in addition have norm $\|\vec{v}\|\,\|\vec{w}\|\sin\varphi = 1$ ($\vec{a} = (0,0,\sin\vartheta, \pm\cos\vartheta)^t$ for arbitrary $\vartheta\in\mathbb{R}$ works).

Recall that for the inner product we proved the formula $\langle \vec{v}\,, \vec{w}\rangle = \|\vec{v}\|\,\|\vec{w}\|\cos\varphi$ where φ is the angle between the two vectors, see Theorem 2.19. In the next theorem we will prove a similar relation for the cross product.

Theorem 2.34. *Let $\vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ and let φ be the angle between them. Then*

$$\boxed{\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\|\sin\varphi}$$

Proof. A long, but straightforward calculations shows that $\|\vec{v}\times\vec{w}\|^2 = \|\vec{u}\|^2\|\vec{w}\|^2 - \langle\vec{v}\,,\vec{w}\rangle^2$. Now it follows from Theorem 2.19 that

$$\begin{aligned}\|\vec{v}\times\vec{w}\|^2 &= \|\vec{u}\|^2\|\vec{w}\|^2 - \langle\vec{v}\,,\vec{w}\rangle^2 = \|\vec{u}\|^2\|\vec{w}\|^2 - \|\vec{v}\|^2\|\vec{w}\|^2(\cos\varphi)^2 \\ &= \|\vec{u}\|^2\|\vec{w}\|^2(1-(\cos\varphi)^2) = \|\vec{u}\|^2\|\vec{w}\|^2(\sin\varphi)^2.\end{aligned}$$

If we take the square root on both sides, we arrive at the claimed formula. (We do not need to worry about taking the absolute value because $\varphi\in[0,\pi]$, hence $\sin\varphi\geq 0$.) $\square$

Exercise. Show that $\|\vec{v}\times\vec{w}\|^2 = \|\vec{u}\|^2\|\vec{w}\|^2 - \langle\vec{v}\,,\vec{w}\rangle^2$.

Application: Area of a parallelogram and volume of a parelellepiped

Area of a parallelogram

Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{R}^3$. Then they define a parallelogram (if the vectors are parallel or one of them is equal to $\vec{0}$, it is a *degenerate parallelogram*).

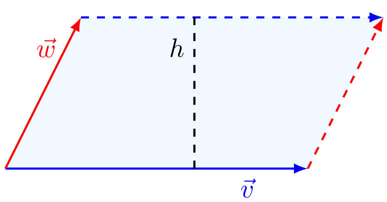


FIGURE 2.16: Parallelogram spanned by $\vec{v}$ and $\vec{w}$.

Proposition 2.35 (Area of a parallelogram). *The area of the parallelogram spanned by the vectors* $\vec{v}$ *and* $\vec{w}$ *is*

$$A = \|\vec{v}\times\vec{w}\|. \tag{2.9}$$

Proof. The area of a parallelogram is the product of the length of its base with the height. We can take $\vec{w}$ as base. Let φ be the angle between $\vec{w}$ and $\vec{v}$. Then we obtain that $h = \|\vec{v}\|\sin\varphi$ and therefore, with the help of Theorem 2.34

$$A = \|\vec{w}\|h = \|\vec{w}\|\|\vec{v}\|\sin\varphi = \|\vec{v}\times\vec{w}\|. \qquad\square$$

Note that in the case when $\vec{v}$ and $\vec{w}$ are parallel, this gives the right answer $A = 0$.

Any three vectors in $\mathbb{R}^3$ define a parallelepiped.

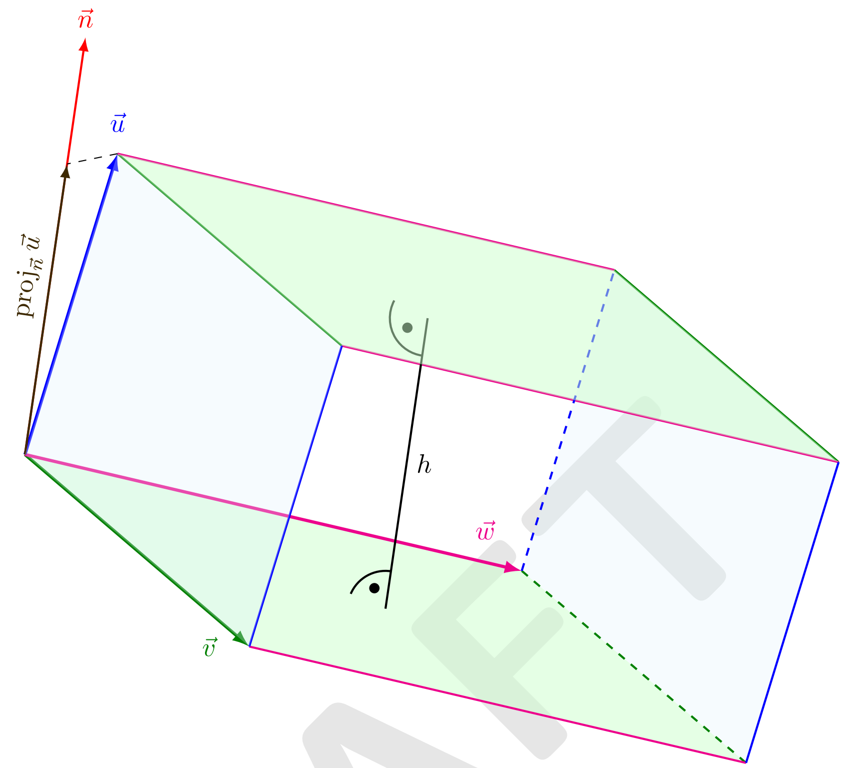


FIGURE 2.17: Parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

Proposition 2.36 (Volume of a parallelepiped). *The volume of the parallelepiped spanned by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is*

$$V = |\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|. \tag{2.10}$$

Proof. The volume of a parallelepiped is the product of the area of its base with the height. Let us take the parallelogram spanned by $\vec{v}, \vec{w}$ as base. If $\vec{v}$ and $\vec{w}$ are parallel or one or them is equal to $\vec{0}$, then (2.10) is true because $V = 0$ and $\vec{v} \times \vec{w} = \vec{0}$ in this case.
Now let us assume that they are not parallel. By Proposition 2.35 we already know that its base has area $A = \|\vec{v} \times \vec{w}\|$. The height is the length of the orthogonal projection of $\vec{u}$ onto the normal vector of the plane spanned by $\vec{v}$ and $\vec{w}$. We already know that $\vec{v} \times \vec{w}$ is such a normal vector. Hence we obtain that

$$h = \|\operatorname{proj}_{\vec{v}\times\vec{w}} \vec{u}\| = \left\| \frac{\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle}{\|\vec{v} \times \vec{w}\|^2} \vec{v} \times \vec{w} \right\| = \frac{|\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|}{\|\vec{v} \times \vec{w}\|^2} \|\vec{v} \times \vec{w}\| = \frac{|\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|}{\|\vec{v} \times \vec{w}\|}.$$

Therefore, the volume of the parallelepiped is

$$V = Ah = \|\vec{v} \times \vec{w}\| \frac{|\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|}{\|\vec{v} \times \vec{w}\|} = |\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|. \qquad \square$$

Corollary 2.37. *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Then*

$$|\langle \vec{u}\,, \vec{v} \times \vec{w} \rangle| = |\langle \vec{v}\,, \vec{w} \times \vec{u} \rangle| = |\langle \vec{w}\,, \vec{u} \times \vec{v} \rangle|.$$

Proof. The formula holds because each of the expressions describes the volume of the parallelepiped spanned by the three given vectors since we can take any of the sides of the parallelogram as its base. □

You should have understood

- the geometric interpretations of the cross product,
- why it exists only in $\mathbb{R}^3$
- ...

You should now be able to

- calculate the cross product,
- use it to say something about the angle between two vectors in $\mathbb{R}^3$,
- use it calculate the area of a parallelogram and the volume of a parallelepiped,
- ...

2.6 Lines and planes in $\mathbb{R}^3$

In this section we discuss lines and planes and how to describe them. In the next section, we will calculate, e.g., intersections between them. We work mostly in $\mathbb{R}^3$ and only give some hints on how the concepts discussed here generalise to $\mathbb{R}^n$ with $n \neq 3$. The special case $n = 2$ should be clear.
The formal definition of lines and planes need will be given in Definition 5.53 because this requires the concept of linear independence. (for the curious: a *line* is an (affine) one-dimensional subspace of a vector space; a *plane* is an (affine) two-dimensional subspace of a vector space; a *hyperplane* is an (affine) $(n-1)$-dimensional subspace of an n-dimensional vector space). In this section we appeal to our knowledge and intuition from elementary geometry.

Lines

Intuitively, it is clear what a line in $\mathbb{R}^3$ should be. In order to describe a line in $\mathbb{R}^3$ completely, it is not necessary to know all its points. It is sufficient to know either

(a) two different points P, Q on the line

or

(b) one point P on the line and the direction of the line.

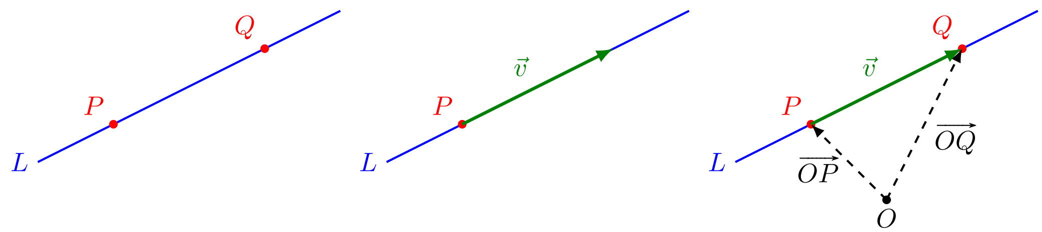


FIGURE 2.18: Line L given by: two points P, Q on L; or by a point P on L and the direction of L.

Clearly, both descriptions are equivalent because: If we have two different points P, Q on the line L, then its direction is given by the vector $\overrightarrow{PQ}$. If on the other hand we are given a point P on L and a vector $\vec{v}$ which is parallel to L, then we easily get another point Q on L by $\overrightarrow{OQ} = \overrightarrow{OP} + \vec{v}$.

Now we want to give formulas for the line.

Vector equation of a line

Given two points $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ with $P \neq Q$, there is exactly one line L which passes through both points. In formulas, this line is described as

$$L = \left\{ \overrightarrow{OP} + t\overrightarrow{PQ} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} p_1 + (q_1 - p_1)t \\ p_2 + (q_2 - p_2)t \\ p_3 + (q_3 - p_3)t \end{pmatrix} : t \in \mathbb{R} \right\} \tag{2.11}$$

If we are given a point $P(p_1, p_2, p_3)$ on L and a vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \neq \vec{0}$ parallel to L, then

$$L = \left\{ \overrightarrow{OP} + t\vec{v} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} p_1 + v_1 t \\ p_2 + v_2 t \\ p_3 + v_3 t \end{pmatrix} : t \in \mathbb{R} \right\} \tag{2.12}$$

The formulas are easy to understand. They say: In order to trace the line, we first move to an arbitrary point on the line (this is the term $\overrightarrow{OP}$) and then we move an amount t along the line. With this procedure we can reach every point on the line, and on the other hand, if we do this, then we are guaranteed to end up on the line.
The formulas (2.11) and (2.12) are called *vector equation* for the line L. Note that they are the same if we set $v_1 = q_1 - p_1$, $v_2 = q_2 - p_2$, $v_3 = q_3 - p_3$. We will mostly use the notation with the v's since it is shorter. The vector $\vec{v}$ is called *directional vector* of the line L.

Question 2.3

Is it true that E passes through the origin if and only if $\overrightarrow{OP} = \vec{0}$?

Remark 2.38. It is important to observe that a given line has many different parametrisations.

- For example, the vector equation that we write down depends on the points we choose on L. Clearly, we have infinitely many possibilities to do so.
- Observe that a given line L has many directional vectors. Indeed, if $\vec{v}$ is a directional vector for L, then $c\vec{v}$ is so too for every $c \in \mathbb{R} \setminus \{0\}$. However, all possible directional vectors are parallel.

Exercise. Check that the following formulas all describe the same line:

(i) $L_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} : t \in \mathbb{R} \right\}$, (ii) $L_2 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 12 \\ 10 \\ 8 \end{pmatrix} : t \in \mathbb{R} \right\}$,

(ii) $L_3 = \left\{ \begin{pmatrix} 13 \\ 12 \\ 11 \end{pmatrix} + t \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} : t \in \mathbb{R} \right\}$.

Question 2.4

- How can you see easily if two given lines are parallel or perpendicular to each other?
- How would you define the angle between two lines? Do they have to intersect so that an angle between them can be defined?

Parametric equation of a line

From the formula (2.12) it is clear that a point (x, y, z) belongs to L if and only if there exists $t \in \mathbb{R}$ such that

$$x = p_1 + tv_1, \quad y = p_2 + tv_2, \quad z = p_3 + tv_3. \tag{2.13}$$

If we had started with (2.11), then we had obtained

$$x = p_1 + t(q_1 - p_1), \quad y = p_2 + t(q_2 - p_2), \quad z = p_3 + t(q_3 - p_3) \tag{2.14}$$

The system of equations (2.13) or (2.14) are called the *parametric equations* of L. Here, t is the parameter.

Symmetric equation of a line

Observe that for $(x, y, z) \in L$, the three equations in (2.13) must hold for the same t. If we assume that $v_1, v_2, v_3 \neq 0$, then we can solve for t and we obtain that

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3} \tag{2.15}$$

If we use (2.14) then we obtain

$$\frac{x - p_1}{q_1 - p_1} = \frac{y - p_2}{q_2 - p_2} = \frac{z - p_3}{q_3 - p_3}. \tag{2.16}$$

The system of equations (2.15) or (2.16) is called the *symmetric equation* of L.

If for instance, $v_1 = 0$ and $v_2, v_3 \neq 0$, then the line is parallel to the yz-plane and its symmetric equation is

$$x = p_1, \quad \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3}.$$

If $v_1 = v_2 = 0$ and $v_3 \neq 0$, then the line is parallel to the z-axis and its symmetric equation is

$$x = p_1, \quad y = p_2, z \in \mathbb{R}.$$

Representations of lines

In $\mathbb{R}^n$, the vector form of a line is

$$L = \left\{ \overrightarrow{OP} + t\vec{v} : t \in \mathbb{R} \right\}$$

for fixed $P \in L$ and a directional vector $\vec{v}$. Its parametric form is

$$x_1 = p_1 + tv_1, \quad x_2 = p_2 + tv_2, \quad \dots, \quad x_n = p_n + tv_n, \qquad t \in \mathbb{R},$$

and, assuming that all v_j are different from 0, its symmetric form is

$$\frac{x_1 - p_1}{v_1} = \frac{x_2 - p_2}{v_2} = \cdots = \frac{x_n - p_n}{v_n}.$$

Question 2.5. Normal form of a line.

In $\mathbb{R}^2$, there is also the *normal form* of a line:

$$L : ax + by = d \tag{2.17}$$

where a, b and d are fixed numbers. This means that L consists of all the points $P(x, y)$ whose coordinates satisfy the equation $ax + by = d$.

(i) Given a line in the form (2.17), find a vector representation.

(ii) Given a line in vector representation, find a normal form (that is, write it as (2.17)).

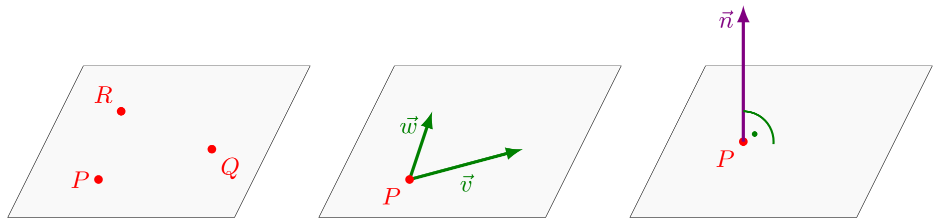
(iii) What is the geometric interpretation of a, b? (Hint: Draw the line L and the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.)

(iv) Can this normal form be extended/generalised to lines in $\mathbb{R}^3$? If it is possible, how can it be done? If it is not possible, why it is not possible.

Planes

In order to know a plane E in $\mathbb{R}^3$ completely, it is sufficient to

(a) three points P, Q on the plane that do not lie on a line,



(a) (b) (c)

FIGURE 2.19: Plane E given by: (a) three points P, Q, R on E, (b) a point P on E and two vectors $\vec{v}, \vec{w}$ parallel to E, (c) a point P on E and a vector $\vec{n}$ perpendicular to E.

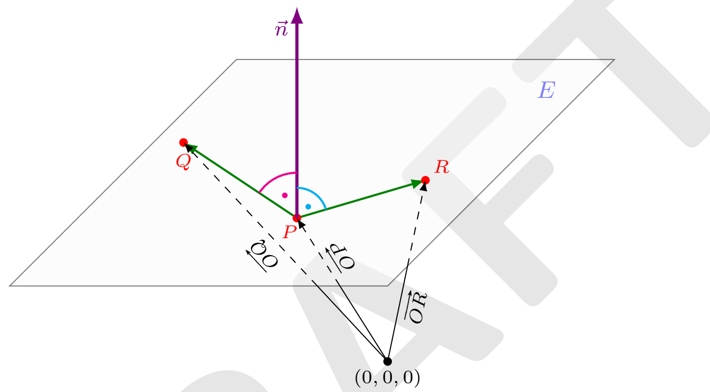


FIGURE 2.20: Plane E given with three points P, Q, R on E, two vectors $\overrightarrow{PQ}, \overrightarrow{PR}$ parallel to E, and a vector $\vec{n}$ perpendicular to E. Note the $\vec{n} \parallel \overrightarrow{PQ} \times \overrightarrow{PR}$.

or

(b) one point P on the plane and two non-parallel vectors $\vec{v}, \vec{w}$ which are both parallel the plane,

or

(c) one point P on the plane and a vector $\vec{n}$ which is perpendicular to the plane,

First, let us see how we can pass from one description to another. Clearly, the descriptions (a) and (b) are equivalent because given three points P, Q, R on E which do not lie on a line, we can form the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Theses vectors are then parallel to the plane E but are not parallel to each other. (Of course, we also could have taken $\overrightarrow{QR}$ and $\overrightarrow{QP}$ or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$.) If, on the other hand, we have one point P on E and two vectors $\vec{v}$ and $\vec{w}$, parallel to E and $\vec{v} \nparallel \vec{w}$, then we can easily get two other points on E, for instance by $\overrightarrow{OQ} = \overrightarrow{OP} + \vec{v}$ and $\overrightarrow{OR} = \overrightarrow{OP} + \vec{w}$. Then the three points P, Q, R lie on E and do not lie on a line.

Vector equation of a plane

In formulas, we can now describe our plane E as

$$E = \left\{ (x, y, z) : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overrightarrow{OP} + s\vec{v} + t\vec{w} \quad \text{for some } s, t \in \mathbb{R} \right\}.$$

As in the case of the vector equation of a line, it is easy to understand the formula. We first move to an arbitrary point on the line (this is the term $\overrightarrow{OP}$) and then we move parallel to the plane as we please (this is the term $s\vec{v} + t\vec{w}$). With this procedure we can reach every point on the plane, and on the other hand, if we do this, then we are guaranteed to end up on the plane.

Question 2.6

Is it true that E passes through the origin if and only if $\overrightarrow{OP} = \vec{0}$?

Normal form of a plane

Now we want to use the normal vector of the plane to describe it. Assume that we are given a point P on E and a vector $\vec{n}$ perpendicular to the plane. This means that every vector which is parallel to the plane E must be perpendicular to $\vec{n}$. If we take an arbitrary point $Q(x, y, z) \in \mathbb{R}^3$, then $Q \in E$ if and only if $\overrightarrow{PQ}$ is parallel to E, that means that $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. Recall that two vectors are perpendicular if and only if their inner product is 0, so $Q \in E$ if and only if

$$\begin{aligned} 0 = \langle n\,, \overrightarrow{PQ} \rangle &= \left\langle \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \begin{pmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{pmatrix} \right\rangle = n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) \\ &= n_1 x + n_2 y + n_3 z - (n_1 p_1 + n_2 p_2 + n_3 - p_3) \end{aligned}$$

If we set $d = n_1 p_1 + n_2 p_2 + n_3 - p_3$, then it follows that a point $Q(x, y, z)$ belongs to E if and only if its coordinates satisfy

$$n_1 x + n_2 y + n_3 z = d. \tag{2.18}$$

Equation (2.18) is called the *normal form* for the plane E and $\vec{n}$ is called a *normal vector* of E.

Notation 2.39. In order to define E, we write $E : n_1 x + n_2 y + n_3 z = d$. As a set, we denote E as $E = \{(x, y, z) : n_1 x + n_2 y + n + 3z = d\}$.

Exercise. Show that E passes through the origin if and only if $d = 0$.

Remark 2.40. As before, note that the normal equation for a plane is not unique. For instance,

$$x + 2y + 3z = 5 \qquad \text{and} \qquad 2x + 4y + 6z = 10$$

describe the same plane. The reason is that "the" normal vector of a plane is not unique. If $\vec{n}$ is normal vector of the plane E, then every $c\vec{n}$ with $c \in \mathbb{R} \setminus \{0\}$ is also a normal vector to the plane.

Definition 2.41. The *angle between two planes* is the angle between their normal vectors.

Note that this definition is consistent with the fact that two planes are parallel if and only if their normal vectors are parallel.

Remark 2.42.

- Assume a plane is given as in (b) (that is, we know a point P on E and two vectors $\vec{v}$ and $\vec{w}$ parallel to E but with $\vec{v} \not\parallel \vec{w}$). In order to find a description as in (c) (that is one point on E and a normal vector), we only have to find a vector $\vec{n}$ that is perpendicular to both $\vec{v}$ and $\vec{w}$. Proposition 2.32(vii) tells us how to do this: we only need to calculate $\vec{v} \times \vec{w}$. Another way to find an appropriate $\vec{n}$ is to find a solution of the linear 3×2 system given by $\{\langle \vec{v}\,, \vec{n}\rangle = 0,\ \langle \vec{w}\,, \vec{n}\rangle = 0\}$.

- Assume a plane is given as in (c) (that is, we know a point P on E and its normal vector). In order to find vectors $\vec{v}$ and $\vec{w}$ as in (b), we can proceed in many ways:

  – Find two solutions of $\vec{x} \times \vec{n} = 0$ which are not parallel.

  – Find two points Q, Ron the plane such that $\overrightarrow{PQ} \not\parallel \overrightarrow{PR}$. Then we can take $\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$.

  – Find one solution $\vec{v} \neq \vec{0}$ of $\vec{n} \times \vec{v} = \vec{0}$ which usually is easy to guess and then calculate $\vec{w} = \vec{v} \times \vec{n}$. Then $\vec{w}$ is perpendicular to $\vec{n}$ and therefore it is parallel to the plane. It is also perpendicular to $\vec{v}$ and therefore it is not parallel to $\vec{v}$. In total, this vector $\vec{w}$ does what we need.

Representations of planes

In $\mathbb{R}^n$, the vector form of plane is

$$E = \left\{ \overrightarrow{OP} + t\vec{v} + s\vec{w} : t \in \mathbb{R} \right\}$$

for fixed $P \in E$ and a two vectors $\vec{v}$, $\vec{w}$ parallel to the plane but not parallel to each other.

Note that there is no normal form of a plane in $\mathbb{R}^n$ for $n \geq 4$. The reason it that for $n \geq 4$, there are more than just one normal directions to a given plane, so a normal form of a plane E must consist of more than one equations (more precisely, it must consist of $n-2$ equations of the form $n_1x_1 + \dots n_nx_n = d$).

You should have understood

- the concept of lines and planes in $\mathbb{R}^3$,
- how they can be described in formulas,
- ...

You should now be able to

- pass easily between the different descriptions of lines and planes,
- ...

2.7 Intersections of lines and planes in $\mathbb{R}^3$

Intersection of lines

Given two lines G and L in $\mathbb{R}^3$, there are three possibilities:

(a) The lines intersect in exactly one point. In this case, they cannot be parallel.

(b) The lines intersect in infinitely many points. In this case, the lines have to be equal. In particular the have to be parallel.

(c) The lines do not intersect. Note that in contrast to the case in $\mathbb{R}^2$, the lines do not have to be parallel for this to happen. For example, the line $L : x = y = 1$ is a line parallel to the z-axis passing through $(1, 1, 0)$, and $G : x = z = 0$ is a line parallel to the y-axis passing through $(0, 0, 0)$, The lines do not intersect and they are not parallel.

Example 2.43. We consider four lines $L_j = \{\vec{p}_j + t\vec{v}_j : t \in \mathbb{R}\}$ with

$$\text{(i)} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{(ii)} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \vec{p}_2 = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix},$$

$$\text{(iii)} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \vec{p}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(iv)} \quad \vec{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \vec{p}_4 = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}.$$

We will calculate their mutual intersections.

$\boxed{L_1 \cap L_2 = L_1}$

Proof. A point $Q(x, y, z)$ belongs to $L_1 \cap L_2$ if and only if it belongs both to L_1 and L_2. This means that there must exist an $s \in \mathbb{R}$ such that $\overrightarrow{OQ} = \vec{p}_1 + s\vec{v}_1$ and there must exist a $t \in \mathbb{R}$ such that $\overrightarrow{OQ} = \vec{p}_2 + t\vec{v}_2$. Note that s and t are different parameters. So we are looking for s and t such that

$$\vec{p}_1 + s\vec{v}_1 = \vec{p}_2 + t\vec{v}_2, \qquad \text{that is} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}. \tag{2.19}$$

Once we have solved (2.19) for s and t, we insert them into the equations for L_1 and L_2 respectively, in order to obtain Q. Note that (2.19) in reality is a system of three equations: one equation for each component of the vector equation. Writing it out and solving each equation for s, we obtain

$$\begin{aligned} 0 + s &= 2 + 2t \\ 0 + 2s &= 4 + 4t \\ 1 + 3s &= 7 + 6t \end{aligned} \qquad \iff \qquad \begin{aligned} s &= 2 + 2t \\ s &= 2 + 2t \\ s &= 2 + 2t. \end{aligned}$$

This means that we have infinitely many solutions: Given any point R on L_1, there is a corresponding $s \in \mathbb{R}$ such that $\overrightarrow{OR} = \vec{p}_1 + s\vec{v}_1$. Now if we choose $t = (s-2)/2$, then $\overrightarrow{OR} = \vec{p}_2 + t\vec{v}_2$ holds, hence $R \in L_2$ too. If on the other hand we have a point $R' \in L_2$, then there is a corresponding $t \in \mathbb{R}$ such that $\overrightarrow{OR'} = \vec{p}_2 + t\vec{v}_2$. Now if we choose $s = 2 + 2t$, then $\overrightarrow{OR'} = \vec{p}_1 + t\vec{v}_1$ holds, hence $R' \in L_2$ too. In summary, we showed that $L_1 = L_2$. □

Remark 2.44. We could also have seen that the directional vectors of L_1 and L_2 are parallel. In fact, $\vec{v}_2 = 2\vec{v}_1$. It then suffices to show that L_1 and L_2 have at least one point in common in order to conclude that the lines are equal.

$\boxed{L_1 \cap L_3 = \{(1,2,4)\}}$

Proof. As before, we need to find $s, t \in \mathbb{R}$ such that

$$\vec{p}_1 + s\vec{v}_1 = \vec{p}_3 + t\vec{v}_3, \qquad \text{that is} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{2.20}$$

We write this as a system of equations, we get

$$\begin{array}{rl} ① & 0 + s = -1 + t \\ ② & 0 + 2s = 0 + t \\ ③ & 1 + 3s = 0 + 2t \end{array} \qquad \iff \qquad \begin{array}{rl} ① & s - t = -1 \\ ② & 2s - t = 0 \\ ③ & 3s - 2t = -1 \end{array}$$

From ① it follows that $s = t - 1$. Inserting in ② gives $0 = 2(t-1) - t = t - 2$, hence $t = 2$. From ① we then obtain that $s = 2 - 1 = 1$. Observe that so far we used only equations ① and ②. In order to see if we really found a solution, we must check if it is consistent with ③. Inserting our candidates for s and t, we find that $3 \cdot 1 - 2 \cdot 2 = -1$ which is consistent with ③.
So $L1$ and L_3 intersect in exactly one point. In order to find it, we put $s = 1$ in the equation for L_1:

$$\overrightarrow{OQ} = \vec{p}_1 + 1 \cdot \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

hence the intersection point is $Q(1,2,4)$.

In order to check if this result is correct, we can put $t = 2$ in the equation for L_3. The result must be the same. The corresponding calculation is:

$$\overrightarrow{OQ} = \vec{p}_3 + 2 \cdot \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

which confirms that the intersection point is $Q(1,2,4)$. □

$\boxed{L_1 \cap L_4 = \varnothing}$

Proof. As before, we need to find $s, t \in \mathbb{R}$ such that

$$\vec{p}_1 + s\vec{v}_1 = \vec{p}_4 + t\vec{v}_4, \qquad \text{that is} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{2.21}$$

We write this as a system of equations and we get

$$\begin{array}{lrcl} ① & s &=& 3 + t \\ ② & 2s &=& t \\ ③ & 1+3s &=& 5+2t \end{array} \qquad\Longleftrightarrow\qquad \begin{array}{lrcl} ① & s - t &=& 3 \\ ② & 2s - t &=& 0 \\ ③ & 3s - 2t &=& 5 \end{array}$$

From ① it follows that $s = t+3$. Inserting in ② gives $0 = 2(t+3) - t = t+6$, hence $t = -6$. From ① we then obtain that $s = -6+3 = -3$. Observe that so far we used only equations ① and ②. In order to see if we really found a solution, we must check if it is consistent with ③. Inserting our candidates for s and t, we find that $3\cdot(-3) - 2\cdot(-6) = 3$ which is inconsistent with ③. Therefore we conclude that there is no pair of real numbers s, t which satisfies all three equations ①–③ simultaneously, so the two lines do not intersect. $\square$

Exercise. Show that $L_3 \cap L_4 = \varnothing$.

Intersection of planes

Given two planes E_1 and E_2 in $\mathbb{R}^3$, there are two possibilities:

(a) The planes intersect. In this case, they necessarily intersect in infinitely many points. The intersection is either a line. In this case E_1 and E_2 are not parallel. Or the intersection is a plane. In this case $E_1 = E_2$.

(b) The planes do not intersect. In this case, the planes must be parallel and not equal.

Example 2.45. We consider the following four planes:

$$E_1 : x+y+2z = 3, \quad E_2 : 2x+2y+4z = -4, \quad E_3 : 2x+2y+4z = 6, \quad E_4 : x+y-2z = 5.$$

We will calculate their mutual intersections.

$\boxed{E_1 \cap E_2 = \varnothing}$

Proof. The set of all points $Q(x, y, z)$ which belong both to E_1 and E_2 is the set of all x, y, z which simultaneously satisfy

$$\begin{array}{lrcl} ① & x + y + 2z &=& 3, \\ ② & 2x + 2y + 4z &=& -4. \end{array}$$

Now clearly, if x, y, z satisfies ①, then it cannot satisfy ② (the right side would be 6). We can see this more formally if we solve ①, e.g., for x and then insert into ②. We obtain from ①: $x = 3 - y - 2z$. Inserting into ② leads to

$$-4 = 2(3-y-2z) + 2y + 4z = 6,$$

which is absurd. $\square$

This result was to be expected since the normal vectors of the planes are $\vec{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{n}_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ respectively. Since they are parallel, the planes are parallel and therefore they either are equal or they have empty intersection. Now we see that for instance $(3, 0, 0) \in E_1$ but $(3, 0, 0) \notin E_2$, so the planes cannot be equal. Therefore they have empty intersection.

$\boxed{E_1 \cap E_3 = E_1}$

Proof. The set of all points $Q(x, y, z)$ which belong both to E_1 and E_3 is the set of all x, y, z which simultaneously satisfy

$$\begin{array}{cr} (1) & x + y + 2z = 3, \\ (2) & 2x + 2y + 4z = 6. \end{array}$$

Clearly, both equations are equivalent: if x, y, z satisfies (1), then it also satisfies (2) and vice versa. Therefore, $E_1 = E_3$. $\square$

$$\boxed{E_1 \cap E_4 = \left\{ \begin{pmatrix} 4 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}.}$$

Proof. First, we notice that the normal vectors $\vec{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{n}_4 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ are not parallel, so we expect that the solution is a line in $\mathbb{R}^3$.

The set of all points $Q(x, y, z)$ which belong both to E_1 and E_4 is the set of all x, y, z which simultaneously satisfy

$$\begin{array}{cr} (1) & x + y + 2z = 3, \\ (2) & x + y - 2z = 5. \end{array}$$

Equation (1) shows that $x = 3 - y - 2z$. Inserting into (2) leads to $5 = 3 - y - 2z + y - 2z = 3 - 4z$, hence $z = -\frac{1}{2}$. Putting this into (1), we find that $x + y = 3 - 2z = 4$. So in summary, the intersection consists of all points (x, y, z) which satisfy

$$z = -\frac{1}{2}, \quad x = 4 - y \quad \text{with} \quad y \in \mathbb{R} \quad \text{arbitrary},$$

in other words,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - y \\ y \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{with} \quad y \in \mathbb{R} \quad \text{arbitrary}.$$ $\square$

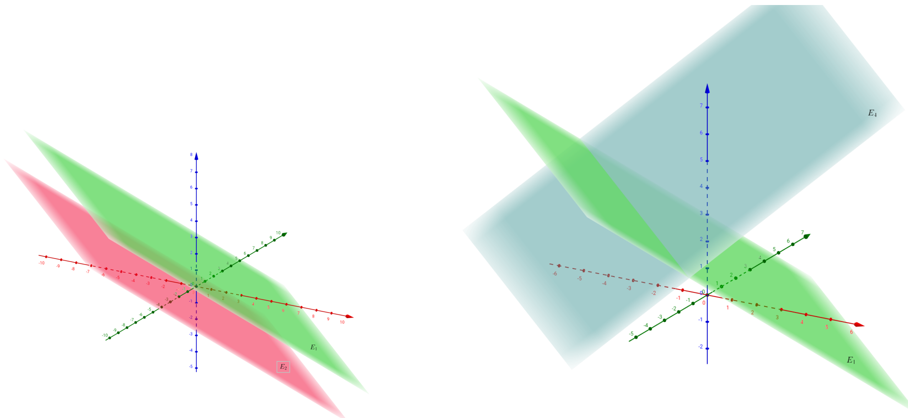


FIGURE 2.21: The left figure shows $E_1 \cap E_2 = \varnothing$, the right figure shows $E_1 \cap E_4$ which is a line.

Intersection of a line with a plane

Finally we want to calculate the intersection of a plane E with a line L. There are three possibilities:

(a) The plane and the line intersect in exactly one point. This happens if and only if L is not parallel to E which is the case if and only if L is not perpendicular to the normal vector of E.

(b) The plane and the line do not intersect. In this case, the E and L must be parallel, that is, L must be perpendicular to the normal vector of E.

(c) The plane and the line intersect in infinitely many points. In this case, L lies in E, that is, E and L must be parallel and they must share at least one point.

As an example we calculate $E_1 \cap L_2$. Since L_2 is clearly not parallel to E_1, we expect that their intersection consists of exactly one point.

$\boxed{E_1 \cap L_2 = \{(1/9,\ 2/9,\ 4/3)\}}$

Proof. The set of all points $Q(x, y, z)$ which belong both to E_1 and L_2 is the set of all x, y, z which simultaneously satisfy

$$x + y + 2z = 3 \qquad \text{and} \qquad x = 2 + 2t,\ y = 4 + 4t, z = 7 + 6t \ \text{ for some } \ t \in \mathbb{R}.$$

Replacing the expression with t from L_2 into the equation of the plane E_1, we obtain the following equation for t:

$$3 = (2 + 2t) + (4 + 4t) + 2(7 + 6t) = 20 + 18t \qquad \Longrightarrow \qquad t = -17/18.$$

Replacing this t into the equation for L_2 gives the point of intersection $Q(1/9,\ 2/9,\ 4/3)$.
In order to check our result, we insert the coordinates in the equation for E_1 and obtain $x+y+2z = 1/9 + 2/9 + 2 \cdot 4/3 = 1/3 + 8/3 = 3$ which shows that $Q \in E_1$. $\square$

Intersection of several lines and planes

If we wanted to intersect for instance, 5 planes in $\mathbb{R}^3$, then we would have to solve a system of 5 equations for 3 unknowns. Or if we wanted to intersect 7 lines in $\mathbb{R}^3$, then we had to solve a system of 3 equations for 7 unknowns. If we solve them as we did here, the process could become quite messy. So the next chapter is devoted to find a systematic and efficient way to solve a system of m linear equations for n unknowns.

You should have understood

- what the possible geometric structures of intersections of lines and planes is and how this depends on their relative orientation,
- the interpretation of a linear system with three unknowns as the intersection of planes in $\mathbb{R}^3$,
- ...

You should now be able to

- calculate the intersection of lines and planes,
- ...

2.8 Summary

The vector space $\mathbb{R}^n$ is given by

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}.$$

For points $P(p_1, \dots, p_n)$, $Q(q_1, \dots, q_n)$, the vector whose initial point is P and final point is Q, is

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ \vdots \\ q_n - p_n \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{OQ} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \quad \text{where } O \text{ denotes the origin.}$$

On $\mathbb{R}^n$, the *sum and product with scalars* are defined by

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad \vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}, \qquad \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad c\vec{v} = \begin{pmatrix} cv_1 \\ \vdots \\ cv_n \end{pmatrix}.$$

The *norm* of a vector is

$$\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2}.$$

If $\vec{v} = \overrightarrow{PQ}$, then $\|\vec{v}\| = \|\overrightarrow{PQ}\| =$ distance between P and Q.

For vectors $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$ their *inner product* is a real number defined by

$$\langle \vec{v}\,, \vec{w} \rangle = \left\langle \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = v_1 w_1 + \cdots + v_n w_n.$$

Important formulas involving the inner product

- $\langle \vec{v}\,, \vec{w} \rangle = \langle \vec{w}\,, \vec{v} \rangle$,
  $\langle \vec{v}\,, c\vec{w} \rangle = c\langle \vec{v}\,, \vec{w} \rangle$,
  $\langle \vec{v}\,, \vec{w} + \vec{u} \rangle = \langle \vec{v}\,, \vec{w} \rangle + \langle \vec{v}\,, \vec{u} \rangle$,
- $\langle \vec{v}\,, \vec{w} \rangle = \|\vec{v}\|\,\|\vec{w}\| \cos\varphi$,
- $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ $\boxed{\text{Triangle inequality}}$
- $\vec{v} \perp \vec{w} \iff \langle \vec{v}\,, \vec{w} \rangle = 0$,
- $\langle \vec{v}\,, \vec{v} \rangle = \|\vec{v}\|^2$.

The *cross product* is defined **only** in $\mathbb{R}^3$. It is a vector defined by

$$\vec{v} \times \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

Important formulas involving the cross product

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$,
  $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$,
  $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$,
- $\langle \vec{u}\,, \vec{v} \times \vec{w} \rangle = \langle \vec{u} \times \vec{v}\,, \vec{w} \rangle$.
- $\vec{u} \parallel \vec{v} \quad \Longrightarrow \quad \vec{u} \times \vec{v} = \vec{0}$.
- $\langle \vec{u}\,, \vec{u} \times \vec{v} \rangle = 0$ and $\langle \vec{v}\,, \vec{u} \times \vec{v} \rangle = 0$, in particular $\boxed{\vec{v} \perp \vec{v} \times \vec{u}, \quad \vec{u} \perp \vec{v} \times \vec{u}}$.
- $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\,\|\vec{w}\| \sin\varphi$.

Applications

- Area of a parallelogram spanned by $\vec{v}, \vec{w} \in \mathbb{R}^3$: $A = \|\vec{v} \times \vec{w}\|$.
- Volume of a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$: $V = |\langle \vec{u}\,, \vec{v} \times \vec{w} \rangle|$.

Representations of lines

- $\boxed{\text{Vector equation}}$ $L = \left\{ \overrightarrow{OP} + t\vec{v} : t \in \mathbb{R} \right\}$.
  P is a point on the line, $\vec{v}$ is called directional vector of L.
- $\boxed{\text{Parametric equation}}$ $x_1 = p_1 + tv_1,\ \ldots,\ x_n = p_n + tv_n,\ t \in \mathbb{R}$.
  Then $P(p_1, \ldots, p_n)$ is a point on L and $\vec{v} = (v_1, \ldots, v_n)^t$ is a directional vector of L.

- $\boxed{\text{Symmetric equation}}$ $\frac{x_1-p_1}{v_1} = \frac{x_2-p_2}{v_2} = \cdots = \frac{x_n-p_n}{v_n}$.
  Then $P(p_1, \ldots, p_n)$ is a point on L and $\vec{v} = (v_1, \ldots, v_n)^t$ is a directional vector of L.

  If one or several of the v_j are equal to 0, then the formula above has to be modified.

Representations of planes

- $\boxed{\text{Vector equation}}$ $E = \left\{\overrightarrow{OP} + t\vec{v} + s\vec{w} : s, t \in \mathbb{R}\right\}$.
  P is a point on the line, $\vec{v}$ and $\vec{w}$ are vectors parallel to E with $\vec{v} \not\parallel \vec{w}$.
- $\boxed{\text{Normal form (only in } \mathbb{R}^3\text{!!)}}$ $E : ax + by + cz = d$.

  The vector $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ formed by coefficients on the left hand side is perpendicular to E.

  Moreover, E passes through the origin if and only if $d = 0$.

The parametrisations are not unique!! (One and the same line or plane has many different parametrisations.)

- The angle between two lines is the angle between their directional vectors.
- Two lines are parallel if and only if their directional vectors are parallel.
  Two lines are perpendicular if and only if their directional vectors are perpendicular.
- The angle between two planes is the angle between their normal vectors.
- Two planes are parallel if and only if their normal vectors are parallel.
  Two planes are perpendicular if and only if their normal vectors are perpendicular.
- A line is parallel to a plane if and only if its directional vector is perpendicular to the plane.
  A line is perpendicular to a plane if and only if its directional vector is parallel to the plane.

2.9 Exercises

1. Sean $P(2,3)$, $Q(-1,4)$ puntos en $\mathbb{R}^2$ y sea $\vec{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ un vector en $\mathbb{R}^2$.

   (a) Calcule $\overrightarrow{PQ}$.

   (b) Calcule $\|\overrightarrow{PQ}\|$.

   (c) Calcule $\overrightarrow{PQ} + \vec{v}$.

   (d) Encuentre todos los vectores que son ortogonales a $\vec{v}$.

2. Sea $\vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \in \mathbb{R}^2$.

   (a) Encuentre todos los vectores unitarios cuya dirección es opuesta a la de $\vec{v}$.

   (b) Encuentre todos los vectores de longitud 3 que tienen la misma dirección que $\vec{v}$.

   (c) Encuentre todos los vectores que tienen la misma dirección que $\vec{v}$ y que tienen doble longitud de $\vec{v}$.

   (d) Encuentre todos los vectores con norma 2 que son ortogonales a $\vec{v}$.

3. Show that the following equations describe the same line:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} : t \in \mathbb{R} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} : t \in \mathbb{R} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$\left\{ \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} : t \in \mathbb{R} \right\}, \quad \frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}, \quad \frac{x+3}{4} = \frac{y+3}{5} = \frac{z+3}{6}.$$

   Find at least one more vector equation and one more symmetric equation. Find at least two different parametric equations.

4. Para los siguientes vectores $\vec{u}$ y $\vec{v}$ decida si son ortogonales, paralelos o ninguno de los dos. Calcule el coseno del ángulo entre ellos. Si son paralelos, encuentre números reales λ y μ tales que $\vec{v} = \lambda\vec{u}$ y $\vec{u} = \mu\vec{v}$.

   (a) $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$,

   (b) $\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

   (c) $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$,

   (d) $\vec{v} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

5. (a) Para las siguientes parejas $\vec{v}$ y $\vec{w}$ encuentre todos los $\alpha \in \mathbb{R}$ tal que $\vec{v}$ y $\vec{w}$ son paralelos:

   (i) $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} \alpha \\ -2 \end{pmatrix}$, (ii) $\vec{v} = \begin{pmatrix} 2 \\ \alpha \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1+\alpha \\ 2 \end{pmatrix}$, (iii) $\vec{v} = \begin{pmatrix} \alpha \\ 5 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1+\alpha \\ 2\alpha \end{pmatrix}$,

(b) Para las siguientes parejas $\vec{v}$ y $\vec{w}$ encuentre todos los $\alpha \in \mathbb{R}$ tal que $\vec{v}$ y $\vec{w}$ son perpendiculares:

(i) $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} \alpha \\ -2 \end{pmatrix}$, (ii) $\vec{v} = \begin{pmatrix} 2 \\ \alpha \end{pmatrix}$, $\vec{w} = \begin{pmatrix} \alpha \\ 2 \end{pmatrix}$, (iii) $\vec{v} = \begin{pmatrix} \alpha \\ 5 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1+\alpha \\ 2 \end{pmatrix}$.

6. Sean $\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ y $\vec{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

(a) Calcule $\operatorname{proj}_{\vec{b}} \vec{a}$ y $\operatorname{proj}_{\vec{a}} \vec{b}$.

(b) Encuentre todos los vectors $\vec{v} \in \mathbb{R}^2$ tal que $\|\operatorname{proj}_{\vec{a}} \vec{v}\| = 0$. Describa este conjunto geométricamente.

(c) Encuentre todos los vectors $\vec{v} \in \mathbb{R}^2$ tal que $\|\operatorname{proj}_{\vec{a}} \vec{v}\| = 2$. Describa este conjunto geométricamente.

(d) ¿Existe un vector $\vec{x}$ tal que $\operatorname{proj}_{\vec{a}} \vec{x} \parallel \vec{b}$?
¿Existe un vector $\vec{x}$ tal que $\operatorname{proj}_{\vec{x}} \vec{a} \parallel \vec{b}$?

7. Sean $\vec{a}, \vec{b} \in \mathbb{R}^2$ con $\vec{a} \neq \vec{0}$.

(a) Demuestre que $\|\operatorname{proj}_{\vec{a}} \vec{b}\| \leq \|\vec{b}\|$.

(b) ¿Qué deben cumplir $\vec{a}$ y $\vec{b}$ para que $\|\operatorname{proj}_{\vec{a}} \vec{b}\| = \|\vec{b}\|$?

8. Sean $\vec{a}, \vec{b} \in \mathbb{R}^n$ con $\vec{b} \neq \vec{0}$.

(a) Demustre que $\|\operatorname{proj}_{\vec{b}} \vec{a}\| \leq \|\vec{a}\|$.

(b) Encuentre condiciones para $\vec{a}$ y $\vec{b}$ para que $\|\operatorname{proj}_{\vec{b}} \vec{a}\| = \|\vec{a}\|$.

(c) ¿Es cierto que $\|\operatorname{proj}_{\vec{b}} \vec{a}\| \leq \|\vec{b}\|$?

9. (a) Calcule el área del paralelogramo cuyos vértices adyacentes $A(1,2,3)$, $B(2,3,4)$, $C(-1,2,-5)$ son y calcule el cuarto vértice.

(b) Calcule el área del triángulo con los vértices. $A(1,2,3)$, $B(2,3,4)$, $C(-1,2,-5)$.

(c) Calcule el volumen del paralelepipedo determinado por los vectores
$\vec{u} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$.

10. (a) Demuestre que no existe un elemento neutral para el producto cruz en $\mathbb{R}^3$. Es decir: Demuestre que no existe ningún vector $\vec{v} \in \mathbb{R}^3$ tal que $\vec{v} \times \vec{w} = \vec{w}$ para todo $\vec{w} \in \mathbb{R}^3$.

(b) Sea $\vec{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$.

(i) Encuentre todos los vectores $\vec{a}, \vec{b} \in \mathbb{R}^3$ tales que $\vec{a} \times \vec{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{b} \times \vec{w} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$,

(ii) Encuentre todos los vectores $\vec{v} \in \mathbb{R}^3$ tales que $\langle \vec{v}, \vec{w} \rangle = 4$.

11. Dados líneas L_1 y L_2 y el punto P, determine:

- si L_1 y L_2 son paralelas,
- si L_1 y L_2 tienen un punto de intersección,
- si P pertenece a L_1 y/o a L_2,
- una recta paralela a L_2 que pase por P.

(a) $L_1 : \vec{r}(t) = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\quad L_2 : \frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{4}$, $\quad P(5, 2, 11)$.

(b) $L_1 : \vec{r}(t) = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\quad L_2 : x = t + 1,\ y = 3t - 4,\ z = -t + 2$, $\quad P(5, 7, 2)$.

12. En $\mathbb{R}^3$ considere el plano E dado por $E : 3x - 2y + 4z = 16$.

(a) Encuentre por lo menos tres puntos que pertenecen a E.

(b) Encuentre un punto en E y dos vectores $\vec{v}$ y $\vec{w}$ en E que no son paralelos entre si.

(c) Encuentre un punto en E y un vector $\vec{n}$ que es ortogonal a E.

(d) Encuentre un punto en E y dos vectores $\vec{a}$ y $\vec{b}$ en E con $\vec{a} \perp \vec{b}$.

13. Para los puntos $P(1, 1, 1)$, $Q(1, 0, -1)$ y los siguientes planos E:

- Encuentre la ecuación del plano.
- Determine si P pertenece al plano.
- Encuentre una recta que esté ortogonal a E y que contenga al punto Q.

(i) E es el plano que contiene al punto $A(1, 0, 1)$ y es paralelo a los vectores $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ y $\vec{w} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

(ii) E es el plano que contiene los puntos $A(1, 0, 1)$, $B(2, 3, 4)$, $C(3, 2, 4)$.

(iii) E es el plano que contiene el punto $A(1, 0, 1)$ y es ortogonal al vector $\vec{n} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

14. Considere el plano $E : 2x - y + 3z = 9$ y la recta $L : x = 3t + 1,\ y = -2t + 3,\ z = 5t$.

(a) Encuentre $E \cap L$.

(b) Encuentre una recta G que no interseque ni al plano E ni a la recta L. Pruebe su afirmación. Cúantas rectas con esta propiedad hay?

15. En $\mathbb{R}^3$ considere el plano E dado por $E : 3x - 2y + 4z = 16$.

(a) Demuestre que los vectores $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ y $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ son paralelos al plano E.

(b) Encuentre números $\lambda, \mu \in \mathbb{R}$ tal que $\lambda\vec{a} + \mu\vec{b} = \vec{v}$.

(c) Demuestre que el vector $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ no es paralelo al plano E y encuentre vectores $c_\parallel$ y $c_\perp$ tal que $c_\parallel$ es paralelo a E, $c_\perp$ es ortogonal a E y $c = c_\parallel + c_\perp$.

16. Sea E un plano en $\mathbb{R}^2$ y sean $\vec{a}$, $\vec{b}$ vectores paralelos a E. Demuestre que para todo $\lambda, \mu \in \mathbb{R}$, el vector $\lambda\vec{a} + \mu\vec{b}$ es paralelo al plano.

17. Sea V un espacio vectorial. Demuestre lo siguiente:

   (a) El elemento neutral es único.
   (b) $0v = \mathbb{O}$ para todo $v \in V$.
   (c) $\lambda\mathbb{O} = \mathbb{O}$ para todo $\lambda \in \mathbb{R}$.
   (d) Dado $v \in V$, su inverso $\widetilde{v}$ es único.
   (e) Dado $v \in V$, su inverso $\widetilde{v}$ cumple $\widetilde{v} = (-1)v$.

18. De todos los siguientes conjuntos decida si es un espacio vectorial con su suma y producto usual.

   (a) $V = \left\{ \begin{pmatrix} a \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$,
   (b) $V = \left\{ \begin{pmatrix} a \\ a^2 \end{pmatrix} : a \in \mathbb{R} \right\}$,
   (c) V es el conjunto de todas las funciones continuas $\mathbb{R} \to \mathbb{R}$.
   (d) V es el conjunto de todas las funciones f continuas $\mathbb{R} \to \mathbb{R}$ con $f(4) = 0$.
   (e) V es el conjunto de todas las funciones f continuas $\mathbb{R} \to \mathbb{R}$ con $f(4) = 1$.

Chapter 3

Linear Systems and Matrices

We will rewrite linear systems as matrix equations in order to solve them systematically and efficiently. We will interpret matrices as linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ which then allows us to define algebraic operations with matrices, specifically we will define the sum and the composition (=multiplication) of matrices which then leads naturally to the concept of the inverse of a matrix. We can interpret a matrix as a system which takes some input (the variables $x_1, \dots, x_n$) and gives us back as output $b_1, \dots, b_m$ via $A\vec{x} = \vec{b}$. Sometimes we are given the input and we want to find the b_j; and sometimes we are given de output $b_1, \dots, b_m$ and we want to find the input $x_1, \dots, x_n$ which produces the desired output. The latter question is usually the harder one. We will see that a unique input for any given output exists if and only if the matrix is invertible. We can refine the concept of invertibility of a matrix. We say that A has a left inverse if for any $\vec{b}$ the equation $A\vec{x} = \vec{b}$ has at most one solution and we say that it has a right inverse $A\vec{x} = \vec{b}$ has at least one solution for any $\vec{b}$.

We will discuss in detail the Gauß and Gauß-Jordan elimination which helps us to find solutions of a given linear system and the inverse of a matrix if it exists. In Section 3.7 we define the transposition of matrices and we have a first look at symmetric matrices. They will become important in Chapter 8. We will also see the interplay of transposing a matrix and the inner product. In the last section of this chapter we define the so-called elementary matrices which can be seen as the building blocks of invertible matrices. We will use them in Chapter 4 to prove important properties of the determinant.

3.1 Linear systems and Gauß and Gauß-Jordan elimination

We start with a linear system as in Definition 1.8:

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ &\cdots + a_{1n}x_n = \ b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \ &\cdots + a_{2n}x_n = \ b_2 \\ \vdots \qquad\qquad &\ \ \vdots \qquad\qquad\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ &\cdots + a_{mn}x_n = b_m. \end{aligned} \tag{3.1}$$

Recall that the system is called *consistent* if it has at least one solution; otherwise it is called *inconsistent*. According to (1.4) and (1.5) its associated *coefficient matrix* and *augmented coefficient matrices* are

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{3.2}$$

and

$$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{array}\right). \tag{3.3}$$

Definition 3.1. The set of all matrices with m rows and n columns is denoted by $M(m \times n)$. If we want to emphasise that the matrix has only real entries, then we write $M(m \times n, \mathbb{R})$ or $M_{\mathbb{R}}(m \times n)$. Another frequently used notation is $M_{m\times n}$. A matrix A is called a *square matrix* if its number of rows is equal to its number of columns.

In order to solve (3.1), we could use the first equation, solve for x_1 and insert this in all the other equations. This gives us a new system with $m-1$ equations for $n-1$ unknowns. Then we solve the next equation for x_2, insert it in the other equations, and we continue like this until we have only one equation left. This of course will fail if for example $a_{11} = 0$ because in this case we cannot solve the first equation for x_1. We could save our algorithm by saying: we solve the first equation for the first unknown whose coefficient is different from 0 (or we could take an equation where the coefficient of x_1 is different from 0 and declare this one to be our first equation. After all, we can order the equations as we please). Even with this modification, the process of solving and replacing is error prone.

Another idea is to manipulate the equations. The question is: Which changes to the equations are allowed without changing the information contained in the system? We don't want to destroy information (thus potentially allowing for more solutions) nor introduce more information (thus potentially eliminating solutions). Or, in more mathematical terms, what changes to the given system of equations result in an equivalent system? Here we call two systems equivalent if they have the same set of solutions.

We can check if the new system is equivalent to the original one, if there is a way to restore the original one.

For example, if we exchange the first and the second row, then nothing really happened and we end up with an equivalent system. We can come back to the original equation by simply exchanging again the first and the second row.
If we multiply both sides of the first equation on both sides by some factor, let's say, by 2, then again nothing changes. Assume for example that the first equation is $x + 3y = 7$. If we multiply both sides by 2, we obtain $2x + 6y = 14$. Clearly, if a pair (x, y) satisfies the first equation, then it satisfies also the second one an vice versa. Given the new equation $3x + 6y = 14$, we can easily restore the old one by simply dividing both sides by 2.

If we take an equation and multiply both of its sides by 0, then we *destroy* information because we end up with $0 = 0$ and there is no way to get back the information that was stored in the original equation. So this is not an allowed operation.

Show that squaring both sides of an equation in general does not give an equivalent equation. Are there cases, when it does?

Squaring an equation or taking the logarithm on both sides or other such things usually are not interesting to us because the resulting equation will no longer be a linear equation.

Let us denote the jth row of our linear system (3.1) by R_j. The following tabel contains the so-called called *elementary row operations*. They are the "allowed" operations because they do not alter the information contained in a given linear system since they are reversible.
The first column describes the operation in words, the second introduces their shorthand notation and in the last row we give the inverse operation which allows us to get back to the original system.

Elementary operation	**Notation**	**Inverse Operation**
① Swap rows j and k.	$R_j \leftrightarrow R_k$	$R_j \leftrightarrow R_k$
② Multiply row j by some $\lambda \neq 0$.	$R_k \to \lambda R_k$	$R_j \to \frac{1}{\lambda} R_j$
③ Replace row k by the sum of row k and λ times R_j and leave row j unchanged.	$R_k \to R_k + \lambda R_j$	$R_k \to R_k - \lambda R_j$

Exercise. Show that the operation in the third column reverses the operation from the second column.

Exercise. Show that instead of the operation ③ it suffices to take ③': $R_k \to R_k + R_j$ because ③ can be written and as a composition of operations of the form ② and ③'. Show how this can be done.

Exercise. Show that in reality ① is not necessary since it can be achieved by a composition of operations of the form ② and ③ (or ② and ③'). Show how this can be done.

Let us see in an example how this works.

Example 3.2.

$$\left.\begin{aligned} x_1 + \ \ x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ 4x_2 + x_3 &= 7 \end{aligned}\right\} \xrightarrow{R_2 \to R_2 - 2R_2} \left\{\begin{aligned} x_1 + x_2 \ \ - \ \ x_3 &= 1 \\ x_2 \ \ + 3x_3 &= 1 \\ 4x_2 + \ \ x_3 &= 7 \end{aligned}\right\} \xrightarrow{R_3 \to R_3 - 4R_2} \left\{\begin{aligned} x_1 + x_2 - \quad x_3 &= 1 \\ x_2 + \ \ 3x_3 &= 1 \\ - 11x_3 &= 3 \end{aligned}\right\}$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \left\{\begin{aligned} x_1 + x_2 - \ \ x_3 &= \qquad 1 \\ x_2 + 3x_3 &= \qquad 1 \\ x_3 &= -3/11. \end{aligned}\right.$$

Here we can stop because it is already quite easy to read off the solution. Proceeding from the bottom to the top, we obtain

$$x_3 = -3/11, \qquad x_2 = 1 - 3x_3 = 20/11, \qquad x_1 = 1 + x_3 - x_2 = -12/11.$$

Note that we could continue our row manipulations to clean up the system even more:

$$\cdots \longrightarrow \begin{Bmatrix} x_1 + x_2 - x_3 = 1 \\ x_2 + 3x_3 = 1 \\ -11x_3 = 3 \end{Bmatrix} \xrightarrow{R_3 \to -1/11 R_3} \begin{Bmatrix} x_1 + x_2 - x_3 = 1 \\ x_2 + 3x_3 = 1 \\ x_3 = -3/11 \end{Bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 3R_3} \begin{Bmatrix} x_1 + x_2 - x_3 = 1 \\ x_2 = 20/11 \\ x_3 = -3/11 \end{Bmatrix} \xrightarrow{R_1 \to R_1 - 1/11 R_3} \begin{Bmatrix} x_1 + x_2 = 8/11 \\ x_2 + = 20/11 \\ x_3 = -3/11 \end{Bmatrix}$$

$$\xrightarrow{\to R_1 R_1 - R_2} \begin{Bmatrix} x_1 + = -12/11 \\ x_2 = 20/11 \\ x_3 = -3/11 \end{Bmatrix}$$

Our strategy was to apply manipulations that successively eliminate the unknowns in the lower equations and we aimed to get to a form of the system of equations where the last one contains the least number of unknowns possible.

Convince yourself that the first step of our reduction process is equivalent to solve hte first equation for x_1 and insert it in the other equations in order to eliminate it there. The next step in the reduction is equivalent to solve the new second equation for x_2 and insert it into the third equation.

It is important to note that there are infinitely many different routes how to get to the final result, but usually some are quicker than others.

Let us analyse what we did. We looked at the *coefficients* of the system and we applied transformations such that they become 0 because this results in removing the corresponding unknowns from the equations. So in the example above we could just as well delete all the x_j, keep only the augmented coefficient matrix and perform the line operations in the matrix. Of course, we have to remember that the numbers in the first columns are the coefficients of x_1, those in the second column are the coefficients of x_2, etc. Then our calculations are translated into the following:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 3 \\ 0 & 4 & 1 & 7 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 1 & 7 \end{array}\right) \xrightarrow{R_3 \to R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -11 & 3 \end{array}\right)$$

$$\xrightarrow{R_3 \to 1/11 R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -3/11 \end{array}\right).$$

If we translate this back into a linear system, we get

$$\begin{aligned} x_1 + x_2 + \ \ x_3 &= \quad\ \ 1 \\ x_2 + 3x_3 &= \quad\ \ 3 \\ x_3 &= -3/11 \end{aligned}$$

which can be easily solved from the bottom up.
We did exactly the same calculations as we did with the system of equations but it looks much tidier in matrix notation since we do not have to write down the unknowns all the time.
If we want to solve a linear system we write it as an augmented matrix and then we perform row operations until we reach a "nice" form where we can read off the solutions if there are any.
But what is a "nice" form? Remember that if a coefficient is 0, then the corresponding unknown does not show up in the equation.

- All rows with only zeros should be at the bottom.
- In the first non-zero equation from the bottom, we want as few unknowns as possible and we want them to be the last unknowns. So as last row we want one that has only zeros in it or one that starts with zeros, until finally we get a non-zero number say in column k. This non-zero number can always be made equal to 1 by dividing the row by it. Now we know how the unknowns $x_k, \dots, x_n$ are related. Note that all the other unknowns $x_1, \dots, x_{k-1}$ have disappeared from the equation since their coefficients are 0.

  If $k = n$ as in our example above, then we there is only one solution for x_n.
- The second non-zero row from the bottom should also start with zeros until we get to a column, say column l, with non-zero entry which we always can make equal to 1. This column should be to the left of the column k (that is we want $l < k$). Because now we can use what we know from the last row about the unknowns $x_k, \dots, x_n$ to say something about the unknowns $x_l, \dots, x_{k-1}$.
- We continue like this until all rows are as we want them.

Note that the form of such a "nice" matrix looks a bit like it had a triangle consisting of only zeros in its lower left part. There may be zeros in the upper right part. If a matrix has the form we just described, we say it is in *row echelon form*. Let us give a precise definition.

Definition 3.3 (Row echelon form). We say that a matrix $A \in M(m \times n)$ is in *row echelon form* if:

- All its zero rows are the last rows.
- The first non-zero entry in a row is 1. It is called the *pivot* of the row.
- The pivot of any row is strictly to the right of the row above.

Definition 3.4 (Reduced row echelon form). We say that a matrix $A \in M(m \times n)$ is in *reduced row echelon form* if:

- A is in row echelon form.
- All the entries in A which are above a pivot are equal to 0.

Let us quickly see some examples.

Examples 3.5.

(a) The following matrices are in reduced row echelon form. The pivots are highlighted.

$$\begin{pmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) The following matrices are in row echelon form but not in reduced row echelon form. The pivots are highlighted.

$$\begin{pmatrix} 1 & 6 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & 3 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) The following matrices are not in row echelon form:

$$\begin{pmatrix} 1 & 6 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 6 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise.
- Say why the matrices in (b) are not in reduced row echelon form and use elementary row operations to transform them into a matrix in reduced row echelon form.
- Say why the matrices in (c) are not in row echelon form and use elementary row operations to transform them into a matrix in row echelon form. Transform them further to obtain a matrix in reduced row echelon form.

Question 3.1

If we interchange to lines in a matrix this corresponds to writing down the given equations in a different order. What is the effect on a linear system if we interchange two columns?

Remember: if we translate a linear system to an augmented coefficient matrix $(A|b)$, perform the row operations to arrive at a (reduced) row echelon form $(A'|b')$, and translate back to a linear system, then this new system contains exactly the same information as the original one but it is "tidied up" and it is easy to determine its solution.
The natural question now is: Can we always transform a matrix into one in (reduced) row echelon form? The answer is that this is always possible and we can even give an algorithm for it.

Gauß elimination. Let $A \in M(m \times n)$. The Gauß elimination is an algorithm that transforms A into a row echelon form. The steps are as follows:

- Find the first column which does not consist entirely of zeros. Interchange rows until the entry in that column in the first row is different from zero.
- Multiply the first row by an appropriate number so that its first non-zero entry is 1.
- Use the first row to eliminate all coefficients below its pivot.
- Now our matrix looks like

$$\left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & * \cdots & * \\ \vdots & & \vdots & 0 & & \\ \vdots & 0 & \vdots & \vdots & & A' \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 0 & 0 & & \end{array}\right).$$

  where $*$ are arbitrary numbers and A' is a matrix with less columns than A and $m-1$ columns. Now repeat the process for A'. Note that in doing say the first columns do not change since we are only manipulating zeros.

Gauß-Jordan elimination. Let $A \in M(m \times n)$. The Gauß-Jordan elimination is an algorithm that transforms A into a reduced row echelon form. The steps are as follows:

- Use the Gauß elimination to obtain a row echelon form of A.
- Use the pivots to eliminate the non-zero coefficients which are columns above a pivot.

Of course, if we do a reduction by hand, then we do not have to follow the steps of the algorithm strictly if it makes calculations easier. However, these algorithms always work and therefore can be programmed so that a computer can perform them.

Definition 3.6. Two $m \times n$ matrices A and B are called *row equivalent* if there are elementary row operations that transform A into B. (Clearly then B can be transformed by row operations into A.)

Remark. Let A be an $m \times n$ matrix.

- A can be transformed into infinitely many different row echelon forms.
- There is only one reduced row echelon form that A can be transformed into.

Prove the assertions above.

Before we give examples, we note that from the row echelon form we can immediately tell how many solutions the corresponding linear system has.

Theorem 3.7. *Let $(A|b)$ be the augmented coefficient matrix of a linear $m \times n$ system and let $(A'|b')$ be a row reduced form.*

(1) *If there is a row of the form $(0\cdots 0|\beta)$ with $\beta \neq 0$, then the system has no solution.*

(2) *If there is no row of the form $(0\cdots 0|\beta)$ with $\beta \neq 0$, then one of the following holds:*

   (2.1) *If there is a pivot in every column then the system has exactly one solution.*

   (2.2) *If there is a column without a pivot, then the system has infinitely many solutions.*

Proof. (1) If $(A'|b')$ has a row of the form $(0\cdots 0|\beta)$ with $\beta \neq 0$, then the corresponding equation is $0x_1 + \cdots + 0x_n = \beta$ which clearly has no solution.

(2) Now assume that $(A'|b')$ has no row of the form $(0\cdots 0|\beta)$ with $\beta \neq 0$. In case (2.1), the transformed matrix is then of the form

$$\left(\begin{array}{cccccc|c}
1 & a'_{12} & a'_{13} & & & a'_{1n} & b'_1\\
0 & 1 & a'_{23} & & & a'_{2n} & b'_2\\
\vdots & & 1 & & & \vdots & \vdots\\
 & & & \ddots & & & \\
 & & & & 1 & a'_{(n-1)n} & b'_{n-1}\\
 & & 0 & & & 1 & b'_n\\
 & & & & & 0 & 0\\
 & & & & & \vdots & 0\\
0 & \cdots & & & \cdots & 0 & 0
\end{array}\right). \tag{3.4}$$

Note that the last zero rows appear only if $n < m$. This system clearly has the unique solution

$$x_n = b'_n, \quad x_{n-1} = b'_{n-1} - a_{(n-1)n}x_n, \quad \ldots, \quad x_1 = b'_1 - a_{1n}x_n - \cdots - a_{12}x_2.$$

In case (2.2), the transformed matrix is then of the form

$$\left(\begin{array}{ccccccccccc|c}
0 & \cdots & 0 & 1 & * & \cdots & & & & \cdots & * & b'_1\\
0 & & & & & 0 & 1 & * & & \cdots & * & b'_2\\
0 & & & & & & & 0 & 1 & * \cdots & * & b'_3\\
\vdots & & & & & & & & & & \vdots & \vdots\\
 & & 0 & & & & & & 0 & 1 \; * & * & b'_k\\
 & & & & & & & & & & 0 & 0\\
\vdots & & & & & & & & & & \vdots & \vdots\\
0 & \cdots & & & & & & & & \cdots & 0 & 0
\end{array}\right). \tag{3.5}$$

where the stars stand for numbers. (If we continue the reduction until we get to the reduced row echelon form, then the numbers over the 1's must be zeros.) Note that we can choose the unknowns which correspond to the columns without a pivot arbitrarily. The unknowns which correspond to the columns with pivots can then always be chosen in a unique way such that the system is satisfied. $\square$

Definition 3.8. The variables wich correspond to columns without pivots are called *free variables*.

We will come back to this theorem later on page 101 (the theorem is stated again in the coloured box).
From the above theorem we get as an immediate consequence the following.

Theorem 3.9. *A linear system has either no, exactly one or infinitely many solutions.*

Now let us see some examples.

Example 3.10 (Example with a unique solution (no free variables)). We consider the linear system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 12, \\ -x_1 + 2x_2 + 3x_3 &= 15, \\ 3x_1 - 3x_3 &= 1. \end{aligned} \tag{3.6}$$

Solution. We form the augmented matrix and perform row reduction.

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 12 \\ -1 & 2 & 3 & 15 \\ 3 & 0 & -3 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 + 2R_2} \left(\begin{array}{ccc|c} 0 & 7 & 7 & 42 \\ -1 & 2 & 3 & 15 \\ 3 & 0 & -3 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|c} 0 & 7 & 7 & 42 \\ -1 & 2 & 3 & 15 \\ 0 & 6 & 8 & 46 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 2 & 3 & 15 \\ 0 & 7 & 7 & 42 \\ 0 & 6 & 8 & 46 \end{array}\right) \xrightarrow[R_2 \to \frac{1}{7} R_2]{R_1 \to -R_1} \left(\begin{array}{ccc|c} 1 & -2 & -3 & -15 \\ 0 & 1 & 1 & 6 \\ 0 & 6 & 8 & 46 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 6R_2} \left(\begin{array}{ccc|c} 1 & -2 & -3 & -15 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 2 & 10 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{2} R_3} \left(\begin{array}{ccc|c} 1 & -2 & -3 & -15 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right).$$

This shows that the system (3.6) is equivalent to the system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -15, \\ x_2 + x_3 &= 6, \\ x_3 &= 5 \end{aligned} \tag{3.7}$$

whose solution is easy to write down:

$$x_3 = 5, \quad x_2 = 6 - x_3 = 1, \quad x_1 = -15 + 2x_2 + 3x_3 = 2. \qquad \diamond$$

Remark. If we continue the reduction process until we reach the reduced row echelon form, then we obtain

$$\dots \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & -15 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|c} 1 & -2 & -3 & -15 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right) \xrightarrow{R_1 \to R_1 + 3R_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right).$$

Therefore the system (3.6) is equivalent to the system

$$\begin{aligned} x_1 \qquad\qquad &= 2, \\ x_2 \qquad &= 1, \\ x_3 &= 5. \end{aligned}$$

whose solution can be read off immediately to be

$$x_3 = 5, \quad x_2 = 1, \quad x_1 = 2.$$

Example 3.11 (Example with two free variables). We consider the linear system

$$\begin{aligned} 3x_1 - 2x_2 + 3x_3 + 3x_4 &= 3, \\ 2x_1 + 6x_2 + 2x_3 - 9x_4 &= 2, \\ x_1 + 2x_3 + \ \ x_3 - 3x_4 &= 1. \end{aligned} \tag{3.8}$$

Solution. We form the augmented matrix and perform row reduction.

$$\left(\begin{array}{rrrr|r} 3 & -2 & 3 & 3 & 3 \\ 2 & 6 & 2 & -9 & 2 \\ 1 & 2 & 1 & -3 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{rrrr|r} 3 & -2 & 3 & 3 & 3 \\ 0 & 2 & 0 & -3 & 0 \\ 1 & 2 & 1 & -3 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - 3R_3} \left(\begin{array}{rrrr|r} 0 & -8 & 0 & 12 & 0 \\ 0 & 2 & 0 & -3 & 0 \\ 1 & 2 & 1 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{rrrr|r} 1 & 2 & 1 & -3 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & -8 & 0 & 12 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 + 4R_2} \left(\begin{array}{rrrr|r} 1 & 2 & 1 & -3 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The 3rd and the 4th column do not have pivots and we see that the system (3.8) is equivalent to the system

$$\begin{aligned} x_1 \qquad - x_3 \qquad &= 1, \\ x_2 \qquad + x_4 &= 0. \end{aligned}$$

Clearly we can choose x_3 and x_4 (the unknowns corresponding to the columns without a pivot) arbitrarily. We will always be able to adjust x_1 and x_2 such that the system is satisfied. In order to make it clear that x_3 and x_4 are our free variables, we sometimes call them $x_3 = t$ and $x_4 = s$. Then every solution of the system (3.8) is of the form

$$x_1 = 1 + t, \quad x_2 = -s, \quad x_3 = t, \quad x_4 = s, \quad \text{for arbitrary} \quad s, t \in \mathbb{R}. \qquad \diamond$$

In *vector form* we can write the solution as follows. A tuple (x_1, x_2, x_3, x_4) is a solution of (3.8) if and only if the corresponding vector is of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1+t \\ -s \\ t \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{for some} \quad s, t \in \mathbb{R}.$$

Remark. Geometrically, the set of all solutions is an affine plane in $\mathbb{R}^4$.

Example 3.12 (Example with no solution). We consider the linear system

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 7, \\ 3x_1 + 2x_2 - 2x_3 &= 7, \\ -x_1 + 3x_2 - 3x_3 &= 2. \end{aligned} \tag{3.9}$$

Solution. We form the augmented matrix and perform row reduction.

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 3 & 2 & -2 & 7 \\ -1 & 3 & -3 & 2 \end{array}\right) \xrightarrow{R_1 \to R_1 + 2R_3} \left(\begin{array}{ccc|c} 0 & 7 & -7 & 11 \\ 3 & 2 & -2 & 7 \\ -1 & 3 & -3 & 2 \end{array}\right) \xrightarrow{R_2 \to R_2 + 3R_3} \left(\begin{array}{ccc|c} 0 & 7 & -7 & 11 \\ 0 & 11 & -11 & 13 \\ -1 & 3 & -3 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -1 & 3 & -3 & 2 \\ 0 & 11 & -11 & 13 \\ 0 & 7 & -7 & 11 \end{array}\right) \xrightarrow{11R_3 \to R_3 - 7R_2} \left(\begin{array}{ccc|c} -1 & 3 & -3 & 2 \\ 0 & 11 & -11 & 13 \\ 0 & 0 & 0 & 30 \end{array}\right).$$

The last line tells us immediately that the system (3.9) has no solution because there is no choice of x_1, x_2, x_3 such that $0x_1 + 0x_2 + 0x_3 = 30$. ◇

You should now have understood

- what it means that two linear systems are equivalent,
- which row operations transform a given system into an equivalent one and why this is so,
- when a matrix is in row echelon and a reduced row echelon form,
- why the linear system associated to a matrix in (reduced) echelon form is easy to solve,
- what the Gauß- and Gauß-Jordan elimination do and why they always work,
- that the Gauß- and Gauß-Jordan elimination is nothing very magical; essentially it is the same as solving for variables and replacing in the remaining equations. It only does so in a systematic way;
- why a given matrix can be transformed into may different row echelon forms, but in only one reduced row echelon form,
- why a linear system always has either no, exactly one or infinitely many solutions,
- ...

You should now be able to

- identify if a matrix is in row echelon or a reduced row echelon form,
- use the Gauß- or Gauß-Jordan elimination to solve linear systems,
- say if a system has no, exactly one or infinitely many solutions if you know its echelon form,
- ...

3.2 Homogeneous linear systems

In this short section we deal with the special case homogeneous linear systems. Recall that a linear system (3.1) is called homogeneous if $b_1 = \cdots = b_n = 0$. Such a system has always at least one solution, the so-called *trivial solution* $x_1 = \cdots = x_n = 0$. This also clear from Theorem 3.7 since no matter what row operations we perform, the right side will always remain equal to 0. Note that if we perform Gauß or Gauß-Jordan elimination, there is no need to write down the right hand side since it always will be 0.
If we adapt Theorem 3.7 to the special case of a homogeneous system, we obtain the following.

Theorem 3.13. *Let A be the coefficient matrix of a homogeneous linear $m \times n$ system and let A' be a row reduced form.*

(i) *If there is a pivot in every column then the system has exactly one solution, namely the trivial solution.*

(ii) *If there is a column with without a pivot, then the system has infinitely many solutions.*

Corollary 3.14. *A homogeneous linear system has either exactly one or infinitely many solutions.*

Let us see an example.

Example 3.15 (Example of a homogeneous system with infinitely many solutions). We consider the linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0, \\ 2x_1 + 3x_2 - 2x_3 &= 0, \\ 3x_1 - x_2 - 3x_3 &= 0. \end{aligned} \tag{3.10}$$

Solution. We form the augmented matrix and perform row reduction.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 3 & -1 & -3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 3 & -1 & -3 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & -7 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{the 2nd column}]{\text{use } R_2 \text{ to clear}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the third variable is free, so we set $x_3 = t$. The solution is

$$x_1 = t, \quad x_2 = 0, \quad x_3 = t \quad \text{for} \quad t \in \mathbb{R}.$$

or in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{for} \quad t \in \mathbb{R}. \qquad \diamond$$

Example 3.16 (Example of a homogeneous system with exactly one solution). We consider the linear system

$$\begin{aligned} x_1 + 2x_2 &= 0,\\ 2x_1 + 3x_2 &= 0, \\ 3x_1 + 5x_2 &= 0. \end{aligned} \tag{3.11}$$

Solution. We form the augmented matrix and perform row reduction.

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{pmatrix} \xrightarrow[\text{the 1st column}]{\text{use } R_1 \text{ to clear}} \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \xrightarrow[\text{the 2nd column}]{\text{use } R_2 \text{ to clear}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So the only possible solution is $x_1 = 0$ and $x_2 = 0$. ◇

In the next section we will see the connection between the set of solutions of a linear system and the corresponding homogeneous linear system.

You should now have understood

- why a homogeneous linear system always has either one or infinitely many solutions,
- ...

You should now be able to

- use the Gauß- or Gauß-Jordan elimination to solve homogeneous linear systems,
- ...

3.3 Matrices and linear systems

So far we were given a linear system with a specific right hand side and we asked ourselves which x_j do we have to feed into the system in order to obtain the given right hand side. Problems of this type are called *inverse problems* since we are given an output (the right hand of the system; the "state" that we want to achieve) and we have to find a suitable input in order to obtain the desired output.

Now we change our perspective a bit and we ask ourselves: If we put certain $x_1, \dots, x_n$ into the system, what do we get as a result on the right hand side? To investigate this question, it is very useful to write the system (3.1) in a short form. First note that we can view it as an equality of the two vectors with m components each:

$$\begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots + a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots + a_{2n}x_n \\ \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \tag{3.12}$$

Let A be the coefficient matrix and $\vec{x}$ the vector whose components are $x_1, \dots, x_n$. Then we write the left hand side of (3.12) as

$$A\vec{x} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots + a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots + a_{2n}x_n \\ \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots + a_{mn}x_n \end{pmatrix}. \tag{3.13}$$

With this notation, the linear system (3.1) can be written very short as

$$A\vec{x} = \vec{b}$$

with $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

A way to remember the formula for the multiplication of a matrix and a vector is that we "multiply each row of the matrix by the column vector", so we calculate *"row by column"*. For example, the jth component of $A\vec{x}$ is "(jth row of A) by (column $\vec{v}$)".

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n \\ \vdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}. \tag{3.14}$$

Definition 3.17. The formula in (3.13) is called the *multiplication of a matrix and a vector.*

An $m \times n$ matrix A takes a vector with n components and gives us back a vector with m components.

Observe that something like $\vec{x}A$ does **not** make sense!

Remark 3.18. Formula (3.13) can be interpreted as follows. If A is an $m \times n$ matrix and $\vec{x}$ is a vector in $\mathbb{R}^n$, then $A\vec{x}$ is the vector in $\mathbb{R}^m$ which is the sum of the columns of A weighted with coefficients given by $\vec{x}$ since

$$\begin{aligned} A\vec{x} &= \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots + a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots + a_{2n}x_n \\ \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{pmatrix} \\ &= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}. \end{aligned} \tag{3.15}$$

Remark 3.19. Recall that $\vec{e}_j$ is the vector which has a 1 is its jth component and has zeros everywhere else. Formula (3.13) shows that for every $j = 1, \dots, n$

$$A\vec{e}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} = j\text{th column of } A. \tag{3.16}$$

Let us prove some easy properties.

Proposition 3.20. *Let A be an $m \times n$ matrix, $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then*

(i) $A(c\vec{x}) = cA\vec{x}$,

(ii) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$,

(iii) $A\vec{0} = \vec{0}$.

Proof. The proofs are not difficult. They follow by using the definitions and carrying out some straightforward calculations as follows.

(i) $$A(c\vec{x}) = A\begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} = \begin{pmatrix} a_{11}cx_1 + & \cdots + a_{1n}cx_n \\ a_{21}cx_1 + & \cdots + a_{2n}cx_n \\ \vdots & \vdots \\ a_{m1}cx_1 + & \cdots + a_{mn}cx_n \end{pmatrix} = c\begin{pmatrix} a_{11}x_1 + & \cdots + a_{1n}x_n \\ a_{21}x_1 + & \cdots + a_{2n}x_n \\ \vdots & \vdots \\ a_{m1}x_1 + & \cdots + a_{mn}x_n \end{pmatrix} = cA\vec{x}.$$

(ii)

$$A(\vec{x} + \vec{y}) = A\begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} = \begin{pmatrix} a_{11}(x_1 + y_1) + & \cdots + a_{1n}(x_n + y_n) \\ a_{21}(x_1 + y_1) + & \cdots + a_{2n}(x_n + y_n) \\ \vdots & \vdots \\ a_{m1}(x_1 + y_1) + & \cdots + a_{mn}(x_n + y_n) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + & \cdots + a_{1n}x_n \\ a_{21}x_1 + & \cdots + a_{2n}x_n \\ \vdots & \vdots \\ a_{m1}x_1 + & \cdots + a_{mn}x_n \end{pmatrix} + \begin{pmatrix} a_{11}y_1 + & \cdots + a_{1n}y_n \\ a_{21}y_1 + & \cdots + a_{2n}y_n \\ \vdots & \vdots \\ a_{m1}y_1 + & \cdots + a_{mn}y_n \end{pmatrix} = A\vec{x} + A\vec{y}.$$

(iii) To show that $A\vec{0} = \vec{0}$, we could simply do the calculation (which is very easy!) or we can use (i):

$$A\vec{0} = A(0\vec{0}) = 0A\vec{0} = \vec{0}.$$ □

Note that in (iii) the $\vec{0}$ on the left hand side is the zero vector in $\mathbb{R}^n$ whereas the $\vec{0}$ on the right hand side is the zero vector in $\mathbb{R}^m$.

Proposition 3.20 gives an important insight into the structure of solutions of linear systems.

Theorem 3.21. (i) *Let $\vec{x}$ and $\vec{y}$ be solutions of the linear system (3.1). Then $\vec{x} - \vec{y}$ is a solution of the associated homogeneous linear system.*

(ii) *Let $\vec{x}$ be a solution of the linear system (3.1) and let $\vec{z}$ be a solution of the associated homogeneous linear system. Then $\vec{x} + \vec{z}$ is solution of the system (3.1).*

Proof. Assume that $\vec{x}$ and $\vec{y}$ are solutions of the (3.1), that is

$$A\vec{x} = \vec{b} \quad \text{and} \quad A\vec{y} = \vec{b}.$$

By Proposition 3.20 (i) and (ii) we have

$$A(\vec{x} - \vec{y}) = A\vec{x} + A(-\vec{y}) = A\vec{x} - A\vec{y} = \vec{b} - \vec{b} = \vec{0}$$

which shows that $\vec{x} - \vec{y}$ solves the homogeneous system $A\vec{v} = \vec{0}$ and thereby proves (i).
In order to show (ii), we proceed similarly. If $\vec{x}$ solves the inhomogeneous system (3.1) and $\vec{z}$ solves the associated homogeneous system, then

$$A\vec{x} = \vec{b} \quad \text{and} \quad A\vec{y} = \vec{0}.$$

Now (ii) follows from

$$A(\vec{x} + \vec{z}) = A\vec{x} + A\vec{z} = \vec{b} + \vec{0} = \vec{b}.$$ □

Corollary 3.22. *Let $\vec{x}$ be an arbitrary solution of the inhomogeneous system (3.1). Then the set of all solutions of (3.1) is*

$$\{\vec{x} + \vec{z} : \vec{z} \ \textit{ is solution of the associated homogeneous system}\}.$$

This means that in order to find all solutions of an inhomogeneous system it suffices to find one particular solution and all solutions of the corresponding homogeneous system.

We will show later that the set of all solutions of a homogeneous system is a vector space. When you study the set of all solutions of linear differential equations, you will encounter the same structure.

Example 3.23.

You should now have understood

- that an $m \times n$ matrix can be viewed as an operator that takes vectors in $\mathbb{R}^n$ and returns a vector in $\mathbb{R}^m$,
- the structure of the set of all solutions of a given linear system,
- ...

You should now be able to

- calculate expressions like $A\vec{x}$,

- relate the solutions of an inhomogeneous system with those of the corresponding homogeneous one,
- ...

3.4 Matrices as functions from $\mathbb{R}^n$ to $\mathbb{R}^m$; composition of matrices

In the previous section we saw that a matrix $A \in M(m \times n)$ takes a vector $\vec{x} \in \mathbb{R}^n$ and gives us back a vector $A\vec{x}$ in $\mathbb{R}^m$. This allows us to view A as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, and therefore we can define the sum and composition of two matrices. Before we do this, let us see a few examples of such matrices. As examples we work with 2×2 because their action on $\mathbb{R}^2$ can be sketched in the plane.

Example 3.24. Let us consider $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This defines a function T_A from $\mathbb{R}^2$ to $\mathbb{R}^2$ by

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_A\vec{x} = A\vec{x}.$$

Remark. We write T_A to denote the function induced by A, but sometimes we will write simply $A : \mathbb{R}^2 \to \mathbb{R}^2$ when it is clear that we consider the matrix A as a function.

We can calculate easily

$$T_A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T_A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad \text{in general} \quad T_A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

So we see that T_A represents the reflection of a vector $\vec{x}$ about the x-axis.

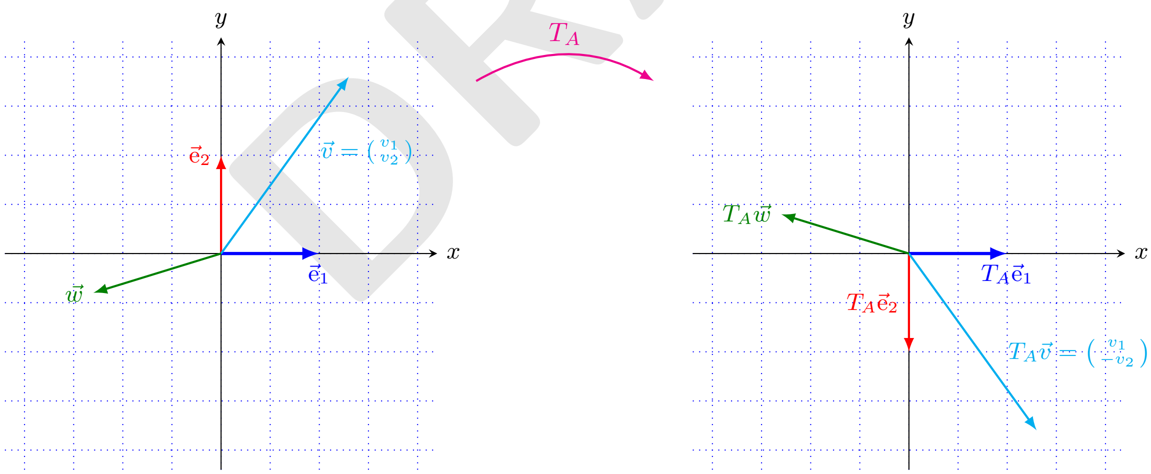


FIGURE 3.1: Reflection on the x-axis.

Example 3.25. Let us consider $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This defines a function T_B from $\mathbb{R}^2$ to $\mathbb{R}^2$ by

$$T_B : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_B\vec{x} = B\vec{x}.$$

We can calculate easily

$$T_B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T_B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{in general} \quad T_B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

So we see that T_B represents the projection of a vector $\vec{x}$ onto the y-axis.

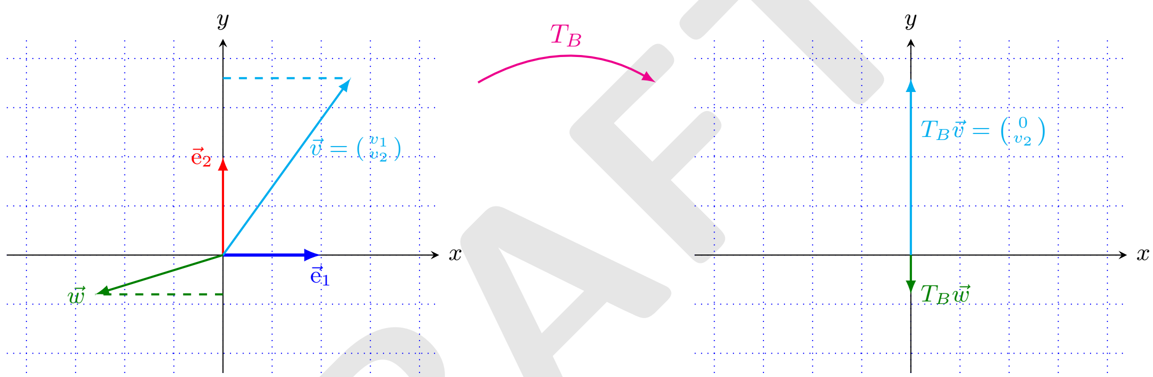


FIGURE 3.2: Orthogonal projection onto the y-axis.

Example 3.26. Let us consider $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This defines a function T_C from $\mathbb{R}^2$ to $\mathbb{R}^2$ by

$$T_C : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_C\vec{x} = C\vec{x}.$$

We can calculate easily

$$T_C\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad T_C\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \text{in general} \quad T_C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

So we see that T_C represents the rotation of a vector $\vec{x}$ about 90° counterclockwise.

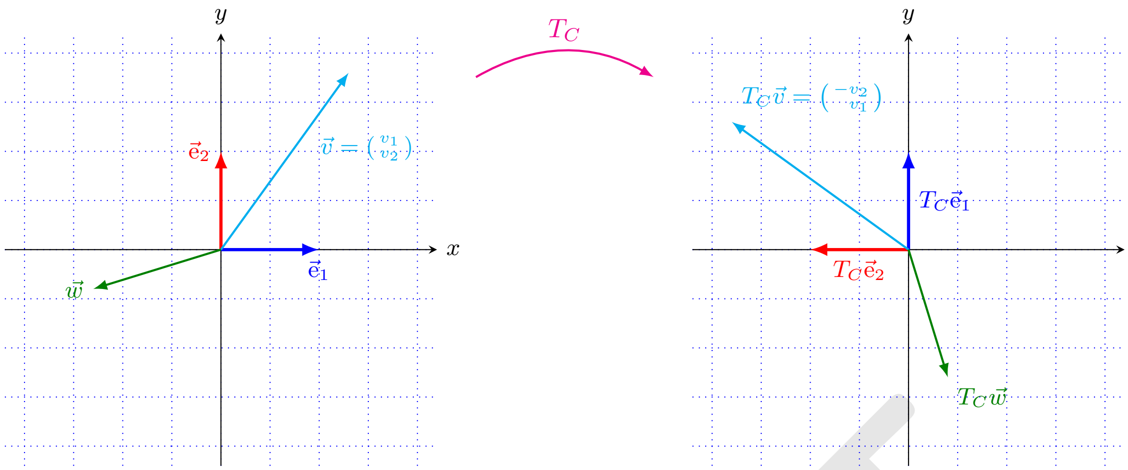


FIGURE 3.3: Rotation about $\pi/2$ counterclockwise.

Just as with other functions, we can sum them or compose them. Remember from your calculus classes, that functions are summed "pointwise". That means, if we have two functions $f, g : \mathbb{R} \to \mathbb{R}$, then the *sum* $f + g$ is a new function which is defined by

$$f + g : \mathbb{R} \to \mathbb{R}, \quad (f + g)(x) = f(x) + g(x). \tag{3.17}$$

The multiplication of a function f with a number c gives the new function cf defined by

$$cf : \mathbb{R} \to \mathbb{R}, \quad (cf)(x) = c(f(x)). \tag{3.18}$$

The composition of functions if defined as

$$f \circ g : \mathbb{R} \to \mathbb{R}, \quad (f \circ g)(x) = f(g(x)). \tag{3.19}$$

Matrix sum

Let us see how this looks like in the case of matrices. Let A and B be matrices. First note that they both must depart from the same space $\mathbb{R}^n$ because we want to apply them to the same $\vec{x}$, that is, both $A\vec{x}$ and $B\vec{x}$ must be defined. Therefore A and B must have the same number of columns. They also must have the same number of rows because we want to be able to sum $A\vec{x}$ and $B\vec{x}$. So let $A, B \in M(m \times n)$ and let $\vec{x} \in \mathbb{R}$. Then, by definition of the sum of two functions, we have

$$(A+B)\vec{x} := A\vec{x} + B\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} + \begin{pmatrix} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n \\ b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n \\ \vdots \\ b_{m1}x_1 + b_{m2}x_2 + \cdots + b_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + b_{m1}x_1 + b_{m2}x_2 + \cdots + b_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}+b_{11})x_1 + (a_{12}+b_{12})x_2 + \cdots + (a_{1n}+b_{mn})x_n \\ (a_{21}+b_{11})x_1 + (a_{22}+b_{12})x_2 + \cdots + (a_{2n}+b_{mn})x_n \\ \vdots \\ (a_{m1}+b_{11})x_1 + (a_{m2}+b_{12})x_2 + \cdots + (a_{mn}+b_{mn})x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{mn} \\ a_{21}+b_{11} & a_{22}+b_{12} & \cdots & a_{2n}+b_{mn} \\ \vdots & \vdots & \vdots & \\ a_{m1}+b_{11} & a_{m2}+b_{12} & \cdots & a_{mn}+b_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

We see that $A + B$ is again a matrix of the same size and that the components of this new matrix are just the sum of the corresponding components of the matrices A and B.

Multiplication of a matrix by a scalar

Now let c be a number and let $A \in M(m \times n)$. Then we have

$$(cA)\vec{x} = c(A\vec{x}) = c\left[\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right] = c\begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} ca_{11}x_1 + \cdots + ca_{1n}x_n \\ \vdots \\ ca_{m1}x_1 + \cdots + ca_{mn}x_n \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

We see that cA is again a matrix and that the components of this new matrix are just the product of the corresponding components of the matrix A with c.

Proposition 3.27. *Let $A, B, C \in M(m \times n)$ let $\mathbb{O}$ be the matrix whose entries are all 0 and let $\lambda, \mu \in \mathbb{R}$. Moreover, let $\widetilde{A}$ be the matrix whose entries are the negative entries of A. Then the following is true.*

(i) **Associativity of the matrix sum:** $(A + B) + C = A + (B + C)$.

(ii) **Commutativity of the matrix sum:** $A + B = B + A$.

(iii) **Additive identity:** $A + \mathbb{O} = A$.

(iv) **Additive inverse** $A + \widetilde{A} = \mathbb{O}$.

(v) $1A = A$.

(vi) $(\lambda + \mu)A = \lambda A + \mu A$ *and* $\lambda(A + B) = \lambda A + \lambda B$.

Proof. The claims of the proposition can be proved by straightforward calculations. □

Prove Proposition 3.27.

From the proposition we obtain immediately the following theorem.

Theorem 3.28. *$M(m \times n)$ is a vector space.*

Composition of two matrices

Now let us calculate the composition of two matrices. This is also called the *product of the matrices*. Assume we have $A \in M(m \times n)$ and we want to calculate AB for some matrix B. Note that A describes a function from $\mathbb{R}^n \to \mathbb{R}^m$. In order for AB to make sense, we need that B goes from some $\mathbb{R}^k$ to $\mathbb{R}^n$, that means that $B \in M(n \times k)$. The resulting function AB will then be a map from $\mathbb{R}^k$ to $\mathbb{R}^m$.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

So let $B \in M(n \times k)$. Then, by the definition of the composition of two functions, we have for every

$\vec{x} \in \mathbb{R}^k$

$$(AB)\vec{x} = A(B\vec{x}) = A\left[\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}\right] = A\begin{pmatrix} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1k}x_k \\ b_{21}x_1 + b_{22}x_2 + \cdots + b_{2k}x_k \\ \vdots \\ b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nk}x_k \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}[b_{11}x_1 + b_{12}x_2 + \cdots + b_{1k}x_k] + a_{12}[b_{21}x_1 + b_{22}x_2 + \cdots + b_{2k}x_k] + \cdots + a_{1n}[b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nk}x_k] \\ a_{21}[b_{11}x_1 + b_{12}x_2 + \cdots + b_{1k}x_k] + a_{22}[b_{21}x_1 + b_{22}x_2 + \cdots + b_{2k}x_k] + \cdots + a_{2n}[b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nk}x_k] \\ \vdots \\ a_{m1}[b_{11}x_1 + b_{12}x_2 + \cdots + b_{1k}x_k] + a_{m2}[b_{21}x_1 + b_{22}x_2 + \cdots + b_{2k}x_k] + \cdots + a_{mn}[b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nk}x_k] \end{pmatrix}$$

$$= \begin{pmatrix} [a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}]x_1 + [a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}]x_2 + \cdots + [a_{11}b_{1k} + a_{12}b_{2k} + \cdots + a_{1n}b_{nk}]x_k \\ [a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1}]x_1 + [a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}]x_2 + \cdots + [a_{21}b_{1k} + a_{22}b_{2k} + \cdots + a_{2n}b_{nk}]x_k \\ \vdots \\ [a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1}]x_1 + [a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2}]x_2 + \cdots + [a_{m1}b_{1k} + a_{m2}b_{2k} + \cdots + a_{mn}b_{nk}]x_k \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1k} + a_{12}b_{2k} + \cdots + a_{1n}b_{nk} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1k} + a_{22}b_{2k} + \cdots + a_{2n}b_{nk} \\ \vdots & & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1k} + a_{m2}b_{2k} + \cdots + a_{mn}b_{nk} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$$

We see that AB is a matrix of the size $m \times k$ as was to be expected since the composition function goes from $\mathbb{R}^k$ to $\mathbb{R}^m$. The component $j\ell$ of the new matrix (the entry in lines j and column ℓ) is

$$c_{j\ell} = \sum_{h=1}^{n} a_{jh}b_{h\ell}.$$

So in order to calculate this entry we need from A only its jth row and from B we only need its ℓth column and we multiply them component by component. You can memorise this again as "row by column", more precisely:

$$c_{j\ell} = \text{component in row } j \text{ and column } \ell \text{ of } AB = (\text{row } j \text{ of } A) \times (\text{column } \ell \text{ of } B) \tag{3.20}$$

as in the case of multiplication of a vector by a matrix. Actually, a vector in $\mathbb{R}^n$ can be seen as an $n \times 1$ matrix (a matrix with n rows and one column).

$$AB = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}\begin{pmatrix} b_{11} & \dots & b_{1\ell} & \dots & b_{1k} \\ b_{j1} & \dots & b_{2\ell} & \dots & b_{jk} \\ \vdots & \dots & \vdots & & \vdots \\ \vdots & \dots & \vdots & & \vdots \\ b_{n1} & \dots & b_{n2} & \dots & b_{nk} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1\ell} & \dots & c_{1k} \\ \vdots & & \vdots & & \vdots \\ c_{j1} & \dots & c_{j\ell} & \dots & c_{jk} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{m2} & \dots & c_{mk} \end{pmatrix}$$

with $c_{j\ell} = a_{j1}b_{1\ell} + a_{j2}b_{2\ell} + \cdots + a_{jn}b_{n\ell}$.

Example 3.29. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 8 & 6 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 1 & 2 & 3 \\ -2 & 0 & 1 & 4 \\ 2 & 6 & -3 & 0 \end{pmatrix}$. Then

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 & 3 \\ 8 & 6 & 4 \end{pmatrix} \begin{pmatrix} 7 & 1 & 2 & 3 \\ -2 & 0 & 1 & 4 \\ 2 & 6 & -3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1\cdot 7 + 2\cdot 2 + 3\cdot 2 & 1\cdot 1 + 2\cdot 0 + 3\cdot 6 & 1\cdot 2 + 2\cdot 1 + 3\cdot -3 & 1\cdot 3 + 2\cdot 4 + 3\cdot 0 \\ 8\cdot 7 + 6\cdot 2 + 4\cdot 2 & 8\cdot 1 + 6\cdot 0 + 4\cdot 6 & 8\cdot 2 + 6\cdot 1 + 4\cdot -3 & 8\cdot 3 + 6\cdot 4 + 4\cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 17 & 19 & -5 & 11 \\ 76 & 32 & 10 & 48 \end{pmatrix}. \end{aligned}$$

Let us see some properties of the algebraic operations for matrices that we just introduced.

Proposition 3.30. *Let $A \in M(m \times n)$, $B, C \in M(k \times m)$, $R, T \in M(n \times k)$ and $S \in M(k \times \ell)$. Then the following is true.*

(i) **Associativity of the matrix product:** *$A(RS) = A(RS)$.*

(ii) **Distributivity:** *$A(S + T) = AS + AT$ and $(B + C)A = BA + CA$.*

Proof. The claims of the proposition can be proved by straightforward calculations. $\square$

Prove Proposition 3.30.

Very important remark.

The matrix multiplication is not commutative, that is, in general

$$AB \neq BA.$$

That matrix multiplication is not commutative is to be expected since it is the composition of two functions (think of functions that you know from your calculus classes. For example, it does make a difference if you first square a variable and then take the arctan or if you first calculate its arctan and then square the result).

Let us see an example. Let B be the matrix from Example 3.25 and C be the matrix from Example 3.26. Recall that B represents the orthogonal projection onto the y-axis and that C represents counterclockwise rotation by $90°$. If we take $\vec{e}_1$ (the unit vector in x-direction), and we first rotate and then project, we get the vector $\vec{e}_2$. If however we project first and rotate then, we get $\vec{0}$. That means, $BC\vec{e}_1 \neq CB\vec{e}_1$, therefore $BC \neq CB$. Let us calculate the products:

$$BC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{first rotation, then projection,}$$

$$CB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad \text{first projection, then rotation.}$$

Let A be the matrix from Example 3.24, B be the matrix from Example 3.25 and C the matrix from Example 3.26. Verify that $AB \neq BA$ and $AC \neq CA$ and understand this result geometrically by following for example where the unit vectors get mapped to.

Note also that usually, when AB is defined, the expression BA is not defined because in general the number of columns of B will be different from the number of rows of A.

We finish this section with the definition of the so-called identity matrix.

Definition 3.31. Let $n \in \mathbb{N}$. Then the $n \times n$ identity matrix is the matrix which has 1s on its diagonal and has zero everywhere else:

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & 1 & \\ & & & & 1 \end{pmatrix}. \tag{3.21}$$

As notation for the identity matrix, the following symbols are used in the literature: E_n, id_n, Id_n, I_n, $\mathbf{1_n}$, $\mathbb{1}_n$. The subscript n can be omitted if it is clear.

Remark 3.32. It can be easily verified that

$$A \,\mathrm{id}_n = A, \quad \mathrm{id}_n B = B, \quad \mathrm{id}_n \vec{x} = \vec{x}$$

for every $A \in M(m \times n)$, for every $B \in M(n \times k)$ and for every $\vec{x} \in \mathbb{R}^n$.

You should now have understood

- what the sum and the composition of two matrices is and where the formulas come from,
- why the composition of matrices is not commutative,
- that $M(m \times n)$ is a vector space,
- ...

You should now be able to

- calculate the sum and product (composition) of two matrices,
- ...

3.5 Inverses of matrices

We will give two motivations why we are interested in inverses of matrices before we give the formal definition.

Inverse of a matrix as a function

The inverse of a given matrix is a matrix that "undoes" what the original matrix did. We will review the matrices from the Examples 3.24, 3.25 and 3.26.

- Assume we are given the matrix A from Example 3.24 which represents reflection on the x-axis and we want to find a matrix that restores a vector after we applied A to it. Clearly, we have to reflect again on the x-axis: reflecting an arbitrary vector $\vec{x}$ twice on the x-axis leaves the vector where it was. Let us check:

  $$AA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_2 \,.$$

  That means that for every $\vec{x} \in \mathbb{R}^2$, we have that $A^2\vec{x} = \vec{x}$, hence A is its own inverse.

- Assume we are given the matrix C from Example 3.26 which represents counterclockwise rotation by $90°$ and we want to find a matrix that restores a vector after we applied C to it. Clearly, we have to rotate clockwise by $90°$. Let us assume that there exists a matrix which represents this rotation and let us call it $C_{-90°}$. By Remark 3.18 it is enough to know how it acts on $\vec{e}_1$ and $\vec{e}_2$ in order to write it down. Clearly $C_{-90°}\vec{e}_1 = -\vec{e}_2$ and $C_{-90°}\vec{e}_2 = \vec{e}_1$, hence $C_{-90°} = (-\vec{e}_2|\vec{e}_1)$.

  Let us check:

  $$C_{-90°}C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_2$$

  and

  $$CC_{-90°} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_2$$

  which was to be expected because rotating first $90°$ clockwise and then $90°$ counterclockwise, leaves any vector where it is.

- Assume we are given the matrix B from Example 3.25 which represents projection onto the y-axis. In this case, we cannot restore a vector $\vec{x}$ after we projected it onto the y-axis. For example, if we know that $B\vec{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, then $\vec{x}$ could have been $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ or any other vector in $\mathbb{R}^2$ whose second component is equal to 2. This shows that B does not have an inverse.

Inverse of a matrix for solving a linear system

Let us consider the following situation. A grocery sells two different packages of fruits. Type A contains 1 peach and 3 mangos and type B contains 2 peaches and 1 mango. We can ask two different type of questions:

(i) Given a certain number of packages of type A and of type B, how many peaches and how many mangos do we get?

(ii) How many packages of each type do we need in order to obtain a given number of peaches and mangos?

The first question is quite easy to answer. Let us write down the information that we are given. If

$$\begin{aligned} a &= \text{number of packages of type } A, & p &= \text{number of peaches} \\ b &= \text{number of packages of type } B, & m &= \text{number of mangos.} \end{aligned}$$

then

$$\begin{aligned} p &= 1a + 2b \\ m &= 3a + 1b. \end{aligned} \tag{3.22}$$

Using vectors and matrices, we can rewrite this as

$$\begin{pmatrix} p \\ m \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$. Then the above becomes simply

$$\begin{pmatrix} p \\ m \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.23}$$

If we know a and b (that is, we know how many packages of each type we bought), then we can find the values of p and m by simply evaluating $A\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ which is relatively easy.

Example 3.33. Assume that we buy 1 package of type A and 3 packages of type B, then we calculate

$$\begin{pmatrix} p \\ m \end{pmatrix} = A \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix},$$

which shows that we have 9 peaches and 7 mangos.

If on the other hand, we know p and m and we are asked find a and b such that (3.22) holds, we have to solve a linear system which is much more cumbersome. Of course, we can solve (3.23) using the Gauß or Gauß-Jordan elimination process, but if we were asked to do this for several pairs p and m, then it would become long quickly. However, if we had a matrix A' such that $A'A = \mathrm{id}_2$, then this task would be quite easy since in this case we could manipulate (3.23) as follows:

$$\begin{pmatrix} p \\ m \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix} \qquad \Longrightarrow \qquad A' \begin{pmatrix} p \\ m \end{pmatrix} = A'A \begin{pmatrix} a \\ b \end{pmatrix} = \mathrm{id}_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

If in addition we knew that $AA' = \mathrm{id}_2$, then we have that

$$\begin{pmatrix} p \\ m \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix} \qquad \Longleftrightarrow \qquad A' \begin{pmatrix} p \\ m \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.24}$$

The task to find a and b again reduces to perform a matrix multiplication. The matrix A', if it exists, is called the *inverse of* A and we will dedicate the rest of this section to give criteria for its existence, investigate its properties and give a recipe for finding it.

Example 3.34.

Exercise. Check that $A' = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$ satisfies $A'A = \mathrm{id}_2$.

Assume that we want to buy 5 peaches and 5 mangos. Then we calculate

$$\begin{pmatrix} a \\ b \end{pmatrix} = A' \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which shows that we have to by 1 package of type A and 2 packages of type B.

Now let us give the precise definition of the inverse of a matrix.

Definition 3.35. A matrix $A \in M(n \times n)$ is called *invertible* if there exists a matrix $A' \in M(n \times n)$ such that

$$AA' = \mathrm{id}_n \qquad \text{and} \qquad A'A = \mathrm{id}_n$$

In this case A' is called the *inverse of* A and it is denoted by A^{-1}. If A is not invertible then it is called *non-invertible* or *singular*.

The reason why in the definition we only admit square matrices (matrices with the same number or rows and columns) is explained in the following remark.

Remark 3.36. (i) Let $A \in M(m\times n)$ and assume that there is a matrix B such that $BA = \mathrm{id}_n$. This means that if for some $\vec{b} \in \mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has a solution, then it is unique because

$$A\vec{x} = \vec{b} \quad \Longrightarrow \quad BA\vec{x} = B\vec{b} \quad \Longrightarrow \quad \vec{x} = B\vec{b}.$$

From the above it is clear that $A \in M(m \times n)$ can have an inverse only if for every $\vec{b} \in \mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has *at most one solution*. We know that if A has more columns than rows, then the number of columns will be larger than the number of pivots. Therefore, $A\vec{x} = \vec{b}$ has either no or infinitely many solutions (see Theorem 3.7). Hence a matrix A with more columns than rows cannot have an inverse.

(ii) Again, let $A \in M(m \times n)$ and assume that there is a matrix B such that $AB = \mathrm{id}_m$. This means that for every $\vec{b} \in \mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ is solved by $\vec{x} = B\vec{b}$ because

$$\mathrm{id}_m \vec{b} = \vec{b} \quad \Longrightarrow \quad AB\vec{b} = \vec{b} \quad \Longrightarrow \quad A(B\vec{b}) = \vec{b}.$$

From the above it is clear that $A \in M(m\times n)$ can have an inverse only if for every $\vec{b} \in \mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has *at least one solution*. Assume that A has more rows than columns. If we apply the Gauß elimination process to the augmented matrix $A|\vec{b})$ then hen the last row of the row-echelon form has to be $(0 \ \cdots \ 0|\beta_m)$. If we chose $\vec{b}$ such that after the reduction $\beta_m \neq 0$, then $A\vec{x} = \vec{b}$ does not have a solution. Such a $\vec{b}$ is easy to find: We only need to take $\vec{e}_m$ (the mth unit vector) and do the steps from the Gauß elimination backwards. If we take this vector as right hand side of our system, then the last row after the reduction will be $(0 \ \dots \ 0|1)$. Therefore, a matrix A with more rows than columns cannot have an inverse because there will always be some $\vec{b}$ such that the equation $A\vec{x} = \vec{b}$ has no solution.

In conlusion we showed that we must have $m = n$ if A ought to have an inverse matrix.

If $A \in M(m \times n)$ with $n \neq m$, then it does not make sense to speak of an inverse of A as explained above. However, we can define the left inverse and the right inverse.

Definition 3.37. Let $A \in M(m \times n)$.

(i) A matrix C is called a *left inverse* of A if $CA = \mathrm{id}_n$.

(ii) A matrix D is called a *right inverse* of A if $AD = \mathrm{id}_m$.

Note that C and D must be $n \times m$ matrices. The following examples show that the left- and right inverses do not need to exist, and if they do, they are not unique.

Examples 3.38. (i) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has neither left- nor right inverse.

(ii) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has no left inverse and has right inverse $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. In fact, for every $x, y \in \mathbb{R}$ the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x & y \end{pmatrix}$ is a right inverse of A.

(iii) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no right inverse and has left inverse $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. In fact, for every $x, y \in \mathbb{R}$ the matrix $\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \end{pmatrix}$ is a left inverse of A.

Remark 3.39. We will show in Theorem 3.44 that a matrix $A \in M(n \times n)$ is invertible if and only if it has a left- and a right inverse.

Examples 3.40.

- From the examples at the beginning of this section we have:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad A^{-1} = A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad C^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad B \text{ is not invertible.}$$

- Let $A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Then we can easily guess that $A^{-1} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$ is an inverse of A. It is easy to check that the product of these matrices gives id_4.
- Let $A \in M(n \times n)$ and assume that the kth row of A consists of only zeros. Then A is not invertible because for any matrix $B \in M(n \times n)$, the kth row of the product matrix AB will be zero, no matter how we choose B. So there is no matrix B such that $AB = \mathrm{id}_n$.
- Let $A \in M(n \times n)$ and assume that the kth column of A consists of only zeros. Then A is not invertible because for any matrix $B \in M(n \times n)$, the kth column of the product matrix BA will be zero, no matter how we choose B. So there is no matrix B such that $BA = \mathrm{id}_n$.

Now let us prove some theorems about the inverse matrices. Recall that $A \in M(n \times n)$ is invertible if and only if there exists a matrix $B \in M(n \times n)$ such that $AB = BA = \mathrm{id}_n$.
First we will show that the inverse matrix, if it exists, is unique.

Theorem 3.41. *Let $A, B \in M(n \times n)$.*

(i) *If A is invertible, then its inverse is unique.*

(ii) *If A is invertible, then its inverse A^{-1} is invertible and its inverse is A.*

(iii) *If A and B are invertible, then their product AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof. (i) Assume that A is invertible and that A' and A'' are inverses of A. Note that this means that

$$AA' = A'A = \mathrm{id}_n \quad \text{and} \quad AA'' = A''A = \mathrm{id}_n\,. \tag{3.25}$$

We have to show that $A' = A''$. This follows from (3.25) and from the associativity of the matrix multiplication because

$$A' = A' \operatorname{id}_n = A'(AA'') = (A'A)A'' = \operatorname{id}_n A'' = A''.$$

(ii) Assume that A is invertible, and let A^{-1} be its inverse. In order to show that A^{-1} is invertible, we need a matrix C such that $CA^{-1} = A^{-1}C = \operatorname{id}_n$. This matrix C is then the inverse of A^{-1}. Clearly, $C = A$ does the trick. Therefore A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(iii) Assume that A and B are invertible. In order to show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$, we only need to verify that $B^{-1}A^{-1}(AB) = (AB)B^{-1}A^{-1} = \operatorname{id}_n$. We see that this is true using the associativity of the matrix product:

$$\begin{aligned} B^{-1}A^{-1}(AB) &= B^{-1}(A^{-1}A)B = B^{-1} \operatorname{id}_n B = B^{-1}B = \operatorname{id}_n, \\ (AB)B^{-1}A^{-1} &= A(BB^{-1})A^{-1} = A^{-1} \operatorname{id}_n A = A^{-1}A = \operatorname{id}_n . \end{aligned}$$ □

Note that in the proof we guessed the formula for $(AB)^{-1}$ and then we verified that it indeed is the inverse of AB. We can also calculate it as follows. Assume that C is a left inverse of AB. Then

$$C(AB) = \operatorname{id}_n \iff CAB = \operatorname{id}_n \iff CA = \operatorname{id}_n B^{-1} = B^{-1} \iff C = B^{-1}A^{-1}$$

If D is a right inverse of AB then

$$(AB)D = \operatorname{id}_n \iff ABD = \operatorname{id}_n \iff BD = A^{-1} \operatorname{id}_n = A^{-1} \iff D = B^{-1}A^{-1}.$$

Since $C = D$, this is the inverse of AB.

Remark 3.42. In general, the sum of invertible matrices is not invertible. For example, both id_n and $-\operatorname{id}_n$ are invertible, but their sum however is the zero matrix which is not invertible.

Theorem 3.43 in the next section will show us how to find the inverse of a invertible matrix; see in particular the section on page 102.

You should now have understood

- what invertibility of a matrix means and why it does not make sense to speak of the invertibility of a matrix which is not a square matrix,
- that invertibility of matrix of $n \times n$-matrix is equivalent to the fact that for every $\vec{b} \in \mathbb{R}^m$ the associated linear system $A\vec{x} = \vec{b}$ has exactly one solution.
- ...

You should now be able to

- guess the inverse of simple invertible matrices, for example of matrices which have a clear geometric interpretation, or of diagonal matrices,
- verify if two given matrices are inverse to each other,
- give examples of invertible and of non-invertible matrices,
- ...

3.6 Matrices and linear systems

Let us recall from Theorem 3.7:

For $A \in M(m \times n)$ and $\vec{b} \in \mathbb{R}^m$ consider the equation

$$A\vec{x} = \vec{b}. \tag{3.26}$$

Then the following is true:

(1)	Equation (3.26) has no solution.	$\iff$	The reduced row echelon form of the augmented system $(A\|\vec{b})$ has a row of the form $(0 \cdots 0\|\beta)$ with some $\beta \neq 0$.
(2)	Equation (3.26) has at least one solution.	$\iff$	The reduced row echelon form of the augmented system $(A\|\vec{b})$ has no row of the form $(0 \cdots 0\|\beta)$ with some $\beta \neq 0$.

In case (2), we have the following two sub-cases:

(2.1)	Equation (3.26) has exactly one solution.	$\iff$	#pivots = #columns.
(2.2)	Equation (3.26) has infinitely many solutions.	$\iff$	#pivots < #columns.

Observe that the case (1), no solution, cannot occur for homogeneous systems.

Theorem 3.43. *Let $A \in M(n \times n)$. Then the following is equivalent:*

(i) *A is invertible.*

(ii) *For every $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has exactly one solution.*

(iii) *The equation $A\vec{x} = \vec{0}$ has exactly one solution.*

(iv) *Every row-reduced echelon form of A has n pivots.*

(v) *A is row-equivalent to id_n.*

We will complete this theorem with one more item in Chapter 4 (Theorem 4.11).

Proof. (ii) $\Rightarrow$ (iii) follows if we choose $\vec{b} = \vec{0}$.

(iii) $\Rightarrow$ (iv) If $A\vec{x} = \vec{0}$ has only one solution, then, by the case (2.1) above (or by Theorem 3.7(2.1)), the number of pivots is equal to n (the number of columns of A) in every row-reduced echelon form of A.

(iv) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (ii) follows from case (2.1) above (or by Theorem 3.7(2.1)) because no row-reduced form of A can have a row consisting of only zeros.

So far we have shown that (ii) - (v) are equivalent. Now we have to connect them to (i).

(i) $\Rightarrow$ (ii) Assume that A is invertible and let $\vec{b} \in \mathbb{R}^n$. Then $A\vec{x} = \vec{b} \iff \vec{x} = A^{-1}\vec{b}$ which shows existence and uniqueness of the solution.

(ii) $\Rightarrow$ (i) Assume that (ii) holds. We will construct A^{-1} as follows (this also tells us how we can calculate A^{-1} if it exists). Recall that we need a matrix C such that $AC = \mathrm{id}_n$. This C will then be our candidate for A^{-1} (we still would have to check that $CA = \mathrm{id}_n$). Let us denote the columns of C by $\vec{c}_j$ for $j = 1, \dots, n$, so that $C = (\vec{c}_1|\cdots|\vec{c}_n)$. Recall that the kth column of AC is A(kth column of C) and that the columns of id_n are exactly the unit vectors $\vec{e}_k$ (the vector with a 1 as kth component and zeros everywhere else). Then $AC = \mathrm{id}_n$ can be written as

$$(A\vec{c}_1|\cdots|A\vec{c}_n) = (\vec{e}_1|\cdots|\vec{e}_n).$$

By (ii) we know that equations of the form $A\vec{x} = \vec{e}_j$ have a unique solution. So we only need to set $\vec{c}_j$ = unique solution of the equation $A\vec{x} = \vec{e}_j$. With this choice we then have indeed that $AC = \mathrm{id}_n$.
It remains to prove that $CA = \mathrm{id}_n$. To this end, note that

$$A = \mathrm{id}_n A \quad \Longrightarrow A = ACA \quad \Longrightarrow A - ACA = 0 \quad \Longrightarrow A(\mathrm{id}_n - CA) = 0$$

This means that $A(\mathrm{id}_n - CA)\vec{x} = \vec{0}$ for every $\vec{x} \in \mathbb{R}^n$. Since by (ii) the equation $A\vec{y} = \vec{0}$ has the unique solution $\vec{y} = \vec{0}$, it follows that $(\mathrm{id}_n - CA)\vec{x} = \vec{0}$ for every $x \in \mathbb{R}^n$. But this means that $\vec{x} = CA\vec{x}$ for every $\vec{x}$, hence CA must be equal to id_n. $\square$

Theorem 3.44. *Let $A \in M(n \times n)$.*

(i) *If A has a left inverse C (that is, if $CA = \mathrm{id}_n$), then A is invertible and $A^{-1} = C$.*

(ii) *If A has a right inverse D (that is, if $AD = \mathrm{id}_n$), then A is invertible and $A^{-1} = D$.*

Proof. (i) By Theorem 3.43 it suffices to show that $A\vec{x} = \vec{0}$ has a the unique solution $\vec{0}$. So assume that $\vec{x} \in \mathbb{R}^n$ satisfies $A\vec{x} = \vec{0}$. Then $\vec{x} = \mathrm{id}_n \vec{x} = (CA)\vec{x} = C(A\vec{x}) = C\vec{0} = \vec{0}$. This shows that A is invertible. Moreover, $C = C(\mathrm{id}_n) = C(AA^{-1}) = (CA)A^{-1} = \mathrm{id}_n A^{-1} = A^{-1}$, hence $C = A^{-1}$.

(ii) By (i) applied to D, it follows that D has an inverse and that $D^{-1} = A$, so by Theorem 3.41 (ii), A is invertible and $A^{-1} = (D^{-1})^{-1} = D$. $\square$

Calculation of the inverse of a given square matrix

Let A be a square matrix. The proof of Theorem 3.43 tells us how to find its inverse if it exists. We only need to solve $A\vec{x} = \vec{e}_k$ for $k = 1, \dots, n$. This might be cumbersome and long, but we already know that if these equations have solutions, then we can find them with the Gauß-Jordan elimination. We only need to form the augmented matrix $(A|\vec{e}_k)$, apply row operations until we get to $(\mathrm{id}_n|\vec{c}_k)$. Then $\vec{c}_k$ is the solution of $A\vec{x} = \vec{e}_k$ and we obtain the matrix A^{-1} as the matrix whose columns are the vectors $\vec{c}_1, \dots, \vec{c}_n$. If it is not possible to reduce A to the identity matrix, then it is not invertible.
Note that the steps that we have to perform to reduce A to the identiy matrix, depend only on the coefficients in A and not on the right hand side. So we can calculate the n vectors $\vec{c}_1, \dots \vec{c}_n$ with only one (big) Gauß-Jordan elimination if we augment our given matrix A by the n vectors $\vec{e}_1, \dots, \vec{e}_n$. But the matrix $(\vec{e}_1|\cdots|\vec{e}_n)$ is nothing else than the identity matrix id_n. So if we take $(A|\mathrm{id}_n)$ and apply the Gauß-Jordan elimination and if we can reduce A to the identity matrix, then the columns on the right are the columns of the inverse matrix A^{-1}. If we cannot get to the identity matrix, then A is not invertible.

Examples 3.45. (i) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let us show that A is invertible by reducing the augmented matrix $(A|\,\mathrm{id}_2)$:

$$(A|\,\mathrm{id}_2) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3R_1 \to R_2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{-1/2 R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right).$$

Hence A is invertible and $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

We can check your result by calculating

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -2+3 & -4+4 \\ 3/2-3/2 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}$. Let us show that A is not invertible by reducing the augmented matrix $(A|\,\mathrm{id}_2)$:

$$(A|\,\mathrm{id}_2) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 + 2R_1 \to R_2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right).$$

Since there is a zero row in the left matrix, we conclude that A is not invertible.

(iii) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$. Let us show that A is invertible by reducing the augmented matrix $(A|\,\mathrm{id}_3)$:

$$(A|\,\mathrm{id}_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 5R_1 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{4R_2 + 3R_3 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 8 & 0 & -15 & 4 & 3 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right) \xrightarrow{4R_1 + R_3 \to R_1} \left(\begin{array}{ccc|ccc} 4 & 4 & 0 & -1 & 0 & 1 \\ 0 & 8 & 0 & -15 & 4 & 3 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{2R_1 - R_2 \to R_1} \left(\begin{array}{ccc|ccc} 8 & 0 & 0 & 13 & -4 & -1 \\ 0 & 8 & 0 & -15 & 4 & 3 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{2R_1 - R_2 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & 1/4 \end{array}\right)$$

Hence A is invertible and $A^{-1} = \begin{pmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & 1/4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 13 & -4 & -1 \\ -15 & 4 & 3 \\ 10 & 0 & 2 \end{pmatrix}$

We can check your result by calculating

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}\begin{pmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & 1/4 \end{pmatrix} = \dots = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix} = \dots = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Special case: Inverse of a 2×2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We already know that A is invertible if and only if its associated homogeneous linear system has exactly one solution. By Theorem 1.11 this is the case if and only if $\det A \neq 0$. Recall that $\det A = ad - bc$. So let us assume here that $\det A \neq 0$.

$\boxed{\text{Case 1. } a \neq 0.}$

$$(A|\,\mathrm{id}_2) = \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow{aR_2 - cR_1 \to R_2} \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{b}{ad-bc}R_2 \to R_1} \left(\begin{array}{cc|cc} a & 0 & 1 - \frac{bc}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & ad-bc & -c & a \end{array}\right) = \left(\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & ad-bc & -c & a \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{b}{ad-bc}R_2 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right).$$

It follows that

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{3.27}$$

$\boxed{\text{Case 1. } a = 0.}$ Since $0 \neq \det A = ad - bc = bc$ in this case, it follows that $c \neq 0$ and calculations as above again lead to formula (3.27).

You should now have understood

- the relation between the invertibility of a square matrix A and the existence and uniqueness of solution of $A\vec{x} = \vec{b}$,
- that inverting a matrix is the same as solving a linear system,
- ...

You should now be able to

- calculate the inverse of a square matrix if it exists,
- use the inverse of a square matrix if it exists to solve the associated linear system,
- ...

3.7 The transpose of a matrix

Definition 3.46. Let $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in M(m \times n)$. Then its transpose A^t is the $n \times m$ matrix whose columns are the rows of A and whose rows are the columns of A, that is,

$$A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} \in M(n \times m).$$

If we denote $A^t = (\widetilde{a}_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$, then $\widetilde{a}_{ij} = a_{ji}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$.

Examples 3.47. The transposes of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \\ 3 & 2 & 3 \end{pmatrix}$$

are

$$A^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \quad C^t = \begin{pmatrix} 1 & 4 & 7 & 3 \\ 2 & 5 & 7 & 2 \\ 3 & 6 & 7 & 3 \end{pmatrix}.$$

Proposition 3.48. *Let $A \in M(m \times n)$. Then $(A^t)^t = A$.*

Proof. Clear. □

Theorem 3.49. *Let $A \in M(m \times n)$ and $B \in M(n \times k)$. Then $(AB)^t = B^t A^t$.*

Proof. Note that both $(AB)^t$ and $B^t A^t$ are $m \times k$ matrices. In order to show that they are equal, we only need to show that they are equal in every entry. Let $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k\}$.

Then

$$\begin{aligned}\text{component } ij \text{ of } (AB)^t &= \text{component } ji \text{ of } AB \\ &= [\text{row } j \text{ of } A] \times [\text{column } i \text{ of } B] \\ &= [\text{column } j \text{ of } A^t] \times [\text{row } i \text{ of } B^t] \\ &= [\text{row } i \text{ of } B^t] \times [\text{column } j \text{ of } A^t] \\ &= \text{component } ij \text{ of } B^tA^t.\end{aligned}$$ □

Theorem 3.50. *Let $A \in M(n \times n)$. Then A is invertible if and only if A^t is invertible. In this case, $(A^t)^{-1} = (A^{-1})^t$.*

Proof. Assume that A is invertible. Then $AA^{-1} = \mathrm{id}$. Taking the transpose on both sides, we find

$$\mathrm{id} = \mathrm{id}^t = (AA^{-1})^t = (A^{-1})^tA^t.$$

This shows that A^t is invertible and its inverse is $(A^{-1})^t$, see Theorem 3.44. Now assume that A^t is invertible. From what we just showed, it follows that then also its transpose $(A^t)^t = A$ is invertible. □

Next we show an important relation between transposition of a matrix and the inner product on $\mathbb{R}^n$.

Theorem 3.51. *Let $A \in M(m \times n)$.*

(i) *$\langle A\vec{x}\,, \vec{y}\rangle = \langle \vec{x}\,, A^t\vec{y}\rangle$ for all $\vec{x} \in \mathbb{R}^n$ and all $\vec{y} \in \mathbb{R}^m$.*

(ii) *If $\langle A\vec{x}\,, \vec{y}\rangle = \langle \vec{x}\,, B\vec{y}\rangle$ for all $\vec{x} \in \mathbb{R}^n$ and all $\vec{y} \in \mathbb{R}^m$, then $B = A^t$.*

Proof. Let $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ and $B = (b_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$.

(i) Observe that the kth component of $A\vec{x}$ is $(A\vec{x})_k = \sum_{j=1}^n a_{kj}x_j$. and that the ℓth coordinate of $B\vec{y}$ is $(A^t\vec{y})_\ell = \sum_{j=1}^m a_{j\ell}y_j$. Then

$$\langle A\vec{x}\,, \vec{y}\rangle = \sum_{k=1}^m (A\vec{x})_k y_k = \sum_{k=1}^m\sum_{j=1}^n a_{kj}x_jy_k = \sum_{j=1}^n\sum_{k=1}^m a_{kj}y_kx_j = \sum_{j=1}^n (A^t\vec{y})_jx_j = \langle \vec{x}\,, A^t\vec{y}\rangle.$$

(ii) We have to show: For all $i = 1,\dots,m$ and $j = 1,\dots,n$ we have that $a_{ij} = b_{ji}$. Take $\vec{x} = \vec{\mathrm{e}}_j \in \mathbb{R}^n$ and $\vec{y} = \vec{\mathrm{e}}_i \in \mathbb{R}^m$. If we take the inner product of $A\vec{\mathrm{e}}_j$ with $\vec{\mathrm{e}}_i$, then obtain the ith component of $A\vec{\mathrm{e}}_j$. Recall that $A\vec{\mathrm{e}}_j$ is the jth column of A, hence

$$\langle A\vec{\mathrm{e}}_j\,, \vec{\mathrm{e}}_i\rangle = a_{ij}.$$

Similarly if we take the inner product of $B\vec{\mathrm{e}}_i$ with $\vec{\mathrm{e}}_j$, then obtain the jth component of $B\vec{\mathrm{e}}_i$. Since $B\vec{\mathrm{e}}_i$ is the jth column of B it follows that

$$\langle \vec{\mathrm{e}}_j\,, B\vec{\mathrm{e}}_i\rangle = b_{ji}.$$

Since $\langle A\vec{\mathrm{e}}_j\,, \vec{\mathrm{e}}_i\rangle = \langle \vec{\mathrm{e}}_j\,, B\vec{\mathrm{e}}_i\rangle$ by assumption, it follows that $a_{ij} = b_{ji}$, hence $B = A^t$. □

Definition 3.52. Let $A = (a_{ij})_{i,j=1}^n \in M(n \times n)$ be a square matrix.

(i) A is called *upper triangular* if $a_{ij} = 0$ if $i > j$.

(ii) A is called *lower triangular* if $a_{ij} = 0$ if $i < j$.

(iii) A is called *diagonal* if $a_{ij} = 0$ if $i \neq j$. Diagonal matrices are sometimes denoted by $\operatorname{diag}(c_1, \dots, c_n)$ where the $c_1, \dots, c_n$ are the numbers on the diagonal of the matrix.

That means that for an upper triangular matrix all entries below the diagonal are cero, for a lower triangular matrix all entries above the diagonal are cero and for a diagonal matrix, all entries except the ones on the diagonal must be cero. These matrices look as follows:

$$\begin{pmatrix} a_{11} & & & \\ & a_{22} & & \huge{*} \\ & & \ddots & \\ \huge{0} & & & a_{nn} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \huge{0} \\ & & \ddots & \\ \huge{*} & & & a_{nn} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \huge{0} \\ & & \ddots & \\ \huge{0} & & & a_{nn} \end{pmatrix}$$

upper triangular matrix, lower triangular matrix, diagonal matrix $\operatorname{diag}(a_{11}, \dots, a_{nn})$.

Remark 3.53. A matrix is both upper and lower triangular if and only if it is diagonal.

Examples 3.54.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 2 & 4 & 2 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 0 & 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrices A, B, D, E are upper triangular, C, D, E are lower triangular, D, E are diagonal.

Definition 3.55. (i) A matrix $A \in M(n \times n)$ is called *symmetric* if $A^t = A$. The set of all symmetric $n \times n$ matrices is denoted by $M_{sym}(n \times n)$.

(ii) A matrix $A \in M(n \times n)$ is called *antisymmetric* if $A^t = -A$. The set of all symmetric $n \times n$ matrices is denoted by $M_{\mathrm{asym}}(n \times n)$.

Examples 3.56.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 2 & 5 \\ 4 & 5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 & -5 \\ -2 & 0 & -3 \\ 5 & 3 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

The matrices A and B are symmetric, C is antisymmetric and D is neither.

Clearly, every diagonal matrix is symmetric.

Exercise 3.57.

- Let $A \in M(n \times n)$. Show that $A + A^t$ is symmetric and that $A - A^t$ is antisymmetric.
- Show that every matrix $A \in M(n \times n)$ can be written as the sum of symmetric and an antisymmetric matrix.

Question 3.2

How many possibilities are there to express a given matrix $A \in M(n\times n)$ as sum of a symmetric and an antisymmetric matrix?

Exercise 3.58. Show that the diagonal entries of an antisymmetric matrix are 0.

You should now have understood

- why $(AB)^t = B^t A^t$,
- what the transpose of a matrix has to do with the inner product,
- …

You should now be able to

- calculate the transpose of a given matrix,
- check if a matrix is symmetric, antisymmetric or none,
- …

3.8 Elementary matrices

In this section we study three special types of matrices. They are called *elementary matrices*. Let us define them.

Definition 3.59. For $n \in \mathbb{N}$ we define the following matrices in $M(n \times n)$:

(i) $S_j(c) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ for $j = 1, \dots, n$ and $c \neq 0$.

(ii) $Q_{jk}(c) = \begin{array}{cc} \text{column } k & \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & c & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \text{row } j \\ \\ \\ \\ \end{array} \end{array}$ for $j, k = 1, \dots, n$ with $j \neq k$ and $c \in \mathbb{R}$. The number c is in row j and column k.

(iii) $$P_{jk}(c) = \begin{array}{cc} \begin{matrix} & & \text{col. } k & & \text{col. } j & & \end{matrix} & \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{matrix} \\ \\ \text{row } k \\ \\ \text{row } j \\ \\ \\ \end{matrix} \end{array}$$ for $j, k = 1, \dots, n$. This matrix is obtained from the identity matrix by swapping rows j and k (or, equivalently, by swapping columns j and k).

Examples 3.60. Let us see some examples for $n = 2$.

$$S_1(5) = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_{21}(3) = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad P_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Some examples for $n = 3$:

$$S_3(-2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad Q_{23}(4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{31} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad P_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us see how these matrices act on other $n \times n$ matrices. Let $A = (a_{ij})_{i,j=1}^n \in M(n \times n)$. We want to calculate EA where E is an elementary matrix.

- $$S_j(c)A = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ca_{j1} & ca_{j2} & \cdots & ca_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- $$Q_{jk}(c)A = \begin{pmatrix} 1 & & & \\ & \ddots & c & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} + ca_{k1} & a_{j2} + ca_{k2} & \cdots & a_{jn} + ca_{kn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- $$P_{jk}A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots\cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots\cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots\cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots\cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix}.$$

In summary, we see that

Proposition 3.61.
- *$S_j(c)$ multiplies the jth row of A by c.*
- *$Q_{jk}(c)$ sums c times the kth row to the jth row of A.*
- *P_{jk} swaps the kth and the jth row of A.*

These are exactly the row operations from the Gauß or Gauß-Jordan elimination! So we see that every row operation can be achieved by multiplying from the left by an appropriate elementary matrix.

Remark 3.62. The form of the elementary matrices is quite easy to remember if you recall that $E\,\mathrm{id}_n = E$ for every matrix E, in particular for an elementary matrix. So, if you want to remember e.g. how the 5×5 matrix looks like which sums 3 times the 2nd row to the 4th, just remember that this matrix is

$$E = E\,\mathrm{id}_5 = (\text{take } \mathrm{id}_5 \text{ and sum 3 times its 2nd row to its 4th row}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which is $Q_{42}(3)$.

Question 3.3

How do elementary matrices act on other matrices if we multiply them from the right?

Hint. There are two ways to find the answer. One is to carry out the matrix multiplication as we did on page 109. Or you could use that $AE = [(AE)^t]^t = [E^tA^t]^t$. If E is an elementary matrix, then so is E^t, see Proposition 3.64. Since you know how E^tA^t looks like, you can then deduce how its transpose looks like.

Since the action of an elementary matrix can be "undone" (since the corresponding row operation can be undone), we expect them to be invertible. The next proposition shows that they indeed are and that their inverse is again an elementary matrix of the same type.

Proposition 3.63. *Every elementary $n \times n$ matrix is invertible. More precisely, for $j, k = 1, \dots, n$ with $j \neq k$ the following holds:*

(i) $(S_j(c))^{-1} = S_j(c^{-1})$ *for* $c \neq 0$.

(ii) $(Q_{jk}(c))^{-1} = Q_{jk}(-c)$.

(iii) $(P_{jk})^{-1} = P_{jk}$.

Proof. Straightforward calculations. □

Show that Proposition 3.63 is true. Convince yourself that it is true using their interpretation as row operations.

Proposition 3.64. *The transpose of an elementary $n \times n$ matrix is again an elementary matrix. More precisely, for $j, k = 1, \dots, n$ with $j \neq k$ the following holds:*

(i) $(S_j(c))^t = S_j(c)$ *for* $c \neq 0$.

(ii) $(Q_{jk}(c))^t = Q_{kj}(c)$.

(iii) $(P_{jk})^t = P_{jk}$.

Proof. Straightforward calculations. □

Exercise 3.65. Show that $Q_{jk}(c) = S_k(c^{-1})Q_{jk}(1)S_k(c)$ for $c \neq 0$. Interpret the formuls in terms of row operations.

Exercise. Show that P_{jk} can be written as product of matrices of the form $Q_{jk}(c)$ and $S_j(c)$.

Let us come back to the relation of elementary matrices and the Gauß-Jordan elimination process.

Proposition 3.66. *Let $A \in M(n \times n)$ and let A' be a row echelon form of A. Then there exist elementary matrices $E_1, \dots, E_k$ such that*

$$A = E_1 E_2 \cdots E_k A'.$$

Proof. We know that we can arrive at A' by applying suitable row operations to A. By Proposition 3.61 they correspond to multiplication of A from the left by suitable elementary matrices $F_k, F_{k-1}, \dots, F_2, F_1$, that is

$$A' = F_k F_{k-1} \cdots F_2 F_1 A.$$

We know that all the F_j are invertible, hence their product is invertible and we obtain

$$A = [F_k F_{k-1} \cdots F_2 F_1]^{-1} A' = F_1^{-1} F_2^{-1} \cdots F_{k-1}^{-1} F_k^{-1} A'.$$

We know that the inverse of every elementary matrix F_j is again an elementary matrix, so if we set $E_j = F_j^{-1}$ for $j = 1, \dots, k$, the proposition is proved. □

Corollary 3.67. *Let $A \in M(n \times n)$. Then there exist elementary matrices $E_1, \dots, E_k$ and an upper triangular matrix U such that*

$$A = E_1 E_2 \cdots E_k U.$$

Proof. This follows immediately from Proposition 3.66 if we recall that every row reduced echelon form of A is an upper triangular matrix. $\square$

The next theorem shows that every invertible matrix is "composed" of elementary matrices.

Theorem 3.68. *Let $A \in M(n \times n)$. Then A is invertible if and only if it can be written as product of elementary matrices.*

Proof. Assume that A is invertible. Then the reduced row echelon form of A is id_n. Therefore, by Proposition 3.66, there exist elementary matrices $E_1, \dots, E_k$ such that $A = E_1 \cdots E_k \,\mathrm{id}_n = E_1 \cdots E_k$.

If, on the other hand, we know that A is the product of elementary matrices, say, $A = F_1 \cdots F_\ell$, then clearly A is invertible since each elementary matrix F_j is invertible and the product of invertible matrices is invertible. $\square$

We finish this section with an exercise where write an invertible 2×2 matrix as product of elementary matrices. Notice that there are infinitely many ways to write it as product of elementary matrices just as there are infinitely many ways of performing row reduction to get to the identity matrix.

Example 3.69. Write the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as product of elementary matrices.

Solution. We use the idea of the proof of Theorem 3.43: we apply the Gauß-Jordan elimination process and write the corresponding row transformations as elementary matrices.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow[Q_{21}(-3)]{R_2 \to R_2 - 3R_1} \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}}_{=Q_{21}(-3)A} \xrightarrow[Q_{12}(1)]{R_1 \to R_1 + R_2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}}_{=Q_{21}(1)Q_{21}(-3)A} \xrightarrow[S_2(-\frac{1}{2})]{R_2 \to -\frac{1}{2}R_2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=S_2(-\frac{1}{2})Q_{21}(1)Q_{21}(-3)A}$$

So we obtain that

$$\mathrm{id}_2 = S_2(-\tfrac{1}{2})Q_{21}(1)Q_{21}(-3)A. \tag{3.28}$$

Since the elementary matrices are invertible, we can solve for A and obtain

$$\begin{aligned} A &= [S_2(-\tfrac{1}{2})Q_{21}(1)Q_{21}(-3)]^{-1}\,\mathrm{id}_2 = [S_2(-\tfrac{1}{2})Q_{21}(1)Q_{21}(-3)]^{-1} \\ &= [Q_{21}(-3)]^{-1}[Q_{21}(1)]^{-1}[S_2(-\tfrac{1}{2})]^{-1} \\ &= Q_{21}(3)Q_{21}(-1)S_2(-2). \end{aligned}$$ $\diamond$

Note that from (3.28) we get the factorisation for A^{-1} for free. Clearly, we must have

$$A^{-1} = S_2(-\tfrac{1}{2})Q_{21}(1)Q_{21}(-3). \tag{3.29}$$

If we wanted to we could now use (3.29) to calculate A^{-1}. It is by no means a surprise that we actually get first the factorisation of A^{-1} because the Gauß-Jordan elimination leads to the inverse of A. So A^{-1} is the composition of the matrices which lead from A the identity matrix. (To get from the identity matrix to A, we need to reverse these steps.)

You should now have understood

- the relation of the elementary matrices with the Gauß-Jordan process,
- why a matrix is invertible if and only if it is the product of elementary matrices,
- ...

You should now be able to

- express an invertible matrix as product of elementary matrices,
- ...

3.9 Summary

Elementary row operations (= operations which lead to an equivalent system) for solving a linear system.

Elementary operation	Notation	Inverse Operation
① Swap rows j and k.	$R_j \leftrightarrow R_k$	$R_j \leftrightarrow R_k$
② Multiply row j by some $\lambda \in \mathbb{R} \setminus \{0\}$	$R_j \to \lambda R_k$	$R_j \to \frac{1}{\lambda} R_j$
③ Replace row k by the sum of row k and λ times R_j and keep row j unchanged.	$R_k \to R_k + \lambda R_j$	$R_k \to R_k - \lambda R_j$

On the solutions of a linear system.

- A linear system has either no, exactly one or infinitely many solutions.
- If the system is homogeneous, then it has either exactly one or infinitely many solutions. It always has at least one solution, namely the trivial one.
- The set of all solutions of a homogeneous linear equations is a vector space.
- The set of all solutions of a inhomogeneous linear equations is an affine vector space.

For $A \in M(m \times n)$ and $\vec{b} \in \mathbb{R}^m$ consider the equation $A\vec{x} = \vec{b}$. Then the following is true:

(1) No solution $\iff$ The reduced row echelon form of the augmented system $(A|\vec{b})$ has a row of the form $(0 \cdots 0|\beta)$ with some $\beta \neq 0$.

(2) At least one solution $\iff$ The reduced row echelon form of the augmented system $(A|\vec{b})$ has no row of the form $(0 \cdots 0|\beta)$ with some $\beta \neq 0$.

In case (2), we have the following two sub-cases:

(2.1) Exactly one solution $\iff$ # pivots= # columns.

(2.2) Infinitely many solutions $\iff$ # pivots< # columns.

Algebra with matrices and vectors

A matrix $A \in M(m \times n)$ can be viewed as a function $A : \mathbb{R}^n \to \mathbb{R}^m$.

Definition.

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ a_{12}x_2 + \ \cdots + a_{1n}x_n \\ a_{21}x_1 + \ a_{22}x_2 + \ \cdots + a_{2n}x_n \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix},$$

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b12 & \cdots & a_{1n} + b1n \\ a_{21} + b_{21} & a_{22} + b22 & \cdots & a_{2n} + b2n \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + bm2 & \cdots & a_{mn} + bmn \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} AB &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1k} + a_{12}b_{2k} + \cdots + a_{1n}b_{nk} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & \cdots & a_{21}b_{1k} + a_{22}b_{2k} + \cdots + a_{2n}b_{nk} \\ \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1k} + a_{m2}b_{2k} + \cdots + a_{mn}b_{nk} \end{pmatrix} \\ &= (c_{j\ell})_{j\ell} \end{aligned}$$

with

$$c_{j\ell} = \sum_{h=1}^{n} a_{jh}b_{h\ell}.$$

- Sum of matrices: componentwise,
- Product of matrices with vector or matrix with matrix: *"multiply row by column"*.

Properties. Let $A_1, A_2, A_2 \in M(m \times n)$, $B \in M(n \times k)$, $C \in M(k \times r)$ be matrices, $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\vec{z} \in \mathbb{R}^k$ and $c \in \mathbb{K}$.

- $A_1 + A_2 = A_2 + A_1$,
- $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$,

- $(AB)C = A(BC)$,
- in general, $AB \neq BA$,
- $A(\vec{x} + c\vec{y}) = A\vec{x} + cA\vec{y}$,
- $(A_1 + cA_2)\vec{x} = A_1\vec{x} + cA_2\vec{x}$,
- $(AB)\vec{z} = A(B\vec{z})$,

Transposition of matrices

Let $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1\dots,n}} \in M(m \times n)$. Then its *transpose* is the matrix $A^t = (\widetilde{a}_{ij})_{\substack{i=1,\dots,n\\ j=1\dots,m}} \in M(n \times m)$ with $\widetilde{a}_{ij} = a_{ji}$.
For $A, B \in M(m \times n)$ and $C \in M(n \times k)$ we have

- $(A^t)^t = A$,
- $(A + B)^t = A^t + B^t$,
- $(AC)^t = C^t A^t$,
- $\langle A\vec{x}\,, \vec{y}\rangle = \langle \vec{x}\,, A^t\vec{y}\rangle$ for all $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$.

A matrix A is called *symmetric* if $A^t = A$ and *antisymmetric* if $A^t = -A$. Note that only square matrices can be symmetric.
A matrix $A = (a_{ij})_{i,j=1,\dots,n} \in M(n \times n)$ is called

- *upper triangular* if $a_{ij} = 0$ whenever $i > j$,
- *lower triangular* if $a_{ij} = 0$ whenever $i < j$,
- *diagonal* if $a_{ij} = 0$ whenever $i \neq j$.

Clearly, a matrix is diagonal if and only if it is upper and lower triangular. The transpose of an upper triangular matrix is lower triangular and vice verse. Every diagonal matrix is symmetric.

Invertibility of matrices

A matrix $A \in M(n \times n)$ is called *invertible* if a there exists a matrix $B \in M(n \times n)$ such that $AB = BA = \mathrm{id}_n$. In this case B is called the *inverse* of A and it is denoted by A^{-1}. If A is not invertible, then it is called *singular*.

- The inverse of an invertible matrix A is unique.
- If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$.
- If A is invertible, then so is A^t and $(A^t)^{-1} = (A^{-1})^t$.
- If A and B are invertible, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem. *Let $A \in M(n \times n)$. Then the following is equivalent:*

(i) *A is invertible.*

(ii) *For every $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has exactly one solution.*

(iii) *The equation $A\vec{x} = \vec{0}$ has exactly one solution.*

(iv) *Every row-reduced echelon form of A has n pivots.*

(v) *A is row-equivalent to id_n.*

Calculation of A^{-1} using Gauß-Jordan elimination

Let $A \in M(n \times n)$. Form the augmented matrix $(A|\,\mathrm{id}_n)$ and use the Gauß-Jordan elimination to reduce A to its reduced row echelon form A': $(A|\,\mathrm{id}_n) \to \cdots \to (A'|B)$. If $A' = \mathrm{id}_n$, then A is invertible and $A^{-1} = B$. If $A' \neq \mathrm{id}_n$, then A is not invertible.

Inverse of a 2×2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det A = ad - bc$. If $\det A = 0$, then A is not invertible. If $\det A \neq 0$, then A is invertible and $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Elementary matrices

We have the following three types of elementary matrices:

- $S_j(c) = (s_{ik})_{i,k=1\dots,n}$ for $c \neq 0$ where $s_{ik} = 0$ if $i \neq k$, $s_{kk} = 1$ for $k \neq j$ and $s_{jj} = c$,
- $Q_{jk}(c) = (q_{i\ell})_{i,\ell=1\dots,n}$ for $j \neq k$, where $q_{jk} = c$, $q_{\ell\ell} = 1$ for all $\ell = 1, \dots, n$ and all other coefficients equal to zero,
- $P_{jk} = (p_{i\ell})_{i,\ell=1\dots,n}$ for $j \neq k$, where $p_{\ell\ell} = 1$ for all $\ell \in \{1, \dots, n\} \setminus \{j, k\}$, $p_{jk} = p_{kj} = 1$ and all other coefficients equal to zero.

$$S_j(c) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad Q_{jk}(c) = \begin{array}{cc} \begin{array}{ccc} & & \text{column } k \end{array} & \\ \begin{pmatrix} 1 & & c \\ & \ddots & \\ & & 1 \end{pmatrix} & \begin{array}{l} \text{row } j \\ \\ \\ \end{array} \end{array}, \quad P_{jk} = \begin{array}{cc} \begin{array}{ccccc} & \text{col. } k & & \text{col. } k & \end{array} & \\ \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \text{row } k \\ \\ \text{row } j \\ \\ \end{array} \end{array}.$$

Relation Elementary matrix - Elemntary row operation

Elementary matrix	**Elementary operation**	**Notation**
P_{jk}	Swap rows j with row k	$R_j \leftrightarrow R_k$
$S_j(c),\ c \neq 0$	Multiply row j by c	$R_j \to cR_k$
$Q_{jk}(c)$	Sum c times row k to row j	$R_k \to R_k + cR_j$

3.10 Exercises

1. Vuelva al Capítulo 1 y haga los ejercicios otra vez utilizando los conocimientos adquiridos en este capítulo.

2. Encuentre un polinomio de grado a lo más 2 que pase por los puntos $(-1, -6)$, $(1, 0)$, $(2, 0)$. ¿Cuántos tales polinomios hay?

3. (a) ¿Existe un polynomio de grado 1 que pase por los tres puntos del Ejercicio 2? ¿Cuántos tales polinomios hay?

   (b) ¿Existe un polynomio de grado 3 que pase por los tres puntos del Ejercicio 2? ¿Cuántos tales polinomios hay? Dé por lo menos dos polinomios de grado 3.

4. Enuentre las fracciones parciales de $\dfrac{2x^2 - 4x + 14}{x(x-2)^2}$.

5. Enuentre un sistema lineal 2×3 cuya solución sea

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

   ¿Existen sistemas 3×3 y 4×3 con las mismas solucioes? Dé ejemplos o diga por qué no existen. ¿Existe un sistema 4×3 con las mismas solucioes? Dé ejemplos o diga por qué no existen.

6. Enuentre un sistema lineal 4×4 cuya solución sea

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + s \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix} + t \begin{pmatrix} 7 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \qquad s, t \in \mathbb{R}.$$

7. Considere el sistema lineal

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 3x_1 - x_2 + 2x_3 &= b_2 \\ 4x_1 + x_2 + x_3 &= b_3. \end{aligned}$$

Encuentre todo los posibles b_1, b_2, b_3, o diga por qué no hay, para que el sistema tenga

(a) exactamente una solución,
(b) ninguna solución,
(c) infinitas soluciones.

8. Calcule todas las posibles combinaciones (matriz)(vector):

$$A = \begin{pmatrix} 1 & 0 & 3 & 6 \\ 4 & 8 & 1 & 0 \\ 1 & 4 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 4 & 8 \\ 1 & 4 \\ 5 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & 6 \\ 4 & 1 & 0 \\ 1 & 4 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 2 & 7 \\ 3 & -2 & 2 \end{pmatrix},$$

$$\vec{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 6 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 5 \\ -1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 4 \\ 3 \\ 5 \\ -1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ -2 \\ \pi \end{pmatrix}.$$

9. Sean $A = \begin{pmatrix} 2 & 6 & -1 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{pmatrix}$ y $\vec{b} = \begin{pmatrix} 17 \\ 6 \\ 4 \end{pmatrix}$. Encuentre todos los vectores $\vec{x} \in \mathbb{R}^3$ tal que $A\vec{x} = \vec{b}$.

10. Sea $M = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

(a) Demuestre que no existe $\vec{y} \neq 0$ tal que $M\vec{y} \perp \vec{y}$.
(b) Encuentre todos los vectores $\vec{x} \neq 0$ tal que $M\vec{x} \| \vec{x}$. Para cada tal $\vec{x}$, encuentre $\lambda \in \mathbb{R}$ tal que $M\vec{x} = \lambda\vec{x}$.

11. Calcule todas las posibles combinaciones (matriz)(matriz):

$$A = \begin{pmatrix} 1 & 0 & 3 & 6 \\ 4 & 8 & 1 & 0 \\ 1 & 4 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 4 & 8 \\ 1 & 4 \\ 5 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & 6 \\ 4 & 1 & 0 \\ 1 & 4 & 3 \end{pmatrix},$$

$$D = \begin{pmatrix} -1 & 2 & 7 \\ 3 & -2 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix}.$$

12. Determine si las matrices son invertibles. Si lo son, encuentre su matriz inversa.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} -14 & 21 \\ 12 & -18 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 3 & 6 \\ 4 & 1 & 0 \\ 1 & 4 & 3 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 1 & 5 \\ 3 & 5 & 11 \end{pmatrix}.$$

13. De las siguientes matrices determine si son invertibles. Si lo son, encuentre su matriz inversa.

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 2 \\ 8 & 6 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 10 \\ 6 & 15 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 3 & 6 \\ 4 & 1 & 0 \\ 1 & 4 & 3 \end{pmatrix}.$$

14. Una tienda vende dos tipos de cajitas de dulces:
Tipo A contiene 1 chocolate y 3 mentas, Tipo B contiene 2 chocolates y 1 menta.

(a) Dé una ecuación de la forma $A\vec{x} = \vec{b}$ que describe lo de arriba. Diga que siginifican los vectores $\vec{x}$ y $\vec{b}$.

(b) Calcule, usando el resultado de (a), cuantos chocolates y cuantas mentas contienen:

(i) 1 caja de tipo A y 3 de tipo B,
(ii) 4 cajas de tipo A y 2 de tipo B,
(iii) 2 caja de tipo A y 6 de tipo B,
(iv) 3 cajas de tipo A y 5 de tipo B.

(c) Determine si es posible conseguir

(i) 5 chocolates y 15 mentas,
(ii) 2 chocolates y 11 mentas,
(iii) 21 chocolates y 23 mentas,
(iv) 14 chocolates y 19 mentas.

comprando cajitas de dulces en la tienda. Si es posible, diga cuántos de cada tipo se necesitan.

15. Sea $A_k = \begin{pmatrix} 1 & 3 \\ 2 & k \end{pmatrix}$ y considere la ecuación

$$A_k\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (*)$$

(a) Encuentre todos los $k \in \mathbb{R}$ tal que $(*)$ tiene exactamente una solución para $\vec{x}$.

(b) Encuentre todos los $k \in \mathbb{R}$ tal que $(*)$ tiene infinitas soluciones para $\vec{x}$.

(c) Encuentre todos los $k \in \mathbb{R}$ tal que $(*)$ tiene ninguna solución para $\vec{x}$.

(d) Haga lo mismo para $A_k\vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ en vez de $(*)$.

(e) Haga los mismo para $A_k\vec{x} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ en vez de $(*)$ donde $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ es un vector arbitrario distinto de $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

16. Escriba las matrices invertibles de los Ejercicios 12 y 13 como producto de matrices elementales.

17. Para las sigientes matrices encuentre matrices elementales $E_1, \dots, E_n$ tal que $E_1 \cdot E_2 \cdot \dots \cdot E_n A$ es de la forma triangular superior.

$$A = \begin{pmatrix} 7 & 4 \\ 3 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 4 & -4 \\ 2 & 1 & 0 \\ 3 & 5 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 2 & 4 & 3 \end{pmatrix}.$$

18. Sea $A \in M(m \times n)$ y sean $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Demuestre que $A(\vec{x} + \lambda\vec{y}) = A\vec{x} + \lambda A\vec{y}$.

19. Demuestre que el espacio $M(m \times n)$ es un espacio vectorial con la suma de matrices y producto con $\lambda \in \mathbb{R}$ usual.

20. Sea $A \in M(n \times n)$.
   (a) Demuestre que $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A^t\vec{y}\rangle$ para todo $\vec{x} \in \mathbb{R}^n$.
   (b) Demuestre que $\langle AA^t\vec{x}, \vec{x}\rangle \geq 0$ para todo $\vec{x} \in \mathbb{R}^n$.

21. Sea $A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \in M(m \times n)$ y sea $\vec{e}_k$ el k-ésimo vector unitario en $\mathbb{R}^n$ (es decir, el vector en $\mathbb{R}^n$ cuya k-ésima entrada es 1 y las demás son cero). Calcule $A\vec{e}_k$ para todo $k = 1, \dots, n$ y describa en palabras la relación del resultado con la matriz A.

22. (a) Sea $A \in M(m \times n)$ y suponga que $A\vec{x} = \vec{0}$ para todo $\vec{x} \in \mathbb{R}^n$. Demuestre que $A = 0$ (la matriz cuyas entradas son 0).
   (b) Sea $x \in \mathbb{R}^n$ y suponga que $A\vec{x} = \vec{0}$ para todo $A \in M(n \times n)$. Demuestre que $\vec{x} = \vec{0}$.
   (c) Encuentre una matriz $A \in M(2 \times 2)$ y $\vec{v} \in \mathbb{R}^2$, ambos distintos de cero, tal que $A\vec{v} = \vec{0}$.
   (d) Encuentre matrices $A, B \in M(2 \times 2)$ tal que $AB = 0$ y $BA \neq 0$.

23. Sean $\vec{v} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ y $\vec{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.
   (a) Encuentre una matriz $A \in M(2 \times 2)$ que mapea el vector $\vec{e}_1$ a $\vec{v}$ y el vector $\vec{e}_2$ a $\vec{w}$.
   (b) Encuentre una matriz $B \in M(2 \times 2)$ que mapea el vector $\vec{v}$ a $\vec{e}_1$ y el vector $\vec{w}$ a $\vec{e}_2$.

24. Encuentre una matriz $A \in M(2 \times 2)$ que describe una rotación por $\pi/3$.

25. Sean $A \in M(m, n)$, $B, C \in M(n, k)$, $D \in M(k, l)$.

(a) Demuestre que $A(B+C) = AB + AC$.

(b) Demuestre que $A(BD) = (AB)D$.

26. Sean $R, S \in M(n, n)$ matrices invertibles. Demuestre que

$$RS = SR \quad \iff R^{-1}S^{-1} = S^{-1}R^{-1}.$$

27. Falso o verdadero? Pruebe sus respuestas.

(a) Si A es una matriz simétrica invertible, entonces A^{-1} es símetrica.

(b) Si A, B son matrices simétricas, entonces AB es símetrica.

(c) Si AB es una matriz simétrica, entonces A, B son matrices simétricas.

(d) Si A, B son matrices simétricas, entonces $A + B$ es símetrica.

(e) Si $A + B$ es una matriz simétrica, entonces A, B son matrices simétricas.

(f) Si A es una matriz simétrica, entonces A^t es símetrica.

(g) $AA^t = A^tA$.

28. Sea $A \in M(m \times n)$. Demuestre que AA^t y A^tA son matrices simétricas.

29. Sea $A \in Mnm \times n)$. Demuestre que $A + A^t$ es simétrica y que Demuestre que $A - A^t$ es antisimétrica.

30. Calcule $(S_j(c))^t, (Q_{ij}(c))^t, (P_{ij})^t$.

31. (a) Sea $P_{12} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in M(2 \times 2)$. Demuestre que P_{12} se deja expresar como producto de matrices elementales de la forma $Q_{ij}(c)$ y $S_k(c)$.

(b) Pruebe el caso general: Sea $P_{ij} \in M(n \times n)$. Demuestre que P_{ij} se deja expresar como producto de matrices elementales de la forma $Q_{kl}(c)$ y $S_m(c)$.

Observación: El ejercicio demuestra que en verdad solo hay dos tipos de matrices elementales ya que el tercero (las permutaciones) se dejan reducir a un producto apropiado de matrices de tipo $Q_{ij}(c)$ y $S_j(c)$.

Chapter 4

Determinants

In this section we will define the determinant of matrices in $M(n \times n)$ for arbitrary n and we will recognise the determinant for $n = 2$ defined in Section 1.2 as a special case of our new definition. We will discuss the main properties of the determinant and we will show that a matrix is invertible if and only if its determinant is different from 0. We will also show a geometric interpretation of the determinant and get a glimpse of its importance in geometry and the theory of integration. Finally we will use the determinant to calculate the inverse of an invertible matrix and we will prove Cramer's rule.

4.1 Determinant of a matrix

Recall that in Section 1.2 on page 19 we defined the determinant of a 2×2 matrix by

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Moreover, we know that a 2×2 matrix A is invertible if and only if its determinant is different from 0 because both statements are equivalent to the associated homogeneous system having only the trivial solution.

In this section we will define the determinant for arbitrary $n \times n$ matrices and we will see that again the determinant tells us if a matrix is invertible or not. We will give several formulas for the determinant. As definition, we use the Leibniz formula because it is non-recursive. First need to know what a permutation is.

Definition 4.1. A *permutation* of a set M is a bijection $M \to M$. The set of all permutations of the set $M = \{1, \dots, n\}$ is denoted by S_n. We denote an element $\sigma \in S_n$ by

$$\begin{Bmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n-1) & \sigma(n). \end{Bmatrix}$$

The *sign* (or *parity*) of a permutation $\sigma \in S_n$ is

$$\operatorname{sign}(\sigma) = (-1)^{\#\text{inversions of } \sigma}$$

where an *inversion of* σ is a pair $i < j$ with $\sigma(i) > \sigma(j)$.

Note that S_n consists of $n!$ permutations.

Examples 4.2. (i) S_2 consists of two permutations:

σ	$\begin{Bmatrix} 1 & 2 \\ 1 & 2 \end{Bmatrix}$	$\begin{Bmatrix} 1 & 2 \\ 2 & 1 \end{Bmatrix}$
sign(σ)	1	-1

(ii) S_3 consists of six permutations:

σ	$\begin{Bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{Bmatrix}$	$\begin{Bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{Bmatrix}$	$\begin{Bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{Bmatrix}$	$\begin{Bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{Bmatrix}$	$\begin{Bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{Bmatrix}$	$\begin{Bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{Bmatrix}$
sign(σ)	1	1	1	-1	-1	-1

Note that for instance the second permutation has two inversions ($1 < 3$ but $\sigma(1) > \sigma(3)$ and $2 < 3$ but $\sigma(2) > \sigma(3)$), the third permutation has two inversions ($1 < 2$ but $\sigma(1) > \sigma(2)$, $1 < 3$ but $\sigma(1) > \sigma(3)$), etc.

Definition 4.3. Let $A = (a_{ij})_{i,j=1,\dots,n} \in M(n \times n)$. Then its determinant is defined by

$$\boxed{\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.} \tag{4.1}$$

The formula in equation (4.1) is called the *Leibniz formula*.

Remark. Another notation for the determinant is $|A|$.

Remark 4.4. Note that according to the formula

(a) the determinant is a sum of $n!$ terms,

(b) each term is a product of n components of A,

(c) in each product, there is exactly one factor from each row and from each column and all such products appear in the formula.

So clearly, the Leibniz formula is computational nightmare ...

Equal rights for rows and columns!

Show that

$$\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}. \tag{4.2}$$

This means: instead of putting the permutation in the column index, we can just as well put them in the row index.

Let us check if this new definition coincides with our old definition for the case $n = 2$.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \sum_{\sigma \in S_2} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} = a_{11}a_{22} - a_{21}a_{12}.$$

which is the same as our old definition.

Now let us see what the formula gives us for the case $n = 3$. Using our table with the permutations in S_3, we find

$$\begin{aligned} \det A = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= \sum_{\sigma \in S_3} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} \end{aligned} \tag{4.3}$$

Now let us group terms with coefficients from the first line of A.

$$\det A = a_{11}\big[a_{22}a_{33} - a_{22}a_{33}\big] - a_{12}\big[a_{23}a_{31} - a_{21}a_{33}\big] + a_{13}\big[a_{21}a_{32} - a_{22}a_{31}\big]. \tag{4.4}$$

We see that the terms in brackets are again determinants:

- a_{11} is multiplied by the determinant of the 2×2 matrix that we obtain from A by deleting row 1 and column 1.
- a_{12} is multiplied by the determinant of the 2×2 matrix that we obtain from A by deleting row 1 and column 2.
- a_{13} is multiplied by the determinant of the 2×2 matrix that we obtain from A by deleting row 1 and column 3.

If we had grouped the terms by coefficients from the second row, we had obtained something analogously: each term a_{2j} would be multiplied by the determinant of the 2×2 matrix obtained from A by deleting row 2 and column j.
Of course we could also group the terms by coefficients all from the first column. Then the formula would become a sum of terms where the a_{j1} are multiplied by the determinants of the matrices obtained from A by deleting row j and column 1.

This motivates the definition of the so-called minors of a matrix.

Definition 4.5. Let $A = (a_{ij})_{i,j=1,\dots,n} \in M(n \times n)$. Then the $(n-1) \times (n-1)$ matrix M_{ij} which is obtained from A by deleting row i and column j of A is called a *minor of* A. The corresponding *cofactor* is $C_{ij} := (-1)^{i+j} \det(M_{ij})$.

With these definitions we can write (4.3) as

$$\det A = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det M_{1j} = \sum_{j=1}^{3} a_{1j} C_{1j}$$

This formula is called the *expansion of the determinant of A along the first row*. We also saw that we can expand along the second or the third row, or along columns, so

$$\det A = \sum_{j=1}^{3}(-1)^{k+j}a_{kj}\det M_{kj} = \sum_{j=1}^{3} a_{kj}C_{kj} \qquad \text{for } k=1,2,3,$$
$$\det A = \sum_{i=1}^{3}(-1)^{i+k}a_{ik}\det M_{ik} = \sum_{i=1}^{3} a_{ik}C_{ik} \qquad \text{for } k=1,2,3.$$

The first formula is called *expansion along the kth row*, and the second formula is called *expansion along the kth column*. With a little more effort we can show that an analogous formula is true for arbitrary n.

Theorem 4.6. *Let $A=(a_{ij})_{i,j=1,\dots,n}\in M(n\times n)$ and let M_{ij} denote its minors. Then*

$$\det A = \sum_{j=1}^{n}(-1)^{k+j}a_{kj}\det M_{kj} = \sum_{j=1}^{n} a_{kj}C_{kj} \qquad \textit{for}\;\; k=1,2,\,\dots,\,n, \tag{4.5}$$
$$\det A = \sum_{i=1}^{n}(-1)^{i+k}a_{ik}\det M_{ik} = \sum_{i=1}^{n} a_{ik}C_{ik} \qquad \textit{for}\;\; k=1,2,\,\dots,\,n. \tag{4.6}$$

These formulas are called Laplace expansion of the determinant*. More precisely,* (4.5) *is called* expansion along the kth row*,* (4.6) *is called* expansion along the kth column*.*

Proof. The formulas can be obtained from the Leibniz formula by straightforward calculations; but they are long and quite messy so we omit them here. □

Note that for calculating the determinant of a, for instance, 5×5 matrix, we have to calculate five 4×4 determinants for each of which we have to calculate four (3×3) determinants, etc. Computationally, it is as long as the Leibniz formula, but at least we do not have to find all permutations in S_n first.

Later, we will see how to calculate the determinant using Gaussian elimination. This is computationally much more efficient, see Remark 4.12.

Example 4.7. We use expansion along the second column to calculate

$$\begin{aligned}\det\begin{pmatrix}3&2&1\\5&6&4\\8&0&7\end{pmatrix} &= -\boxed{2}\det\begin{pmatrix}3&\boxed{2}&1\\5&6&4\\8&0&7\end{pmatrix} + \boxed{6}\det\begin{pmatrix}3&2&1\\5&\boxed{6}&4\\8&0&7\end{pmatrix} - \boxed{0}\det\begin{pmatrix}3&2&1\\5&6&4\\8&\boxed{0}&7\end{pmatrix}\\ &= -2\det\begin{pmatrix}5&4\\8&7\end{pmatrix} + 6\det\begin{pmatrix}3&1\\8&7\end{pmatrix} - 0\det\begin{pmatrix}3&1\\5&4\end{pmatrix}\\ &= -2\big[5\cdot 7-4\cdot 8\big]+6\big[3\cdot 7-1\cdot 8\big] = -2\big[35-32\big]+6\big[21-8\big] = -6+78 = 72.\end{aligned}$$

Example 4.8. We give an example of the calculation of the determinant of a 4×4 matrix. The red arrows indicate along which row or column we expand.

$$\begin{aligned}
\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 0 & 1 \\ 2 & 0 & 7 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} &= \det \begin{pmatrix} 6 & 0 & 1 \\ 0 & 7 & 0 \\ 3 & 0 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 0 & 0 & 1 \\ 2 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 3 \det \begin{pmatrix} 0 & 6 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix} - 4 \det \begin{pmatrix} 0 & 6 & 0 \\ 2 & 0 & 7 \\ 0 & 3 & 0 \end{pmatrix} \\
&= 7 \det \begin{pmatrix} 6 & 1 \\ 3 & 1 \end{pmatrix} - 2 \left[7 \det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right] + 3 \left[-2 \det \begin{pmatrix} 6 & 1 \\ 3 & 1 \end{pmatrix} \right] - 4 \left[-6 \det \begin{pmatrix} 2 & 7 \\ 0 & 0 \end{pmatrix} \right] \\
&= 7[6-3] - 14[0-0] - 6[6-3] + 24[0-0] = 21 - 18 = 3.
\end{aligned}$$

Now we calculate the determinant of the same matrix but choose a row with more zeros in the first step. The advantage is that there are only two 3×3 minors whose determinants we really have to compute.

$$\begin{aligned}
\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 0 & 1 \\ 2 & 0 & 7 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} &= -0 \det \begin{pmatrix} 2 & 3 & 4 \\ 0 & 7 & 0 \\ 3 & 0 & 1 \end{pmatrix} + 6 \det \begin{pmatrix} 1 & 3 & 4 \\ 2 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 0 \det \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix} + \det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 7 \\ 0 & 3 & 0 \end{pmatrix} \\
&= 6 \left[-3 \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + 7 \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \right] + \left[\det \begin{pmatrix} 0 & 7 \\ 3 & 0 \end{pmatrix} - 2 \det \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix} \right] \\
&= 6[-6+7] + [-21+18] = 6 - 3 = 3.
\end{aligned}$$

Rule of Sarrus

We finish this section with the so-called *rule of Sarrus*. From (4.3) we know that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \big[a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} + a_{13}a_{22}a_{31} \big]$$

which can be memorised as follows: Write down the matrix A and append its first and second column to it. Then we sum the products of the three terms lying on diagonals from the to left to the bottom right and subtract the products of the terms lying on diagonals from the top right to the bottom left as in the following picture:

$$\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
a_{31} & a_{32} & a_{33} & a_{31} & a_{32}
\end{array}$$

$$\det A = \underline{a_{11}a_{22}a_{33}} + \underline{a_{12}a_{23}a_{31}} + \underline{a_{13}a_{21}a_{32}} - \big[\underline{a_{13}a_{22}a_{31}} + \underline{a_{11}a_{23}a_{32}} + \underline{a_{12}a_{21}a_{33}} \big].$$

The rule of Sarrus works only for 3×3 matrices!!!

Convince yourself that one could also append the first and the second row below the matrix and make crosses.

Example 4.9 (Rule of Sarrus).

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 8 & 7 \end{pmatrix} = 1\cdot 5\cdot 7 + 2\cdot 6\cdot 0 + 3\cdot 4\cdot 8 - \big[3\cdot 5\cdot 0 + 6\cdot 8\cdot 1 + 7\cdot 2\cdot 4\big]$$
$$= 35 + 96 - [48 + 56] = 131 - 106 = 27.$$

You should now have understood

- what a permutation is,
- how to derive the Laplace expansion formula from the Leibniz formula,
- ...

You should now be able to

- calculate the determinant of an $n\times n$ matrix,
- ...

4.2 Properties of the determinant

In this section we will show properties of the determinant and we will prove that a matrix is invertible if and only if its determinant is different from 0.

(D1) The determinant is linear in its rows.

This means the following. Let $\vec r_1, \dots, \vec r_n$ be the row vectors of the matrix A and assume that $\vec r_j = \vec s_j + \gamma \vec t_j$. Then

$$\det A = \det\begin{pmatrix} \vec r_1 \\ \vdots \\ \vec r_j \\ \vdots \\ \vec r_n \end{pmatrix} = \det\begin{pmatrix} \vec r_1 \\ \vdots \\ \vec s_j + \gamma t_j \\ \vdots \\ \vec r_n \end{pmatrix} = \det\begin{pmatrix} \vec r_1 \\ \vdots \\ \vec s_j \\ \vdots \\ \vec r_n \end{pmatrix} + \gamma \det\begin{pmatrix} \vec r_1 \\ \vdots \\ \vec t_j \\ \vdots \\ \vec r_n \end{pmatrix}.$$

This is proved easily by expanding the determinant along the jth row, or it can be seen from the Leibniz formula as well.

(D1') The determinant is linear in its columns.

This means the following. Let $\vec{c}_1,\ \dots, \vec{c}_n$ be the column vectors of the matrix A and assume that $\vec{c}_j = \vec{s}_j + \gamma\vec{t}_j$. Then

$$\begin{aligned} \det A &= \det(\vec{c}_1|\cdots|\vec{c}_j|\cdots|\vec{c}_n) = \det(\vec{c}_1|\cdots|\vec{s}_j + \gamma t_j|\cdots|\vec{c}_n) \\ &= \det(\vec{c}_1|\cdots|\vec{s}_j|\cdots|\vec{c}_n) + \gamma \det(\vec{c}_1|\cdots|t_j|\cdots|\vec{c}_n). \end{aligned}$$

This is proved easily by expanding the determinant along the jth column, or it can be seen from the Leibniz formula as well.

(D2) The determinant is alternating in its rows.

If two rows in a matrix are swapped, then the determinant changes its sign. This means: Let $\vec{r}_1,\ \dots, \vec{r}_n$ be the row vectors of the matrix A and $i \neq j \in \{1, \dots, n\}$. Then

$$\det A = \det \begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_i \\ \vdots \end{pmatrix} = -\det \begin{pmatrix} \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix}.$$

This is sort of easy to see when the two rows that shall be interchanged are adjacent. For example, assume that $j = i+1$. Let A be the original matrix and let B be the matrix with rows i and $i+1$ swapped. We expand the determinant of A along the ith row and and the determinant of B along the $(i+1)$th row. Note that in both cases the minors are equal, that is, $M^A_{ik} = M^B_{(i+1)k}$ (we use superscripts A and B to distinguish between the minors of A and of B). So we find

$$\det B = \sum_{k=1}^{n} (-1)^{(i+1)+k} M^B_{(i+1)k} = \sum_{k=1}^{n} (-1)(-1)^{i+k} M^A_{ik} = -\sum_{k=1}^{n} (-1)^{i+k} M^A_{ik} = -\det A.$$

This can seen also via the Leibniz formula. Now let us see what happens if i and j are not adjacent rows. Without restriction we may assume that $i < j$. Then we first swap the jth row $(j-i)$ times with the row above until it is in the ith row. The original ith row is now in row $(i+1)$. No we swap it down with its neighbouring rows until it becomes lies in row j. To do this we need $j-(i+1)$ swaps. So in total we swapped $[j-i] + [j-(i+1)] = 2j-2i+1$ times neighbouring rows, so the determinant of the new matrix is

$$\underbrace{(-1)\cdot(-1)\cdot\dots\cdot(-1)}_{2j-2i+1 \text{ times (one factor for each swap)}} \cdot \det A = (-1)^{2j-2i+1} \det A = -\det A.$$

(D2') The determinant is alternating in its columns.

If two columns in a matrix are swapped, then the determinant changes its sign. This means: Let $\vec{c}_1,\ \dots, \vec{c}_n$ be the column vectors of the matrix A and $i \neq j \in \{1, \dots, n\}$. Then

$$\det A = \det(\cdots|\vec{c}_i|\cdots|\vec{c}_j|\cdots) = \det(\cdots|\vec{c}_j|\cdots|\vec{c}_i|\cdots).$$

This follows in the same way as the alternating property for rows.

(D3) $\det \mathrm{id}_n = 1$**.**

Expansion in the first row shows

$$\det \mathrm{id}_n = 1 \det \mathrm{id}_{n-1} = 1^2 \det \mathrm{id}_{n-2} = \cdots = 1^n = 1.$$

Remark 4.10. It can shown: Every function $f : M(n \times n) \to \mathbb{R}$ which satisfies **(D1), (D2) and (D3)** (or **(D1'), (D2') and (D3)**) must be det.

Now let us see some more properties of the determinant.

(D4) $\det A = \det A^t$**.**

This follows easily from the Leibniz formula or from the Laplace expansion (if you expand A along the first row and A^t along the first column, you obtain exactly the same terms). This also shows that (D1') follows from (D1) and that (D2') follows from (D2) and vice versa.

(D5) If one row of A is multiple of another row, or if a column is a multiple of another column, then $\det A = 0$. In particular, if A has two equal rows or two equal columns then $\det A = 0$.

Let $\vec{r}_1, \dots, \vec{r}_n$ denote the rows of the matrix A and assume that $\vec{r}_k = c\vec{r}_j$. Then

$$\det A = \det \begin{pmatrix} \vdots \\ \vec{r}_k \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = \det \begin{pmatrix} \vdots \\ c\vec{r}_j \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} \overset{(D2)}{=} -\det \begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ c\vec{r}_j \\ \vdots \end{pmatrix} \overset{(D1)}{=} -c \det \begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix}$$

$$\overset{(D1)}{=} -\det \begin{pmatrix} \vdots \\ c\vec{r}_j \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = -\det \begin{pmatrix} \vdots \\ \vec{r}_k \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = -\det A.$$

This shows $\det A = -\det A$, and therefore $\det A = 0$. If A has a column which is a multiple of another, then its transpose has a row which is multiple of another row and with the help of (D4) it follows that $\det A = \det A^t = 0$.

(D6) The determinant of an upper or lower triangular matrix is the product of its diagonal entries.

Let A be an upper triangular matrix and let us expand its determinant in the first column. Then only the first term in the Laplace expansion is different from 0 because all coefficients in the first column are equal to 0 except possibly the one in the first row. We repeat this and obtain

$$\det A = \det \begin{pmatrix} c_1 & & & \\ & c_2 & & * \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix} = c_1 \det \begin{pmatrix} c_2 & & & \\ & c_3 & & * \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix} = c_1 c_2 \det \begin{pmatrix} c_3 & & & \\ & c_4 & & * \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix}$$

$$= \cdots = c_1 c_2 \cdots c_{n-2} \det \begin{pmatrix} c_{n-1} & 0 \\ 0 & c_n \end{pmatrix} = c_1 c_2 \cdots c_{n-1} c_n.$$

The claim for lower triangular matrices follows from (D4) and what we just showed because the transpose of an upper triangular matrix is lower triangular and the diagonal entries are the same. Or we could repeat the above proof but this time we would expand always in the first row (or last column).

Next we calculate the determinant of elementary matrices.

(D7) The determinant of elementary matrices.

(i) $\det S_j(c) = c$,

(ii) $\det Q_{ij}(c) = 1$,

(iii) $\det P_{ij} = -1$.

The affirmation about $S_j(c)$ and $Q_{ij}(c)$ follow from (D6) since they are triangular matrices. The claim for P_{ij} follows from (D2) and (D3) because swapping row i and row j in P_{ij} gives us the identity matrix, so $\det P_{ij} = -\det \mathrm{id} = -1$.

Now we calculate the determinant of a product of an elementary matrix with another matrix.

(D8) Let E be an elementary matrix and let $A \in M(n \times n)$. Then $\det(EA) = \det E \det A$.

Let E be an elementary matrix and let us denote the rows of A by $\vec{r}_1, \dots, \vec{r}_n$. We have to distinguish between the three different types of elementary matrices.

$\boxed{\text{Case 1. } E = S_j(c).}$ We know from (D6) that $\det E = \det S_j(c) = c$. Using Proposition 3.61 and (D1) we find that

$$\det(EA) = \det\big(S_j(c)A\big) = \det \begin{pmatrix} \vdots \\ c\vec{r}_j \\ \vdots \end{pmatrix} = c \det \begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = c \det A = \det S_j(c) \det A.$$

$\boxed{\text{Case 2. } E = Q_{ij}(c).}$ We know from (D6) that $\det E = \det Q_{ij}(c) = 1$. Using Proposition 3.61 and

(D1) and (D5) we find that

$$\det(EA) = \det\big(Q_{ij}(c)A\big) = \det\begin{pmatrix} \vdots \\ \vec{r}_i + c\vec{r}_j \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = \det\begin{pmatrix} \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} + c\det\begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = \det\begin{pmatrix} \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix}$$
$$= \det A = \det Q_{ij}(c)\det A$$

$\boxed{\text{Case 3. } E = P_{ij}.}$ We know from (D6) that $\det E = \det P_{jk} = -1$. Using Proposition 3.61 and (D2) we find that

$$\det(EA) = \det\big(P_{jk}A\big) = \det\left(P_{jk}\begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_k \\ \vdots \end{pmatrix}\right) = \det\begin{pmatrix} \vdots \\ \vec{r}_k \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = -\det\begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_k \\ \vdots \end{pmatrix}$$
$$= -\det A = \det P_{jk}\det A.$$

If we repeat (D8), then we obtain

$$\det(E_1 \cdots E_k A) = \det(E_1)\cdots\det(E_k)\det(A)$$

for elementary matrices $E_1, \dots, E_k$.

(D9) Let $A \in M(n\times n)$. Then A is invertible if and only if $\det A \neq 0$.

Let A' be the reduced row echelon form of A. By Proposition 3.66 there exist elementary matrices $E_1, \dots, E_k$ such that $A = E_1 \cdots E_k A'$, hence

$$\det A = \det(E_1 \cdots E_k) = \det(E_1)\cdots\det(E_k)\det A'. \tag{4.7}$$

Recall that the determinant of an elementary matrix is different from cero, so (4.7) shows that $\det A = 0$ if and only if $\det A = 0$.
If A is invertible, then $A' = \mathrm{id}$ hence $\det A' = 1 \neq 0$ and therefore also $\det A \neq 0$. If A is not invertible, then the last row of A' must be cero, hence $\det A' = 0$ and therefore also $\det A = 0$.

Next we show that the determinant is multiplicative.

(D10) Let $A, B \in M(n\times n)$. Then $\det(AB) = \det A \det B$.

As before, let A' be the reduced row echelon form of A. By Proposition 3.66 there exist elementary matrices $E_1, \dots, E_k$ such that $A = E_1 \cdots E_k A'$. It follows from (D9) that

$$\det(AB) = \det(E_1 \cdots E_k A'B) = \det(E_1)\cdots\det(E_k)\det(A'B). \tag{4.8}$$

If A is invertible, then $A' = \mathrm{id}$ and (4.7) shows that

$$\det(AB) = \det(E_1)\cdots\det(E_k)\det(B) = \det(E_1\cdots E_k)\det(B) = \det(A)\det(B).$$

If on the other hand A is not invertible, then $\det A = 0$. Moreover, the last row of A' is cero, so also the last row of $A'B$ is cero, hence $A'B$ is not invertible and therefore $\det A'B = 0$. So we have $\det(AB) = 0$ by (4.7), and also $\det(A)\det(B) = 0\det(B) = 0$, so also in this case $\det(AB) = \det A \det B$.

(D11) Let $A \in M(n\times n)$ be an invertible matrix. Then $\det(A^{-1}) = (\det A)^{-1}$.

If A invertible then $\det A \neq 0$ and it follows from (D10) that

$$1 = \det \mathrm{id}_n = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

Solving for $\det(A^{-1}$ gives the desired formula.

Let $A \in M(n\times n)$. Give two proofs of $\det(cA) = c^n \det A$ using either one of the following:

(i) Apply (D1) or (D1') n times.

(ii) Use that $cA = \mathrm{diag}(c, c, \dots, c)A$ and apply (D10) and (D6).

The determinant is not additive!

Recall that $\det(AB) = \det A \det B$. But in general

$$\boxed{\det(A+B) \neq \det A + \det B.}$$

For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $\det A + \det B = 0 + 0 = 0$, but $\det(A+B) = \det \mathrm{id}_2 = 1$.

The following theorem is Theorem 3.43 together with (D9).

Theorem 4.11. *Let $A \in M(n\times n)$. Then the following is equivalent:*

(i) *A is invertible.*

(ii) *For every $\vec b \in \mathbb{R}^n$, the equation $A\vec x = \vec b$ has exactly one solution.*

(iii) *The equation $A\vec x = \vec 0$ has exactly one solution.*

(iv) *Every row-reduced echelon form of A has n pivots.*

(v) *A is row-equivalent to id_n.*

(vi) $\det A \neq 0$.

On the computational complexity of the determinant.

Remark 4.12. The above properties provide an efficient way to calculate the determinant of an $n \times n$ matrix. Note that both the Leibniz formula and the Laplace expansion require $O(n!)$ steps ($O(n!)$ stands for "order of $n!$". You can think of it as "roughly $n!$" or "up to a constant multiple roughly equal to $n!$". Something like $O(2n!)$ is still the same as $O(n!)$). However, reducing a matrix with the Gauß-Jordan elimination requires only $O(n^3)$ steps until we reach a row echelon form. Since this is always an upper triangular matrix, its determinant can be calculated easily.

If n is big, then n^3 is big, too, but $n!$ is a lot bigger, so the Gauß-Jordan elimination is computationally much more efficient than the Leibniz formula or the Laplace expansion.

Let us illustrate this with an example.

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 1 & 7 & 8 & 9 \\ 1 & 5 & 3 & 4 \end{pmatrix} \overset{(1)}{=} \det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 5 & 5 & 5 \\ 0 & 3 & 0 & 0 \end{pmatrix} \overset{(2)}{=} 5 \det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix} \overset{(3)}{=} 5 \det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

$$\overset{(4)}{=} 5 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix} \overset{(5)}{=} -5 \det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \overset{(7)}{=} -30.$$

(1) We subtract the first row from all the other rows. The determinant does not change.
(2) We factor 5 in the third row.
(3) We subtract 1/3 of the last row from rows 2 and 3. The determinant does not change.
(4) We subtract row 3 from row 2. The determinant does not change.
(5) We swap rows 2 and 4. This gives a factor -1.
(6) Easy calculation.

You should now have understood

- the different properties of the determinant,
- why a matrix is invertible if and only if its determinant is different from 0,
- why the Gauß-Jordan elimination is computationally more efficient than the Laplace expansion formula,
- ...

You should now be able to

- compute determinants using their properties,
- compute abstract determinants,
- use the factorisation of a matrix to compute its determinant,
- ...

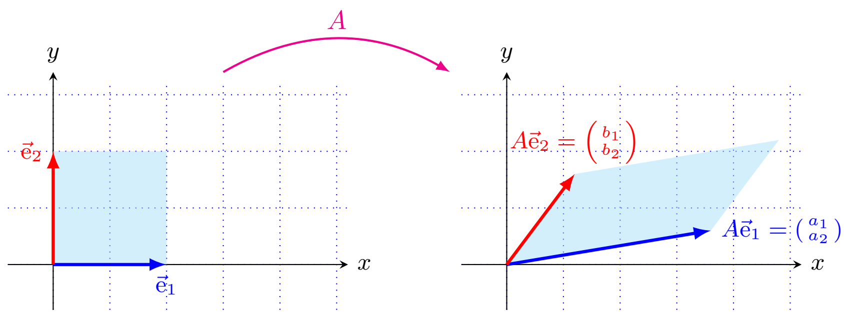


FIGURE 4.1: The figure shows how the area of the unit square transforms under the linear transformation A. The area of the square on left hand side is 1, the area of the parallelogram on the right hand side is $|\det A|$.

4.3 Geometric interpretation of the determinant

In this short section we show a geometric interpretation of the determinant. This is of course only a glimpse of the true importance of the determinant. You will hear more about this in a course on vector calculus when you discuss the transformation formula (the substitution rule for higher dimensional integrals), or in a course on Measure Theory or Differential Geometry. Here we content ourselves with two basic facts.

Area in $\mathbb{R}^2$

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$ and let us consider the matrix $A = (\vec{a}|\vec{b})$ the matrix whose columns are the given vectors. Then

$$A\vec{e}_1 = \vec{a}, \quad A\vec{e}_2 = \vec{b}.$$

That means that A transforms the unit square spanned by the unit vectors $\vec{e}_1$ and $\vec{e}_2$ into the parallelogram spanned by the vectors $\vec{a}$ and $\vec{b}$. Let area$(\vec{a}, \vec{b})$ be the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$. We can view $\vec{a}$ and $\vec{b}$as vectors in $\mathbb{R}^3$ simply by adding a third component. Then (2.9) shows that the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ is equal to

$$\left\| \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} a_1 b_2 - a_2 b_1 \\ 0 \\ 0 \end{pmatrix} \right\| = |a_1 b_2 - a_2 b_1| = |\det A|,$$

hence we obtain the formula

$$\operatorname{area}(\vec{a}, \vec{b}) = |\det A|. \tag{4.9}$$

So while A tells us how the shape of the unit square changes, $|\det A|$ tells us how its area changes, see Figure 4.1.

You should also notice the following: The area of the image of the unit square under A is zero if and only if the two image vectors $\vec{a}$ and $\vec{b}$ are parallel. This is in accordance to the fact that $\det A = 0$ if and only if the two lines described by the associated linear equations are parallel (or if one equations describes the whole plane).

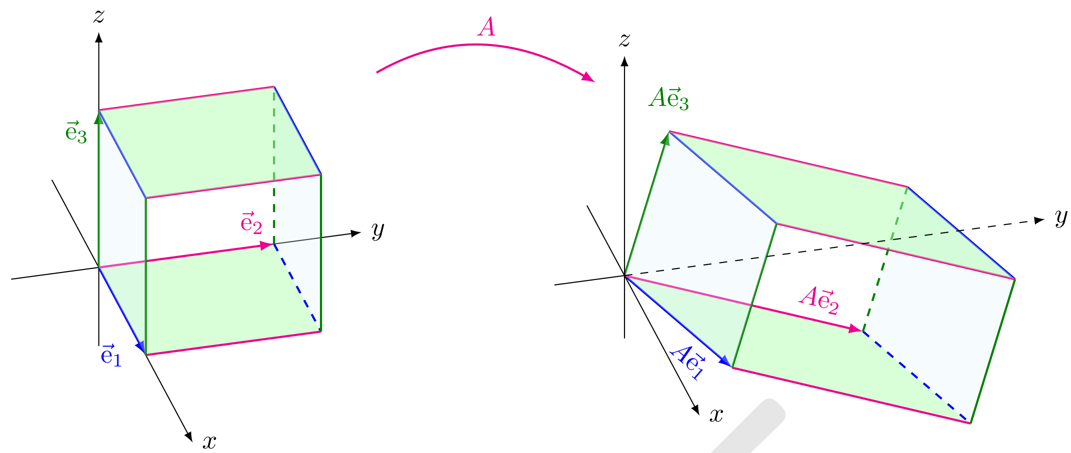


FIGURE 4.2: The figure shows how the volume of the unit cube transforms under the linear transformation A. The volume of the cube on left hand side is 1, the volume of the parallelepiped on the right hand side is $|\det A|$.

Volumes in $\mathbb{R}^3$

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ be vectors in $\mathbb{R}^3$ and let us consider the matrix $A = (\vec{a}\,|\,\vec{b}\,|\,\vec{c})$ the matrix whose columns are the given vectors. Then

$$A\vec{e}_1 = \vec{a}, \quad A\vec{e}_2 = \vec{b}, \quad A\vec{e}_3 = \vec{c}.$$

That means that A transforms the unit cube spanned by the unit vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ into the parallelepiped spanned by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Let $\operatorname{vol}(\vec{a}, \vec{b}, \vec{c})$ be the volume of the parallelepiped spanned by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. According to (2.10), $\operatorname{vol}(\vec{a}, \vec{b}, \vec{c}) = |\langle \vec{a}\,, \vec{b} \times \vec{c} \rangle|$. We calculate

$$\begin{aligned} |\langle \vec{a}\,, \vec{b} \times \vec{c} \rangle| &= \left| \left\langle \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\rangle \right| = \left| \left\langle \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - c_3b_1 \\ b_1c_2 - b_2c_1 \end{pmatrix} \right\rangle \right| \\ &= |a_1(b_2c_3 - b_3c_2) - a_2(c_3b_1 - b_3c_1) + a_3(b_1c_2 - b_2c_1)| \\ &= |\det A| \end{aligned}$$

hence

$$\operatorname{vol}(\vec{b}, \vec{b}, \vec{c}) = |\det A| \tag{4.10}$$

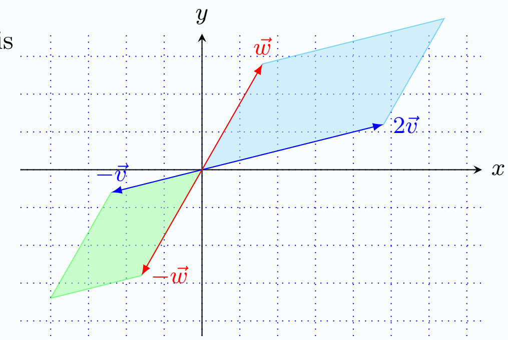
since we recognise the second to last line as the expansion of $\det A$ along the first column. So while A tells us how the shape of the unit cube changes, $|\det A|$ tells us how its volume changes.

You should also notice the following: The volume of the image of the unit cube under A is zero if and only if the three image vectors lie in the same plane. We will see later that this implies that the range of A is not all of $\mathbb{R}^3$, hence A cannot be invertible. For details, see Section 6.2.

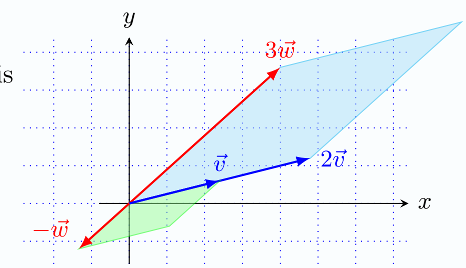
What we saw for $n = 2$ and $n = 3$ can be generalised to $\mathbb{R}^n$ with $n \geq 4$: A matrix $A \in M(n \times n)$ transforms the unit cube in $\mathbb{R}^n$ spanned by the unit vectors $\vec{e}_1, \dots, \vec{e}_n$ into a parallelepiped in $\mathbb{R}^n$ and $|\det A|$ tells us how its volume changes.

Exercise. Give two proofs of the following statements: One using the formula (4.9) and linearity of the determinant in its columns; and another proof using geometry.

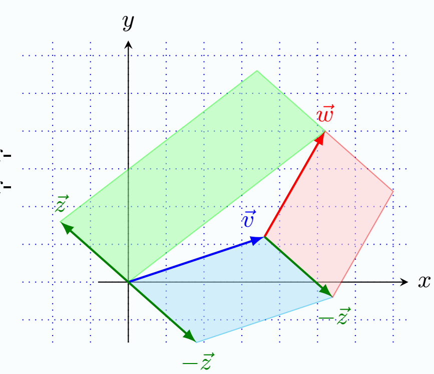
(i) Show that the area of the blue parallelogram is twice the area of the green parallelogram.



(ii) Show that the area of the blue parallelogram is six times the area of the green parallelogram.



(iii) Show that the area of the blue and the red parallelogram is equal to the area of the green parallelogram.



You should now have understood

- the geometric interpretation of the determinant in $\mathbb{R}^2$ and $\mathbb{R}^3$,
- the close relation between the determinant and the cross product in $\mathbb{R}^3$ and that this is the reason why the cross product appears in the formulas for the area of a parallelogram and

the volume of a parallelepiped,

- ...

You should now be able to

- calculate the area of a parallelogram and the volume of a parallelepiped using determinants,
- ...

4.4 Inverse of a matrix

In this section we prove a method to calculate the inverse of an invertible square matrix using determinants. Although the formula might look nice, computationally it is not efficient. Here it goes.

Let $A = (a_{ij})_{i,j=1,\dots,n} \in M(n \times n)$ and let M_{ij} be its minors, see Definition 4.5. We already know from (4.5) that for every fixed $k \in \{1, \dots, n\}$

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj}. \tag{4.11}$$

Now we want to see that happens if the k in a_{kj} and in M_{kj} are different.

Proposition 4.13. *Let $A = (a_{ij})_{i,j=1,\dots,n} \in M(n \times n)$ and let $k, \ell \in \{1, \dots, n\}$ with $k \neq \ell$. Then*

$$\sum_{j=1}^{n} (-1)^{\ell+j} a_{kj} \det M_{\ell j} = 0. \tag{4.12}$$

Proof. We build the new matrix B from A by replacing its ℓth row by the kth row. Then B has two equal rows (row ℓ and row k), hence $\det B = 0$. Note that the matrices A and B are equal everywhere except possibly in the ℓthrow, so their minors along the row ℓ are equal: $M^B_{\ell j} = M^A_{\ell j}$ (we put superscripts A, B in order to distinguish the minors of A and of B). If we expand $\det B$ along the ℓth row then we find

$$0 = \det B = \sum_{j=1}^{n} (-1)^{\ell+j} b_{\ell j} \det M^B_{\ell j} = \sum_{j=1}^{n} (-1)^{\ell+j} a_{kj} \det M^A_{\ell j}. \qquad \square$$

Using the cofactors C_{ij} of A (see Definition 4.5), formulas (4.11) and (4.12) can be written as

$$\sum_{j=1}^{n} (-1)^{\ell+j} a_{kj} \det M^A_{\ell j} = \sum_{j=1}^{n} a_{kj} C_{\ell j} := \begin{cases} \det A & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases} \tag{4.13}$$

Definition 4.14. For $A \in M(n \times n)$ we define its *adjugate matrix* $\operatorname{adj} A$ as the transpose of its cofactor matrix:

$$\operatorname{adj} A := \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^t = \begin{pmatrix} C_{11} & C_{22} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}.$$

Theorem 4.15. *Let $A \in M(n \times n)$ be an invertible matrix. Then*

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A. \tag{4.14}$$

Proof. Let us calculate $A \operatorname{adj} A$. By definition of $\operatorname{adj} A$ the coefficient $c_{k\ell}$ in the matrix product $A \operatorname{adj} A$ is exactly $c_{k\ell} = \sum_{j=1}^{n} (-1)^{\ell+j} a_{kj} \det M_{\ell j}$, so by (4.13) it follows that

$$A \operatorname{adj} A = \begin{pmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det A \end{pmatrix} = (\det A) \operatorname{id}_n .$$

Rearranging, we obtain that $A^{-1} = \frac{1}{\det A} \operatorname{adj} A \operatorname{id}_n^{-1} = \frac{1}{\det A} \operatorname{adj} A$. □

Remark 4.16. Note that the proof of Theorem 4.15 shows that $A \operatorname{adj} A = \det A \operatorname{id}_n$ is true for every $A \in M(n \times n)$, even if it is not invertible (in this case, both sides of the formula are equal to the cero matrix).

Formula (4.14) might look quite nice and innocent, however bear in mind that in order to calculate A^{-1} with it you have to calculate one $n \times n$ determinant and n^2 determinants of the $(n-1) \times (n-1)$ minors of A. This is a lot more than the $O(n^3)$ steps needed in the Gauß-Jordan elimination.

Finally, we state Cramer's rule for finding the solution of a linear system if the corresponding matrix is invertible.

Theorem 4.17. *Let $A \in M(n \times n)$ be an invertible matrix and let $\vec{B} \in \mathbb{R}^n$. Then the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ is given by*

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} \det A_1 \\ \det A_2 \\ \vdots \\ \det A_n \end{pmatrix} \tag{4.15}$$

where $A_j^{\vec{b}}$ is the matrix obtained from the matrix A if we replace its jth column by the vector $\vec{b}$.

Proof. As usual we write C_{ij} for the cofactors of A and M_{ij} for its minors. Since A is invertible, we know that $\vec{x} = A^{-1}\vec{b} = \frac{1}{\det A} \operatorname{adj} A\vec{b}$. Therefore we find for $j = 1, \dots, n$ that

$$x_j = \frac{1}{\det A} \sum_{k=1}^{n} C_{kj} b_k = \frac{1}{\det A} \sum_{k=1}^{n} (-1)^{k+j} b_k C_{kj} = \frac{1}{\det A} \det A_j^{\vec{b}}.$$

The last equality is true because the second to last sum is the expansion of the determinant of $A_j^{\vec{b}}$ along the kth column. □

Note that, even if (4.15) might look quite nice, it involves the computation of $n + 1$ determinants of $n \times n$ matrices, so it involves $O((n + 1)!)$ steps.

Example 4.18. Let us calculate the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 8 & 7 \end{pmatrix}$ from Example 4.9. We already know that $\det A = 27$. Its cofactors are

$$\begin{aligned}
C_{11} &= \det \begin{pmatrix} 5 & 6 \\ 8 & 7 \end{pmatrix} = -13, & C_{12} &= -\det \begin{pmatrix} 4 & 6 \\ 0 & 7 \end{pmatrix} = -28, & C_{13} &= \det \begin{pmatrix} 4 & 5 \\ 0 & 8 \end{pmatrix} = 32, \\
C_{21} &= -\det \begin{pmatrix} 2 & 3 \\ 8 & 7 \end{pmatrix} = 10, & C_{22} &= \det \begin{pmatrix} 1 & 3 \\ 9 & 7 \end{pmatrix} = 7, & C_{23} &= -\det \begin{pmatrix} 1 & 2 \\ 0 & 8 \end{pmatrix} = -8, \\
C_{31} &= \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = -3, & C_{32} &= -\det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = 6, & C_{33} &= \det \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = -3.
\end{aligned}$$

Therefore

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -13 & 10 & -3 \\ -28 & 7 & 6 \\ 32 & -8 & -3 \end{pmatrix}.$$

You should now have understood

- what the adjugate matrix is and why it can be used to calculate the inverse of a matrix,
- ...

You should now be able to

- calculate A^{-1} using $\operatorname{adj} A$.
- ...

4.5 Summary

The determinant is a function from the square matrices to the real or complex numbers. Let $A = (a_{ij})_{i,j-1}^n \in M(n \times n)$.

Formulas for the determinant.

$$\begin{aligned}
\det A &= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} && \text{Leibniz formula} \\
&= \sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj} = \sum_{j=1}^n a_{kj} C_{kj} && \text{Laplace expansion along the } k\text{th row} \\
&= \sum_{i=1}^n (-1)^{i+k} a_{ik} \det M_{ik} = \sum_{i=1}^n a_{ik} C_{ik} && \text{Laplace expansion along the } k\text{th column}
\end{aligned}$$

with the following notation

- S_n is the set of all permutations of $\{1, \dots, n\}$,
- M_{ij} are the minors of A ($(n-1) \times (n-1)$ matrices obtained from A by deleting row i and column j),
- $C_{ij} = (-1)^{i+j} \det M_{ij}$ are the cofactors of A.

Inverse of a matrix using the adjugate matrix

If $A \in M(n \times n)$ is invertible then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{22} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}.$$

Geometric interpretation

The determinant of a matrix A gives the oriented volume of the image of the unit cube under A.

- in $\mathbb{R}^2$: area of parallelogram spanned by $\vec{a}$ and $\vec{b} = |\det A|$,
- in $\mathbb{R}^3$: volume of parallelepiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c} = |\det A|$.

Properties of the determinant.

- The determinant is linear in its rows and columns.
- The determinant is alternating in its rows and columns.
- $\det \mathrm{id}_n = 1$.
- $\det A = \det A^t$.
- If one row of A is multiple of another row, or if a column is a multiple of another column, then $\det A = 0$. In particular, if A has two equal rows or two equal columns then $\det A = 0$.
- The determinant of an upper or lower triangular matrix is the product of its diagonal entries.
- The determinants of the elementary matrices are

$$\det S_j(c) = c, \quad \det Q_{ij}(c) = 1, \quad \det P_{ij} = -1.$$

- Let $A \in M(n \times n)$. Then A is invertible if and only if $\det A \neq 0$.
- Let $A, B \in M(n \times n)$. Then $\det(AB) = \det A \det B$.
- If $A \in M(n \times n)$ is invertible, then $\det(A^{-1}) = (\det A)^{-1}$.

Note however that in general $\det(A+B) \neq \det A + \det B$.

Theorem. *Let $A \in M(n \times n)$. Then the following is equivalent:*

(i) $\det A \neq 0$.

(ii) *A is invertible.*

(iii) *For every $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has exactly one solution.*

(iv) *The equation $A\vec{x} = \vec{0}$ has exactly one solution.*

(v) *Every row-reduced echelon form of A has n pivots.*

(vi) *A is row-equivalent to id_n.*

4.6 Exercises

1. De las siguientes matrices calcule la determinante. Determine si las matrices son invertibles. Si lo son, encuentre su matriz inversa y la determinante de la inversa.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} -14 & 21 \\ 12 & -18 \end{pmatrix}. \qquad D = \begin{pmatrix} 1 & 3 & 6 \\ 4 & 1 & 0 \\ 1 & 4 & 3 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 1 & 5 \\ 3 & 5 & 11 \end{pmatrix}.$$

2. De las siguientes matrices calcule el determinante. Determine si las matrices son invertibles. Si lo son, encuentre su matriz inversa y el determinante de la inversa.

$$A = \begin{pmatrix} \pi & 3 \\ 5 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 3 & 1 \\ 4 & 3 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 4 & 0 & 3 \\ 1 & 1 & 5 & 4 \end{pmatrix}.$$

3. Encuentre por lo menos cuatro matrices 3×3 cuyo determinante es 18.

4. Use las factorizaciones encontrados en los Ejercicios 16 y 16 en Capítulo 3 para calcular sus determinantes.

5. Escribe la matriz $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 6 \\ -2 & -2 & -6 \end{pmatrix}$ como producto de matrices elementales y calcule el determinante de A usando las matrices elementales encontradas.

6. Determine todos los $x \in \mathbb{R}$ tal que las siguientes matrices son invertibles:

$$A = \begin{pmatrix} x & 2 \\ 1 & x-3 \end{pmatrix}, \qquad B = \begin{pmatrix} x & x & 3 \\ 1 & 2 & 6 \\ -2 & 2 & -6 \end{pmatrix}, \qquad C = \begin{pmatrix} 11-x & 5 & -50 \\ 3 & -x & -15 \\ 2 & 1 & -x-9 \end{pmatrix}.$$

7. Suponga que una función y satisface $y^{[n]} = b_{n-1}y^{[n-1]} + \cdots b_1 y' + b_0 y$ donde $b_0, \dots, b_{n-1}$ son coeficientes constantes y $y^{[j]}$ denota la derivada j-ésima de y.

   Verifique que $Y' = AY$ donde

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ b_1 & b_2 & \dots & \dots & \dots & b_{n-1} \end{pmatrix}, \qquad Y = \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{[n-1]} \end{pmatrix}$$

   y calcule el determinante de A.

8. Sin usar fórmulas de expansión para determinantes, encuentre para cada una de las matrices dadas parámetros x, y tales que el determinante de las siguientes matrices es igual a cero y explique por qué los parametros encontrado sirven.

$$N_1 = \begin{pmatrix} x & 2 & 6 \\ 2 & 5 & 1 \\ 3 & 4 & y \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & x & y & 2 \\ x & 0 & 1 & y \\ x & 5 & 3 & y \\ 4 & x & y & 8 \end{pmatrix}.$$

9. (a) Calcule $\det B_n$ donde B_n es la matriz en $M(n \times n)$ cuyas entradas en la diagonal son 0 y todas las demás entradas son 1, es decir:

$$B_1 = 0, \;\; B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; B_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \; B_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \text{ etc.}$$

   ¿Cómo cambia la respuesta si en vez de 0 hay x en la diagonal?

   (b) Calcule $\det B_n$ donde B_n es la matriz en $M(n \times n)$ cuyas entradas en la diagonal son 0 y todas las demás entradas satisfacen $b_{ij} = (-1)^{i+j}$, es decir:

$$B_1 = 0, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, B_4 = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix},$$
$$B_5 = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix}, \text{ etc.}$$

¿Cómo cambia la respuesta si en vez de 0 hay x en la diagonal? Compare con el Ejercicio 8.

Chapter 5

Vector spaces

In the following, $\mathbb{K}$ always denotes a field. In this chapter, you may always think of $\mathbb{K} = \mathbb{R}$, though almost everything is true also for other fields, like $\mathbb{C}$, $\mathbb{Q}$ or $\mathbb{F}_p$ where p is a prime number. Later, in Chapter 8 it will be more useful to work with $\mathbb{K} = \mathbb{C}$.

In this chapter we will work with abstract vector spaces. We will first discuss their basic properties. Then, in Section 5.2 we will define subspaces. These are subsets of vector space which are themselves vector spaces. In Section 5.4 we will introduce basis and dimension of a vector space. These concepts are fundamental in linear algebra since they allow us to classify all finite dimensional vector spaces. In a certain sense, all n dimensional vector spaces over the same field $\mathbb{K}$ are equal. In Chapter 6 we will study linear maps between vector spaces.

5.1 Definitions and basic properties

First we recall the definition of an abstract vector space from Chapter 2 (p. 30).

Definition 5.1. Let V be a set together with two operations

$$\begin{aligned} \text{vector sum} \quad & + : V \times V \to V, \quad (v, w) \mapsto v + w, \\ \text{product of a scalar and a vector} \quad & \cdot : \mathbb{K} \times V \to V, (\lambda, v) \mapsto \lambda \cdot v. \end{aligned}$$

Note that we will usually write λv instead of $\lambda \cdot v$. Then V (or more precisely, $(V, +, \cdot)$) is called a *vector space* if for all $u, v, w \in V$ and all $\lambda, \mu \in \mathbb{K}$ the following holds:

(a) **Associativity:** $(u + v) + w = u + (v + w)$ for every $u, v, w \in V$.

(b) **Commutativity:** $v + w = w + v$ for every $u, v \in V$.

(c) **Identity element of addition:** There exists an element $\mathbb{O} \in V$, called the *additive identity* such that $\mathbb{O} + v = v + \mathbb{O} = v$ for every $v \in V$.

(d) **Inverse element:** For every $v \in V$, there exists an inverse element v' such that $v + v' = \mathbb{O}$.

(e) **Identity element of multiplication by scalar:** For every $v \in V$, we have that $1v = v$.

(f) **Compatibility:** For every $v \in V$ and $\lambda, \mu \in \mathbb{K}$, we have that $(\lambda\mu)v = \lambda(\mu v)$.

(g) **Distributivity laws:** For all $v, w \in V$ and $\lambda, \mu \in \mathbb{K}$, we have

$$(\lambda + \mu)v = \lambda v + \mu v \quad \text{and} \quad \lambda(v + w) = \lambda v + \lambda w.$$

We already know that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Remark 5.2. (i) Note that we use the notation $\vec{v}$ with an arrow only for the special case of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$. Vectors in abstract vector spaces are usually denoted without an arrow.

(ii) If $\mathbb{K} = \mathbb{R}$, then V is called a *real vector space*. If $\mathbb{K} = \mathbb{C}$, then V is called a *complex vector space*.

Before we give examples of vector spaces, we first show some basic properties of vector spaces.

Properties 5.3. (i) *The identity element is unique.* (Note that in the vector space axioms we only asked for *existence* of an additive identity element; we did not ask for uniqueness. So one could think that there may be several elements which satisfy (c) in Definition 5.1. However, this is not possible as the following proof shows.)

Proof. Assume there are two neutral elements $\mathbb{O}$ and $\mathbb{O}'$. Then we know that for every v and w in V the following is true:

$$v = v + \mathbb{O}, \qquad w = w + \mathbb{O}'.$$

Now let us take $v = \mathbb{O}'$ and $w = \mathbb{O}$. Then, using commutativity, we obtain

$$\mathbb{O}' = \mathbb{O}' + \mathbb{O} = \mathbb{O} + \mathbb{O}' = \mathbb{O}.$$ □

(ii) For every $v \in V$, its inverse element is unique. (Note that in the vector space axioms we only asked for *existence* of an additive inverse for every element $x \in V$; we did not ask for uniqueness. So one could think that there may be several elements which satisfy (d) in Definition 5.1. However, this is not possible as the following proof shows.)

Proof. Let $v \in V$ and assume that there are elements v', v'' in V such that

$$v + v' = \mathbb{O}, \quad v + v'' = \mathbb{O}.$$

We have to show that $v' = v''$. This follows from

$$v' = v' + \mathbb{O} = v' + (v + v'') = (v' + v) + v'' = \mathbb{O} + v'' = v''.$$ □

(iii) For every $\lambda \in \mathbb{K}$ we have $\lambda\mathbb{O} = \mathbb{O}$.

Proof. Observe that
$$\lambda\mathbb{O} = \lambda(\mathbb{O} + \mathbb{O}) = \lambda\mathbb{O} + \lambda\mathbb{O}.$$
Now let $(\lambda\mathbb{O})'$ be the inverse of $\lambda\mathbb{O}$ and sum it to both sides of the equation. We obtain
$$\begin{aligned} & \lambda\mathbb{O} + (\lambda\mathbb{O})' = (\lambda\mathbb{O} + \lambda\mathbb{O}) + (\lambda\mathbb{O})' \\ \implies & \qquad \mathbb{O} = \lambda\mathbb{O} + (\lambda\mathbb{O} + (\lambda\mathbb{O})') \\ \implies & \qquad \mathbb{O} = \lambda\mathbb{O} + \mathbb{O} \\ \implies & \qquad \mathbb{O} = \lambda\mathbb{O}. \end{aligned}$$
□

(iv) For every $v \in V$ we have that $0v = \mathbb{O}$.

Proof. The proof is similar to the one above. Observe that
$$0v = (0 + 0)v = 0v + 0v.$$
Now let $(0v)'$ be the inverse of $0v$ and sum it to both sides of the equation. We obtain
$$\begin{aligned} & 0v + (0v)' = (0v + 0v) + (0v)' \\ \implies & \qquad \mathbb{O} = 0v + (0v + (0v)') \\ \implies & \qquad \mathbb{O} = 0v + \mathbb{O} \\ \implies & \qquad \mathbb{O} = 0v. \end{aligned}$$
□

(v) If $\lambda v = \mathbb{O}$, then either $\lambda = 0$ or $v = \mathbb{O}$.

Proof. If $\lambda = 0$, then there is nothing to prove. Now assume that $\lambda \neq 0$. Then v is $\mathbb{O}$ because
$$v = \frac{1}{\lambda}(\lambda v) = \frac{1}{\lambda}\mathbb{O} = \mathbb{O}.$$
□

(vi) For every $v \in V$, its inverse is $(-1)v$.

Proof. Let $v \in V$. Observe that by (v), we have that $0v = \mathbb{O}$. Therefore
$$\mathbb{O} = 0v = (1 + (-1))x = v + (-1)v.$$
Hence $(-1)v$ is an additive inverse of v. By (ii), the inverse of v is unique, therefore $(-1)v$ is the inverse of v. □

Remark 5.4. From now on, we write $-v$ for the additive inverse of a vector. This notation is justified by Property 5.3 (vi).

Examples 5.5. We give some important examples of vector spaces.

- $\mathbb{R}$ is a real vector space. More generally, $\mathbb{R}^n$ is a real vector space. The proof is the same as for $\mathbb{R}^2$ in Chapter 2. Associativity and commutativity are clear. The identity element is the vector whose entries are all equal to zero: $\vec{0} = (0, \dots, 0)^t$. The inverse for a given vector $\vec{x} = (x_1, \dots, x_n)^t$ is $(-x_1, \dots, -x_n)^t$. The distributivity laws are clear, as is the fact that $1\vec{x} = \vec{x}$ for every $\vec{x} \in \mathbb{R}^n$.

- $\mathbb{C}$ is a complex vector space. More generally, $\mathbb{C}^n$ is a complex space. The proof is as for $\mathbb{R}^n$.
- $\mathbb{C}$ can also be viewed as a real vector space.
- $\mathbb{R}$ is **not** a complex vector space with the usual definition of the algebraic operations. If it was, then the vectors would be real numbers and the scalars would be complex numbers. But then if we take $1 \in \mathbb{R}$ and $\mathrm{i} \in \mathbb{C}$, then the product $\mathrm{i}1$ must be a vector, that is, a real number, which is not the case.
- $\mathbb{R}$ can be seen as a $\mathbb{Q}$-vector space.
- For every $n, m \in \mathbb{N}$, the space $M(m \times n)$ of all $m \times n$ matrices with real coefficients is a real vector space.

  Proof. Note that in this case the vectors are matrices. Associativity and commutativity are easy to check. The identity element is the matrix whose entries are all equal to zero. Given a matrix $A = (a_{ij})_{\substack{i=1,\dots,m,\\ j=1,\dots,n}}$ its (additive) inverse is the matrix $-A = (-a_{ij})_{\substack{i=1,\dots,m.\\ j=1,\dots,n}}$ The distributivity laws are clear, as is the fact that $1A = A$ for every $A \in M(m \times n)$. ☐

- For every $n, m \in \mathbb{N}$, the space $M(m \times n, \mathbb{C})$ of all $m \times n$ matrices with complex coefficients, is a complex vector space.

  Proof. As in the case of real matrices. ☐

- Let $C(\mathbb{R})$ be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. We define the sum of two functions f and g in the usual way as the new function

  $$f + g : \mathbb{R} \to \mathbb{R}, \quad (f + g)(x) = f(x) + g(x).$$

  The product of a function f with a real number λ gives the new function λf defined by

  $$\lambda f : \mathbb{R} \to \mathbb{R}, \quad (\lambda f)(x) = \lambda f(x).$$

  Then $C(\mathbb{R})$ is a vector space with these new operations.

  Proof. It is clear that these operations satisfy associativity, commutativity and distributivity and that $1f = f$ for every function $f \in C(\mathbb{R})$. The additive identity is the zero function (the function which is constant to zero). For a given function f, its (additive) inverse is the function $-f$. ☐

Prove that $\mathbb{C}$ is a vector space over $\mathbb{R}$ and that $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Observe that the sets $M(m \times n)$ and $C(\mathbb{R})$ admit more operations, for example we can multiply functions, or we can multiply matrices or we can calculate $\det A$ for a square matrix. However, all these operations have nothing to do with the question whether they are vector spaces or not. It is important to note that for a vector space we only need the sum of two vectors and the product of a scalar with vector and that they satisfy the axioms from Definition 5.1.

We give more examples.

Examples 5.6. • Consider $\mathbb{R}^2$ but we change the usual sum to the new sum $\oplus$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a \\ 0 \end{pmatrix}.$$

With this new sum, $\mathbb{R}^2$ is **not** a vector space. The reason is that there is no additive identity. To see this, assume that we had an additive identity, say $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Then we must have

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

However, for example,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

• Consider $\mathbb{R}^2$ but we change the usual sum to the new sum $\oplus$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+b \\ y+b \end{pmatrix}.$$

With this new sum, $\mathbb{R}^2$ is **not** a vector space. One of the reasons is that the sum is not commutative. For example

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad \text{but} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Show that there is no additive identity $\mathbb{O}$ which satisfies $\vec{x} \oplus \mathbb{O} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

• Let $V = \mathbb{R}_+ = (0, \infty)$. We make V a real vector space with the following operations: Let $x, y \in V$ and $\lambda \in \mathbb{R}$. We define

$$x \oplus y = xy \qquad \text{and} \qquad \lambda \odot x = x^\lambda.$$

Then $(V, \oplus, \odot)$ is a real vector space.

Proof. Let $u, v, w \in V$ and let $\lambda \in \mathbb{R}$. Then:

(a) **Associativity:** $(u \oplus v) \oplus w = (uv) \oplus w = (uv)w = u(vw) = u(v \oplus w) = u \oplus (v \oplus w)$.

(b) **Commutativity:** $v \oplus w = vw = wv = w \oplus v$.

(c) The **additive identity** of $\oplus$ is 1 because for every $x \in V$ we have that $1 \oplus x = 1x = x$.

(d) **Inverse element:** For every $x \in V$, its inverse element is x^{-1} because $x \oplus x^{-1} = xx^{-1} = 1$ which is the identity element. (Note that this is in accordance with Properties 5.3 (v) since $(-1) \odot x = x^{-1}$.)

(e) **Identity element of multiplication by scalar:** For every $x \in V$, we have that $1 \odot x = 1x = x$.

(f) **Compatibility:** For every $x \in V$ and $\lambda, \mu \in \mathbb{R}$, we have that

$$(\lambda\mu) \odot v = v^{\lambda\mu} = (v^\lambda)^\mu = \mu \odot (v^\lambda) = \lambda \odot (\mu \odot v).$$

(g) **Distributivity laws:** For all $x, y \in V$ and $\lambda, \mu \in \mathbb{R}$, we have

$$(\lambda + \mu) \odot x = x^{\lambda+\mu} = x^\lambda x^\mu = (\lambda \odot v)(\mu \odot v) = (\lambda \odot v) \oplus (\mu \odot v)$$

and

$$\lambda \odot (v \oplus w) = (v \oplus w)^\lambda = (vw)^\lambda = v^\lambda w^\lambda = v^\lambda \oplus w^\lambda = (\lambda \odot v) \oplus (\lambda \odot w). \qquad \square$$

- The example above can be generalised: Let $f : \mathbb{R} \to (a, b)$ be an injective function. Then the interval (a, b) becomes a real vector space if we define the sum of two vectors $x, y \in (a, b)$ by

  $$x \oplus y = f(f^{-1}(x) + f^{-1}(y))$$

  and the product of a scalar $\lambda \in \mathbb{R}$ and a vector $x \in (a, b)$ by

  $$\lambda \odot x = f(\lambda f^{-1}(x)).$$

  Note that in the example above we had $(a, b) = (0, \infty)$ and $f = \exp$ (that is: $f(x) = \mathrm{e}^x$).

You should have understood

- the concept of an abstract vector space,
- that the spaces $\mathbb{R}^n$ are examples of vector spaces, but there are many more,
- that "vectors" not necessarily can be written as columns (think of the vector space of all polynomials, etc.)
- ...

You should now be able to

- give examples of vector spaces different from $\mathbb{R}^n$ or $\mathbb{C}^n$,
- check if a given set with a given addition and multiplication with scalars is a vector space,
- recite the vector space axioms when woken in the middle of the night,
- ...

5.2 Subspaces

In this section, we work mostly with real vector spaces for definiteness sake. However, all the statements are also true for complex vector spaces. We only have to replace everywhere $\mathbb{R}$ by $\mathbb{C}$ and the word *real* by *complex*.

In this section we will investigate when a subset of a given vector space is itself a vector space.

Definition 5.7. Let V be a vector space and let $W \subseteq V$ be a subset of V. Then W is called a *subspace* of V if W itself is a vector space with the sum and product with scalars inherited from V. A subspace W is called a *proper subspace* if $W \neq \{0\}$ and $W \neq V$.

First we observe the following basic facts.

Remark 5.8. Let V be a vector space.

- V always contains the following subspaces: $\{0\}$ and V itself. However, they are not proper subspaces.
- If V is a vector space, W is a subspace of V and U is a subspace of W, then U is a subspace of V.

  Prove these statements.

Remark 5.9. Let W be a subspace of a vector space V. Let $\mathbb{O}$ be the neutral element in V. Then $\mathbb{O} \in W$ and it is the neutral element of W.

Proof. Since W is a vector space, it must have a neutral element $\mathbb{O}_W$. A priori, it is not clear that $\mathbb{O}_W = \mathbb{O}$. However, since $\mathbb{O}_W \in W \subset V$, we know that $0\mathbb{O}_W = \mathbb{O}$. On the other hand, since W is a vector space, it is closed under product with scalars, so $\mathbb{O} = 0\mathbb{O}_W \in W$. Clearly, $\mathbb{O}$ is a neutral element in W. So it follows that $\mathbb{O} = \mathbb{O}_W$ by uniqueness of the neutral element of W, see Properties 5.3(i). □

Now assume that we are given a vector space V and in it a subset $W \subseteq V$ and we would like to check if W is a vector space. In principle we would have to check all seven vector space axioms from Definition 5.1. However, if W is a subset of V, then we get some of the vector space axioms for free. More precisely, the axioms (a), (b), (e), (f) and (g) hold automatically. For example, to prove (b), we take two elements $w_1, w_2 \in W$. They belong also to V since $W \subseteq V$, and therefore they commute: $w_1 + w_2 = w_2 + w_1$.
We can even show the following proposition:

Proposition 5.10. *Let V be a real vector space and $W \subseteq V$ a subset. Then W is a subspace of V if and only if the following three properties hold:*

(i) *$W \neq \varnothing$, that is, W is not empty.*

(ii) *W is* closed under sums*, that is, if we take w_1 and w_2 in W, then their sum $w_1 + w_2$ belongs to W.*

(iii) *W is* closed under product with scalars*, that is, if we take $w \in W$ and $\lambda \in \mathbb{R}$, then it must follow that $\lambda w \in W$.*

Note that (ii) and (iii) can be summarised in the following:

(iv) W is *closed under sums and product with scalars*, that is, if we take $w_1, w_2 \in W$ and $\lambda \in \mathbb{R}$, then $\lambda w_1 + w_2 \in W$.

Proof of 5.10. Assume that W is a subspace, then clearly (ii) and (iii) hold. (i) holds because every vector space must contain at least the additive identity $\mathbb{O}$.

Now suppose that W is a subset of V such that the properties (i), (ii) and (iii) are satisfied. In order to show that W is a subspace of V, we need to verify the vector space axioms (a) - (f) from Definition 5.1. By assumptions (ii) and (iii), the sum and product with scalars are well defined in W. Moreover, we already convinced ourselves that (a), (b), (e), (f) and (g) hold. Now, for the existence of an additive identity, we take an arbitrary $w \in W$ (such a w exists because W is not empty by assumption (i)). Hence $\mathbb{O} = 0w \in W$ where $\mathbb{O}$ is the additive identity in V. This then is also the additive identity in W. Finally, given $w \in W \subseteq V$, we know from Properties 5.3 (v) that its additive inverse is $(-1)w$, which, by our assumption (iii), belongs to W. So we have verified that W satisfies all vector space axioms, so it is a vector space. □

The proposition is also true if V is a complex vector space. We only have to replace $\mathbb{R}$ everywhere by $\mathbb{C}$.

In order to verify that a given $W \subseteq V$ is a subspace, one only has to verify (i), (ii) and (iii) from the preceding proposition. In order to verify that W is not empty, one typically checks if it contains $\mathbb{O}$.

The following definition is very important in many applications.

Definition 5.11. Let V be a vector space and $W \subseteq V$ a subset. The W is called an *affine subspace* if there exists an $v_0 \in V$ such that set

$$v_0 + W := \{v_0 + w : w \in W\}$$

is a subspace of V.

Clearly, every subspace is also an affine subspace (take $v_0 = \mathbb{O}$).

Let us see examples of subspaces and affine subspaces.

Examples 5.12. Let V be a vector space. We assume that V is a real vector space, but everything works also for a complex vector space (we only have to replace $\mathbb{R}$ everywhere by $\mathbb{C}$.)

(i) $\{0\}$ is a subspace of V. It is called the *trivial subspace* of V.

(ii) V itself is a subspace of V.

(iii) Fix $z \in V$. Then the set $W := \{\lambda z : \lambda \in \mathbb{R}\}$ is a subspace of V.

(iv) More generally, if we fix $z_1, \dots z_k \in V$, then the set $W := \{\lambda_1 z_1 + \cdots \lambda_k z_k : \lambda_1, \dots, \lambda_k \in \mathbb{R}\}$ is a subspace of V.

(v) If we fix v_0 and $z_1, \dots z_k \in V$, then the set $W := \{v_0 + \lambda_1 z_1 + \cdots \lambda_k z_k : \lambda_1, \dots, \lambda_k \in \mathbb{R}\}$ is an affine subspace of V. In general it will not be a subspace.

Exercise. Show that $W := \{v_0 + \lambda_1 z_1 + \cdots \lambda_k z_k : \lambda_1, \dots, \lambda_k \in \mathbb{R}\}$ is an affine subspace of V. Show that it is a subspace if and only if $v_0 \in \operatorname{span}\{z_1, \dots, z_k\}$.

(vi) If W is a subspace of V, then $V \setminus W$ is not a subspace. This can be easily seen if we recall that W must contain $\mathbb{O}$. But then $V \setminus W$ cannot contain $\mathbb{O}$, hence it cannot be a vector space.

Some more examples:

Examples 5.13.

- The set of all solutions of a homogeneous system of linear equations is a vector space.
- The set of all solutions of an inhomogeneous system of linear equations is an affine vector space.
- The set of all solutions of a homogeneous linear differential equation is a vector space.
- The set of all solutions of an inhomogeneous linear differential equation is an affine vector space.

Examples 5.14 (Examples and non-examples of subspaces of $\mathbb{R}^2$).

- $W = \left\{ \begin{pmatrix} \lambda \\ 0 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$. This is actually a subspace of the form (iii) from Example 5.12 with $z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Note that geometrically W is a line (it is the x-axis).

- For fixed $v_1, v_2 \in \mathbb{R}$ let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and let $W = \{\lambda\vec{v} : \lambda \in \mathbb{R}\}$. Then W is a subspace of $\mathbb{R}^2$. Geometrically, W is the trivial subspace $\{\vec{0}\}$ if $\vec{v} = \vec{0}$. Otherwise it is the line in $\mathbb{R}^2$ passing through the origin which is parallel to the vector $\vec{v}$.

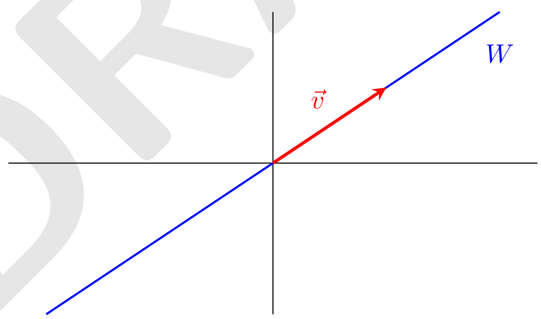


FIGURE 5.1: The subspace W generated by the vector $\vec{v}$.

- For fixed a_1, a_2, v_1, $v_2 \in \mathbb{R}$ let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Let us assume that $\vec{v} \neq \vec{0}$ and set $W = \{\vec{a} + \lambda\vec{v} : \lambda \in \mathbb{R}\}$. Then W is an affine subspace. Geometrically, W represents a line in $\mathbb{R}^2$ parallel to $\vec{v}$ which passes through the point (a_1, a_2). Note that W is a subspace if and only if $\vec{a}$ and $\vec{v}$ are parallel.

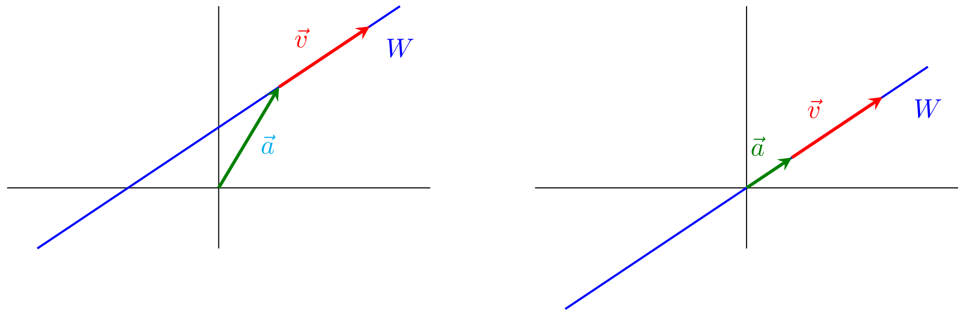


Figure 5.2: Sketches of $W = \{\vec{a} + \lambda\vec{v} : \lambda \in \mathbb{R}\}$. In the figure on the left hand side, $\vec{a} \not\parallel \vec{v}$, so W is an affine subspace of $\mathbb{R}^@$ but not a subspace. In the figure on the right hand side, $\vec{a} \parallel \vec{v}$ and therefore W is a subspace of $\mathbb{R}^2$.

- $U = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \geq 3\}$ is not a subspace of $\mathbb{R}^2$ since it does not contain $\vec{0}$.

- $V = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \leq 2\}$ is not a subspace of $\mathbb{R}^2$. For example, take $\vec{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $\vec{z} \in W$, however $3\vec{z} \notin V$.

- $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0 \right\}$. Then W is not a vector space. For example, $\vec{z} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \in W$, but $(-1)\vec{z} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \notin W$.

  Note that geometrically W is a right half plane in $\mathbb{R}^2$.

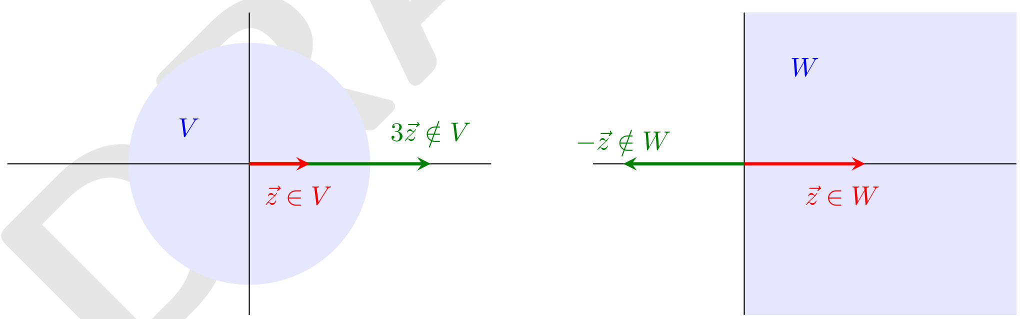


Figure 5.3: The sets V and W in the figures are not subspaces of $\mathbb{R}^2$.

Examples 5.15 (Examples and non-examples of subspaces of $\mathbb{R}^3$).

- For fixed $x_0, y_0, z_0 \in \mathbb{R}$ let $W = \left\{ \lambda \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$. Then W is a subspace of $\mathbb{R}^3$. Geometrically, W is a line in $\mathbb{R}^2$ passing through the origin which is parallel to the vector $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$.

- For fixed $a, b, c \in \mathbb{R}$ the set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : ax + by + cz = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

  Proof. We use Proposition 5.10 to verify that W is a subspace of $\mathbb{R}^3$. Clearly, $\vec{0} \in W$ since $0a+0b+0c = 0$. Now let $\vec{w}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{w}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ in W and let $\lambda \in \mathbb{R}$. Then $\vec{w}_1 + \vec{w}_2 \in W$ because

  $$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) = 0 + 0 = 0.$$

  Also $\lambda\vec{w}_1 \in W$ because

  $$a(\lambda x_1) + b(\lambda y_1) + c(\lambda z_1) = \lambda(ax_1 + by_1 + cz_1) = \lambda 0 = 0.$$

  Hence W is closed under sum and product with scalars, so it is a subspace of $\mathbb{R}$. □

  Remark. If $a = b = c = 0$, then $W = \mathbb{R}^3$. If at least one of the numbers $a, b, c \in \mathbb{R}$ is different from zero, then W is a plane in $\mathbb{R}^3$ which passes through the origin and has normal vector $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

- For fixed $a, b, c, d \in \mathbb{R}$ with $d \neq 0$, the set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : ax + by + cz = d \right\}$ is **not** a subspace of $\mathbb{R}^3$, see Figure 5.4.

  Proof. Let $\vec{w}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{w}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ in W. Then $\vec{w}_1 + \vec{w}_2 \notin W$ because

  $$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) = d + d = 2d \neq d.$$

  (Alternatively, we could have shown that if $\vec{w} \in W$ and $\lambda \in \mathbb{R} \setminus \{1\}$, then $\lambda\vec{w} \notin W$; or we could have shown that $\vec{0} \notin W$.) □

  Remark. If at least one of the numbers $a, b, c \in \mathbb{R}$ is different from zero, then W is a plane in $\mathbb{R}^3$ which has normal vector $\vec{n} = (a, b, c)^t$ but does not pass through the origin. If $a = b = c = 0$, then $W = \varnothing$.

- $W = \{\vec{x} \in \mathbb{R}^3 : \vec{x} \geq 5\}$ is not a subspace of $\mathbb{R}^3$ since it does not contain $\vec{0}$.

- $W = \{\vec{x} \in \mathbb{R}^3 : \vec{x} \leq 9\}$ is not a subspace of $\mathbb{R}^3$. For example, take $\vec{z} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$. Then $\vec{z} \in W$, however, for example, $7\vec{z} \notin W$ (or¿ $\vec{z} + \vec{z} \notin W$).

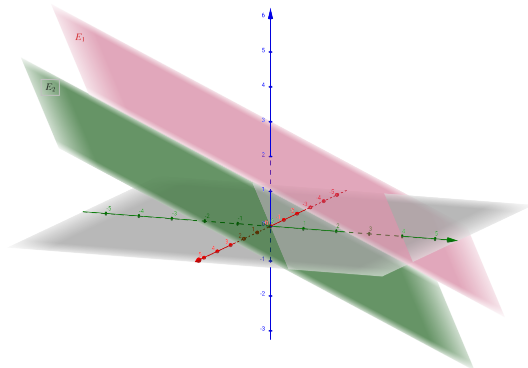


FIGURE 5.4: The green plane passes through the origin and is a subspace of $\mathbb{R}^3$. The red plane does not pass through the origin and therefore it is an *affine* subspace of $\mathbb{R}^3$.

- $W = \left\{ \begin{pmatrix} x \\ x^2 \\ x^3 \end{pmatrix} : x \in \mathbb{R} \right\}$. Then W is not a vector space. For example, $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in W$, but $2\vec{a} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \notin W$.

Examples 5.16 (Examples and non-examples of subspaces of $M(m \times n)$. The following sets are examples for subspaces of $M(m \times n)$:

- The set all matrices with $a_{11} = 0$.
- The set all matrices with $a_{11} = 5a_{12}$.
- The set all matrices such that its first row is equal to its last row.

If $m = n$, then also the following sets are subspaces of $M(n \times n)$:

- The set all symmetric matrices.
- The set all antisymmetric matrices.
- The set all diagonal matrices.
- The set all upper triangular matrices.
- The set all lower triangular matrices.

The following sets are **not** subspaces of $M(n \times n)$:

- The set all invertible matrices.

- The set all non-invertible matrices.
- The set all matrices with determinant equal to 1. The set all functions f with $f(7) = 13$.

Examples 5.17 (Examples and non-examples of subspaces of the set all functions from $\mathbb{R}$ to $\mathbb{R}$). Let V be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Then V clearly is a real vector space. The following sets are examples for subspaces of V:

- The set all continuous functions.
- The set all differential functions.
- The set all bounded functions.
- The set all polynomials.
- The set all polynomials with degree ≤ 5.
- The set all functions f with $f(7) = 0$.
- The set all even functions.
- The set all odd functions.

The following sets are **not** subspaces of V:

- The set all polynomials with degree 3.
- The set all polynomials with degree ≥ 3.
- The set all functions f with $f(7) = 13$.
- The set all functions f with degree $f(7) \geq 0$.

Prove the claims above.

Definition 5.18. For $n \in \mathbb{N}_0$ let P_n be the set of all polynomials of degree less than or equal to n.

Remark 5.19. P_n is a vector space.

Proof. Clearly, the zero function belongs to P_n (it is the polynomial of degree 0). For polynomials $p, q \in P_n$ and numbers $\lambda \in \mathbb{R}$ we clearly have that $p + q$ and λp are again polynomials of degree at most n, so they belong to P_n. By Proposition 5.10, P_n is a subspace of the space of all real functions, hence it is a vector space. $\square$

You should have understood

- the concept of a subspace of a given vector space,
- why we only have to check it a given subset of a vector space is non-empty, closed under sum and closed under multiplication with scalars if we want to see if it is a subspace,
- ...

You should now be able to

- give examples and non-examples of subspaces of vector spaces,
- check if a given subset of a vector space is a subspace,
- ...

5.3 Linear Combinations and linear independence

In this section, we work with real vector spaces for definiteness sake. However, all the statements are also true for complex vector spaces. We only have to replace everywhere $\mathbb{R}$ by $\mathbb{C}$ and the word *real* by *complex*.

We start with a definition.

Definition 5.20. Let V be a real vector space and let $v_1, \dots, v_k \in V$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. Then every vector of the form

$$v = \alpha_1 v_1 + \cdots \alpha_k v_k \tag{5.1}$$

is called a *linear combination of the vectors* $v_1, \dots, v_k \in V$.

Examples 5.21.

- Let $V = \mathbb{R}^3$ and let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $\vec{a} = \begin{pmatrix} 9 \\ 12 \\ 15 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

  Then $\vec{a}$ and $\vec{b}$ are linear combinations of $\vec{v}_1$ and $\vec{v}_2$ because $\vec{a} = \vec{v}_1 + 2\vec{v}_2$ and $\vec{b} = -\vec{v}_1 + \vec{v}_2$.

- Let $V = M(2 \times 2)$ and let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 5 & 7 \\ -7 & 5 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$.

  Then R is a linear combination of A and B because $R = 5A+7B$. S is **not** a linear combination of A and B. because clearly every linear combination of A and B is of the form

  $$\alpha A + \beta B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

  so it can never be equal to S since S has two different elements on its diagonal.

Definition and Theorem 5.22. *Let V be a real vector space and let $v_1, \dots, v_k \in V$. Then the set of all their possible linear combinations is denoted by*

$$\operatorname{span}\{v_1, \dots, v_k\} := \{\alpha_1 v_1 + \cdots + \alpha_k v_k \; : \; \alpha_1, \dots, \alpha_k \in \mathbb{R}\}.$$

It is a subspace of V and it is called the linear span of the $v_1, \dots, v_k$. *The vectors $v_1, \dots, v_k$ are called* generators *of the subspace* $\operatorname{span}\{v_1, \dots, v_k\}$.

Remark. Other names for “linear span” that are commonly used, are *subspace generated by the $v_1, \dots, v_k$* or *subspace spanned by the $v_1, \dots, v_k$*. Instead of $\operatorname{span}\{v_1, \dots, v_k\}$ the notation $\operatorname{gen}\{v_1, \dots, v_k\}$ is used frequently. All these names and notations mean exactly the same.

Proof of Theorem 5.22. We have to show that $W := \operatorname{span}\{v_1, \dots, v_k\}$ is a subspace of V. To this end, we use again Proposition 5.10. Clearly, W is not empty since at least $\mathbb{O} \in W$ (we only need to choose all the $\alpha_j = 0$). Now let $u, w \in W$ and $\lambda \in \mathbb{R}$. We have to show that $\lambda u + w \in W$. Since $u, w \in W$, there are real numbers $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$ such that $u = \alpha_1 v_1 + \dots, \alpha_k v_k$ and $w = \beta_1 w_1 + \dots + \beta_k v_k$. Then

$$\begin{aligned}\lambda u + v &= \lambda(\alpha_1 v_1 + \dots + \alpha_k v_k) + \beta_1 w_1 + \dots + \beta_k v_k \\ &= \lambda\alpha_1 v_1 + \dots + \lambda\alpha_k v_k) + \beta_1 w_1 + \dots + \beta_k v_k \\ &= (\lambda\alpha_1 + \beta_1)v_1 + \dots + (\lambda\alpha_k + \beta_k)v_k\end{aligned}$$

which belongs to W since it is a linear combination of the $v_1, \dots, v_k$. □

Remark. The generators of a given subspace are not unique.

For example, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then

$$\begin{aligned}\operatorname{span}\{A, B\} &= \{\alpha A + \beta B : \alpha, \beta \in \mathbb{R}\} &&= \left\{ \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}, \\ \operatorname{span}\{A, B, C\} &= \{\alpha A + \beta B + \gamma C : \alpha, \beta, \gamma \in \mathbb{R}\} &&= \left\{ \begin{pmatrix} \alpha + \gamma & -(\beta + \gamma) \\ \beta + \gamma & \alpha + \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}, \\ \operatorname{span}\{A, C\} &= \{\alpha A + \gamma C : \alpha, \gamma \in \mathbb{R}\} &&= \left\{ \begin{pmatrix} \alpha + \gamma & -\gamma \\ \gamma & \alpha + \gamma \end{pmatrix} : \alpha, \gamma \in \mathbb{R} \right\}.\end{aligned}$$

We see that $\operatorname{span}\{A, B\} = \operatorname{span}\{A, B, C\} = \operatorname{span}\{A, C\}$ (in all cases it consists of exactly those matrices whose diagonal entries are equal and the off-diagonal entries differ by a minus sign). So we see that neither the generators nor their number is unique.

Remark. If a vector is a linear combination of other vectors, then the coefficients in the linear combination are not necessarily unique.

For example, if A, B, C are the matrices above, then $A + B + C = 2A + 2B = 2C$ or $A + 2B + 3C = 4A + 5B = B + 4C$, etc.

Remark 5.23. Let V be a vector space and let $v_1, \dots, v_n$ and $w_1, \dots, w_m$ be vectors in V. Then the following is equivalent:

(i) $\operatorname{span}\{v_1, \dots, v_n\} = \operatorname{span}\{w_1, \dots, w_m\}$.

(ii) $v_j \in \operatorname{span}\{w_1, \dots, w_m\}$ for every $j = 1, \dots, n$ and $w_k \in \operatorname{span}\{v_1, \dots, v_n\}$ for every $k = 1, \dots, m$.

Proof. (i) $\Longrightarrow$ (ii) is clear.
(ii) $\Longrightarrow$ (i): Note that $v_j \in \operatorname{span}\{w_1, \dots, w_m\}$ for every $j = 1, \dots, n$ implies that every v_j can be written as a linear combination of the $w_1, \dots, w_m$. Then also every linear combination of $v_1, \dots, v_n$ is a linear combination of $w_1, \dots, w_m$. This implies that $\operatorname{span}\{v_1, \dots, v_n\} \subseteq \operatorname{span}\{w_1, \dots, w_m\}$. The converse inclusion $\operatorname{span}\{w_1, \dots, w_m\} \subseteq \operatorname{span}\{v_1, \dots, v_n\}$ can be shown analogously. Both inclusions together show that we must have equality. □

Examples 5.24. (i) A set of generators of P_n is for example $\{1, X, X^2, \dots, X^{n-1}, X^n\}$ since every vector in P_n is a polynomial of the form $p = \alpha_n X^n + \alpha_{n-1}X^{n-1} + \dots + \alpha_1 X + \alpha_0$, so it is a linear combination of the polynomials X^n, X^{n-1}, $\dots$, X, 1.

> **Exercise.** Show that $\{1,\ 1+X,\ X+X^2,\ \dots,\ X^{n-1}+X^n\}$ is also a set of generators of P_n.

(ii) The set of all antisymmetric 2×2 matrices is generated by $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$.

(iii) Let $V = \mathbb{R}^3$ and let $\vec v, \vec w \in \mathbb{R}^3 \setminus \{\vec 0\}$.

- $\operatorname{span}\{\vec v\}$ is a line which passes through the origin and is parallel to $\vec v$.
- If $\vec v \not\parallel \vec w$, then $\operatorname{span}\{\vec v, \vec w\}$ is a plane which passes through the origin and is parallel to $\vec v$ and $\vec w$. If $\vec v \parallel \vec w$, then it is a line which passes through the origin and is parallel to $\vec v$.

Example 5.25. Let $p_1 = X^2 - X + 1, p_2 = X^2 - 2X + 5 \in P_2$, and let $U = \operatorname{span}\{p_1,\ p_2\}$. Check if $q = 2X^2 - X - 2$ and $r = X^2 + X - 3$ belong to U.

Solution. • Let us check if $q \in U$. To this end we have to check if we can find α, β such that $q = \alpha p_1 + \beta p_2$. Inserting the expressions for p_1, p_2, q we obtain

$$2X^2 - X - 2 = \alpha(X^2 - X + 1) + \beta(X^2 - 2X + 5) = X^2(\alpha+\beta) + X(-\alpha - 2\beta) + \alpha + 5\beta.$$

Comparing coefficients of the different powers of X, we obtain the system of equations

$$\begin{aligned} \alpha + \ \ \beta &= 2 \\ -\alpha - 2\beta &= -1 \\ \alpha + 5\beta &= -2. \end{aligned}$$

We use the Gauß-Jordan process to solve the system.

$$A = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & -2 & -1 \\ 1 & 5 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right).$$

We see that $\alpha = 3$ and $\beta = -1$ is a solution, and therfore $q = 2p_1 - p_2$ which shows that $q \in U$.

- Let us check if $r \in U$. To this end we have to check if we can find α, β such that $r = \alpha p_1 + \beta p_2$. Inserting the expressions for p_1, p_2, q we obtain

$$X^2 + X - 3 = \alpha(X^2 - X + 1) + \beta(X^2 - 2X + 5) = X^2(\alpha+\beta) + X(-\alpha - 2\beta) + \alpha + 5\beta.$$

Comparing coefficients of the different powers of X, we obtain the system of equations

$$\begin{aligned} \alpha + \ \ \beta &= 1 \\ -\alpha - 2\beta &= 1 \\ \alpha + 5\beta &= -3. \end{aligned}$$

We use the Gauß-Jordan process to solve the system.

$$A = \left(\begin{array}{rr|r} 1 & 1 & 1 \\ -1 & -2 & 1 \\ 1 & 5 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{rr|r} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 4 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{array}\right).$$

We see that the system is inconsistent. Therefore r is not a linear combination of p_1 and p_2, hence $r \notin U$. $\diamond$

Definition 5.26. A vector space V is called *finitely generated* if is has a finite set of generators.

Examples 5.27. Important finitely generated vector spaces are

- $\mathbb{R}^n$ because clearly gen$\{\vec{e}_1, \dots, \vec{e}_n\}$ where $\vec{e}_j$ is the jth unit vector.
- $M(m \times n)$ because it is generated by the set of all possible matrices which are 0 everywhere except a 1 in exactly one entry.
- P_n is finitely generated as was shown in Example 5.24.
- Let P be the vector space of all real polynomials. Then P is not finitely generated.

  Proof. Assume that P is finitely generated and let $q_1, \dots, q_k$ be a system of generators of P. Note that the q_j are polynomials. We will denote their degrees by $m_j = \deg q_j$ and we set $M = \max\{m_1, \dots, m_k\}$. No matter who we choose the coefficients, any linear combination of them will be a polynomial of degree at most M. However, there are elements in P which have higher degree, for example X^{m+1}. Therefore $q_1, \dots, q_k$ cannot generate all of P. $\square$

  Another proof using the concept of dimension will be given in Example 5.52 (f).

Later, in Lemma 5.49, we will see that every subspace of a finitely generated vector space is again finitely generated.

Now we ask ourselves how many vectors we need at least in order to generate $\mathbb{R}^n$. We know that for example $\mathbb{R}^n = \operatorname{span}\{\vec{e}_1, \dots, \vec{e}_n\}$. So in this case we have n vectors that generate $\mathbb{R}^n$. Could it be that less vectors are sufficient? Clearly, if we take away one of the $\vec{e}_j$, then the remaining system no longer generates $\mathbb{R}^n$ since "one coordinate is missing". However, could we maybe find other vectors so that $n-1$ or less vectors are enough to generate all of $\mathbb{R}^n$? The next proposition says that this is not possible.

Proposition 5.28. *Let $\vec{v}_1, \dots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. If $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$, then $k \geq n$.*

Proof. Let $A = (\vec{v}_1|\dots|\vec{v}_k)$ be the matrix whose columns are the given vectors. We know that there exists an invertible matrix E such that $A' = EA$ is in reduced echelon form (the matrix E is the product of elementary matrices which correspond to the steps in the Gauß-Jordan process to arrive at the reduced echelon form). Now, if $k < n$, then we know that A' must have at least one row which consists of zeros only. If we can find a vector $\vec{w}$ such that it is transformed to $\vec{e}_n$ under the Gauß-Jordan process, then we would have that $A\vec{x} = \vec{w}$ is inconsistent, which means that $\vec{w} \notin \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$. How do we find such a vector $\vec{w}$? Well, we only have to start with $\vec{e}_n$

and "do the Gauß-Jordan process backwards". In other words, we can take $\vec{w} = E^{-1}\vec{e}_n$. Now if we apply the Gauß-Jordan process to the augmented matrix $(A|\vec{w})$, we arrive at $(EA|E\vec{w}) = (A'|\vec{e}_n)$ which we already know is inconsistent.
Therefore, $k < n$ is not possible and therefore we must have that $k \geq n$. □

Note that the proof above is basically the same as the one in Remark 3.36 because if we want to check if a system of vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ is a set of generators for $\mathbb{R}^n$, then we have to check that the equation $A\vec{y} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$.

Now we will answer the question when the coefficients of a linear combination are unique. The following remark shows us that we have to answer this question only for the zero vector.

Remark 5.29. Let V be a vector space, let $v_1, \dots, v_k \in V$ and let $w \in \operatorname{span}\{v_1, \dots, v_k\}$. Then there are unique $\beta_1, \dots, \beta_k \in \mathbb{R}$ such that

$$\beta_1 v_1 + \cdots + \beta_k v_k = w \tag{5.2}$$

if and only if there are unique $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = \mathbb{O}. \tag{5.3}$$

Proof. First note that (5.3) always has at least one solution, namely $\alpha_1 = \cdots = \alpha_k = 0$. This solution is called the *trivial solution*.
Let us assume that (5.2) has two different solutions, so that there are $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ such that for at least one $j = 1, \dots, k$ we have that $\beta_j \neq \gamma_j$ and

$$\gamma_1 v_1 + \cdots + \gamma_k v_k = w. \tag{5.2'}$$

Subtracting (5.2) and (5.2') gives

$$(\beta_1 - \gamma_1)v_1 + \cdots + (\beta_k - \gamma_k)v_k = w - w = \mathbb{O}$$

where at least one coefficient is different from zero. Therefore also (5.3) has more than one solution.
On the other hand, let us assume that (5.3) has a non-trivial solution, that is, at least one of the α_j in (5.3) is different from zero. But then, if we sum (5.2) and (5.3) we obtain another solution for (5.2) because

$$(\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_1 + \beta_k)v_k = \mathbb{O} + w = w.$$ □

The proof shows that there are as many solutions of (5.2) as there are of (5.3).

It should also be noted that if (5.3) has one non-trivial solution, then it has automatically infinitely many solutions, because if $\alpha_1, \dots, \alpha_k$ is a solution, then also $c\alpha_1, \dots, c\alpha_k$ is a solution for arbitrary $c \in \mathbb{R}$ since

$$c\alpha_1 v_1 + \cdots + c\alpha_k v_k = c(\alpha_1 v_1 + \cdots + \alpha_k v_k) = c\,\mathbb{O} = \mathbb{O}.$$

In fact, the discussion above should remind you of the relation between solutions of an inhomogeneous system and the solutions of its associated homogeneous system in Theorem 3.21. Note that just as in the case of linear systems, (5.2) could have no solution. This happens if and only if $w \notin \operatorname{span}\{v_1, \dots, v_k\}$.

So we see that only one of the following two cases can occur: (5.4) as exactly one solution (namely the trivial one) or it has infinitely many solutions. Note that this is analogous to the situation of the solutions of homogeneous linear systems: They have either only the trivial solution or they have infinitely many solutions. The following definition distinguishes between the two cases.

Definition 5.30. Let V be a vector space. The vectors $v_1, \dots, v_k$ in V are called *linearly independent* if

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = \mathbb{O}. \tag{5.4}$$

has only the trivial solution. They are called *linearly dependent* if (5.4) has more than one solution.

Before we continue with the theory, we give a few examples.

Examples. (i) The vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} -4 \\ -8 \end{pmatrix} \in \mathbb{R}^2$ are linearly dependent because $4\vec{v_1} + \vec{v_2} = \vec{0}$.

(ii) The vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ are linearly independent.

Proof. Consider the equation $\alpha\vec{v_1} + \beta\vec{v_2} = \vec{0}$. This equation is equivalent to the following system of linear equations for α and β:

$$\begin{aligned} \alpha + 3\beta &= 0 \\ 2\alpha + 0\beta &= 0. \end{aligned}$$

We can use the Gauß-Jordan process to obtain all solutions. However, in this case we easily see that $\alpha = 0$ (from the second line) and then that $\beta = -\frac{1}{3}\alpha = 0$. Note that we could also have calculated $\det(\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = -6 \neq 0$ to conclude that the homogeneous system above has only the trivial solution. Observe that the columns of the matrix are exactly the given vectors. □

(iii) The vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^2$ are linearly independent.

Proof. Consider the equation $\alpha\vec{v_1} + \beta\vec{v_2} = \vec{0}$. This equation is equivalent to the following system of linear equations for α and β:

$$\begin{aligned} \alpha + 2\beta &= 0 \\ \alpha + 3\beta &= 0 \\ \alpha + 4\beta &= 0. \end{aligned}$$

If we subtract the first from the second equation, we obtain $\beta = 0$ and then $\alpha = -2\beta = 0$. So again, this system has only the trivial solution and therefore the vectors $\vec{v_1}$ and $\vec{v_2}$ are linearly independent. □

(iv) Let $\vec{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ $\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v_4} = \begin{pmatrix} 0 \\ 6 \\ 8 \end{pmatrix} \in \mathbb{R}^2$ Then

(a) The system $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

(b) The system $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ is linearly dependent.

Proof. (a) Consider the equation $\alpha\vec{v_1} + \beta\vec{v_2} + \gamma\vec{v_3} = \vec{0}$. This equation is equivalent to the following system of linear equations for α, β and γ:

$$\begin{aligned} \alpha - 1\beta + 0\gamma &= 0 \\ \alpha + 2\beta + 0\gamma &= 0 \\ \alpha + 3\beta + 1\gamma &= 0. \end{aligned}$$

We use the Gauß-Jordan process to solve the system. Note that the columns of the matrix associated to the above system are exactly the given vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the unique solution is $\alpha = \beta = \gamma = 0$ and consequently the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

Observe that we also could have calculated $\det A = 3 \neq 0$ to conclude that the homogeneous system has only the trivial solution.

(b) Consider the equation $\alpha\vec{v_1} + \beta\vec{v_2} + \delta\vec{v_4} = \vec{0}$. This equation is equivalent to the following system of linear equations for α, β and δ:

$$\begin{aligned} \alpha - 1\beta + 0\delta &= 0 \\ \alpha + 2\beta + 6\delta &= 0 \\ \alpha + 3\beta + 8\delta &= 0. \end{aligned}$$

We use the Gauß-Jordan process to solve the system. Note that the columns of the matrix associated to the above system, are exactly the given vectors.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 6 \\ 1 & 3 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 6 \\ 0 & 4 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the unique solution is $\alpha = \beta = \gamma = 0$ and consequently the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent. So there are infinitely many solutions. If we take $\delta = t$, then $\alpha = \beta = -2t$. Consequently the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent, because, for example, $-2\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = 0$ (taking $t = 1$).

Observe that we also could have calculated $\det A = 0$ to conclude that the system has infinite solutions. $\square$

(v) The matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are linearly independent in $M(2 \times 2)$.

(vi) The matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are linearly dependent in $M(2 \times 2)$.

After these examples we will proceed with some facts on linear independence. We start with the special case when we have only two vectors.

Proposition 5.31. *Let v_1, v_2 be vectors in a vector space V. Then v_1, v_2 are linearly dependent if and only if one vector is a multiple of the other.*

Proof. Assume that v_1, v_2 are linearly dependent. Then there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 v_1 + \alpha_2 v_2 = 0$ and at least one of the α_1 and α_2 is different from zero, say $\alpha_1 \neq 0$. Then we have $v_1 + \frac{\alpha_2}{\alpha_1} v_2 = 0$, hence $v_1 = -\frac{\alpha_2}{\alpha_1} v_2$.
Now assume on the other hand that, e.g., v_1 is a multiple of v_2, that is $v_1 = \lambda v_2$ for some $\lambda \in \mathbb{R}$. Then $v_1 - \lambda v_2 = 0$ which is a nontrivial solution of $\alpha_1 v_1 + \alpha_2 v_2 = 0$ because we can take $\alpha_1 = 1 \neq 0$ (note that λ may be zero). □

The proposition above cannot be extended to the case of three or more vectors. For instance, the vectors $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly dependent because $\vec{a} + \vec{b} - \vec{c} = \vec{0}$, but none of them is a multiple of any of the other two vectors.

Proposition 5.32. *Let V be a vector space.*

(i) *Every system of vectors which contains $\mathbb{O}$ is linearly dependent.*

(ii) *Let $v_1, \dots, v_k \in V$ and assume that there are $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $\alpha_1 v_1 + \dots + \alpha_k v_k = \mathbb{O}$. If $\alpha_\ell \neq 0$, then v_ℓ is a linear combination of the other v_j.*

(iii) *If the vectors $v_1, \dots, v_k \in V$ are linearly dependent, then for every $w \in V$, the vectors $v_1, \dots, v_k, w$ are linearly dependent.*

(iv) *If the vectors $v_1, \dots, v_k \in V$ are linearly independent, then every subset of them is linearly independent.*

(v) *If $v_1, \dots, v_k$ are vectors in V and w is a linear combination of them, then $v_1, \dots, v_k, w$ are linearly dependent.*

Proof. (i) Let $v_1, \dots, v_k \in V$. Clearly $1\mathbb{O} + 0v_1 + \dots + 0v_k = \mathbb{O}$ is a non-trivial linear combination which gives $\mathbb{O}$. Therefore the system $\{v_1, \dots, v_k, \mathbb{O}\}$ is linearly dependent.

(ii) If $\alpha_\ell \neq 0$, then we can solve for v_ℓ: $v_\ell = -\frac{\alpha_1}{\alpha_\ell} v_1 - \dots - \frac{\alpha_{\ell-1}}{\alpha_\ell} v_{\ell-1} - \frac{\alpha_{\ell+1}}{\alpha_\ell} v_{\ell+1} - \dots - \frac{\alpha_k}{\alpha_\ell} v_k$.

(iii) If the vectors $v_1, \dots, v_k \in V$ are linearly dependent, then there exist $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that at least one of them is different from zero and $\alpha_1 v_1 + \dots + \alpha_k v_k = \mathbb{O}$. But then also $\alpha_1 v_1 + \dots + \alpha_k v_k + 0w = \mathbb{O}$ which shows that the system $\{v_1, \dots, v_k, w\}$ is linearly dependent.

(iv) Suppose that a subsystem of $v_1, \dots, v_k \in V$ are linearly dependent. Then, by (iii) every system in which it is contained, must be linearly dependent too. In particular, the original system of vectors must be linearly dependent which contradicts our assumption.

(v) Assume that w is a linear combination of $v_1, \dots, v_k$. Then there exist $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $w = \alpha_1 v_1 + \dots + \alpha_k v_k$. Therefore we obtain $w - \alpha_1 v_1 - \dots - \alpha_k v_k = \mathbb{O}$ which is a non-trivial linear combination since the coefficient of w is 1.

□

Now we specialise to the case when $V = \mathbb{R}^n$. Let us take vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ and let us write $(\vec{v}_1|\cdots|\vec{v}_k)$ for the $n \times k$ matrix whose columns are the vectors $\vec{v}_1, \dots, \vec{v}_k$.

Lemma 5.33. *With the above notation, the following statements are equivalent:*

(i) *$\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent.*

(ii) *There exist $\alpha_1, \dots, \alpha_k$ not all equal to zero, such that $\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k = 0$.*

(iii) *There exists a vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} \neq \vec{0}$ such that $(\vec{v}_1|\cdots|\vec{v}_k) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = \vec{0}$.*

(iv) *The homogeneous system corresponding to the matrix $(\vec{v}_1|\cdots|\vec{v}_k)$ has at least one non-trivial (and therefore infinitely many) solutions.*

Proof. (i) $\implies$ (ii) is simply the definition of linear independence. (ii) $\implies$ (iii) is only rewriting the vector equation in matrix form. (iv) only says in word what the equation in (iii) means. And finally (iv) $\implies$ (i) holds because every non trivial solution the inhomogeneous system associated to $(\vec{v}_1|\cdots|\vec{v}_k)$ gives a non-trivial solution of $\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k$. □

Since we know that a homogeneous linear system with more unknowns than equations has infinitely many solutions, we immediately obtain the following corollary.

Corollary 5.34. *Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$.*

(i) *If $k > n$, then the vectors $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent.*

(ii) *If the vectors $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, then $k \leq n$.*

Observe that (ii) does **not** say that if $k \leq n$, then the vectors $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent. It only says that they have a chance to be linearly independent whereas a system with more than n vectors always is linearly dependent.

Now we specialise further to the case when $k = n$.

Theorem 5.35. *Let $\vec{v}_1, \dots, \vec{v}_n$ be vectors in $\mathbb{R}^n$. Then the following is equivalent:*

(i) *$\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.*

(ii) *The only solution of* $(\vec{v}_1|\cdots|\vec{v}_n)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix} = \vec{0}$ *is the zero vector* $\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix} = \vec{0}$.

(iii) *The matrix* $(\vec{v}_1|\cdots|\vec{v}_n)$ *is invertible.*

(iv) $\det(\vec{v}_1|\cdots|\vec{v}_n) \neq 0$.

Proof. The equivalence of (i) and (ii) follows from Lemma 5.33. The equivalence of (ii), (iii) and (iv) follows from Theorem 4.11. □

Formulate an analogous theorem for linearly dependent vectors.

Now we can state when a system n vectors in $\mathbb{R}^n$ is generating $\mathbb{R}^n$.

Theorem 5.36. *Let* $\vec{v}_1, \dots, \vec{v}_n$ *be vectors in* $\mathbb{R}^n$*. and let* $A = (\vec{v}_1|\cdots|\vec{v}_n)$ *be the matrix whose columns are the given vectors* $\vec{v}_1, \cdots, \vec{v}_n$*. Then the following is equivalent:*

(i) $\vec{v}_1, \dots, \vec{v}_n$ *are linearly independent.*

(ii) $\mathbb{R}^n = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$.

(iii) $\det A \neq 0$.

Proof. (i) $\iff$ (iii) is shown in Theorem 5.35.
(ii) $\iff$ (iii): The vectors $\vec{v}_1, \dots, \vec{v}_n$ generate $\mathbb{R}^n$ if and only if for every $\vec{w} \in \mathbb{R}^n$ there exist numbers $\beta_1, \dots, \beta_n$ such that $\beta_1\vec{v}_1 + \cdots + \beta_n v_n = \vec{w}$. In matrix form that means that $A\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix} = \vec{w}$. By Theorem 3.43 we know that this has a solution for every vector $\vec{w}$ if and only if A is invertible (because if we apply Gauß-Jordan to A, we must get to the identity matrix). □

The proof of the preceding theorem basically goes like this: We consider the equation $A\vec{\beta} = \vec{w}$. When are the vectors $\vec{v}_1, \dots, \vec{v}_n$ linearly independent? – They are linearly independent if and only if for $\vec{w} = \vec{0}$ the system has only the trivial solution. This happens if and only if the reduced echelon form of A is the identity matrix. And this happens if and only if $\det A \neq 0$.
When do the vectors $\vec{v}_1, \dots, \vec{v}_n$ generate $\mathbb{R}^n$? – They do, if and only if for every given vector $\vec{w} \in \mathbb{R}^n$ the system has at least one solution. This happens if and only if the reduced echelon form of A is the identity matrix. And this happens if and only if $\det A \neq 0$.

Since a square matrix A in invertible if and only if its transpose A^t is invertible, Theorem 5.36 leads immediately to the following corollary.

Corollary 5.37. *For a matrix* $A \in M(n \times n)$ *the following is equivalent:*

(i) A *is invertible.*

(ii) *The columns of* A *are linearly independent.*

(iii) *The rows of* A *are linearly independent.*

We end this section with more examples.

Examples.

- Recall that P_n is the vector space of all polynomials of degree $\leq n$.

  In P_3, we consider the vectors $p_1 = X^3 - 1$, $p_2 = X^2 - 1$, $p_3 = X - 1$. These vectors are linearly independent.

  Proof. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$. This means that

$$\begin{aligned} 0 &= \alpha_1(X^3 - 1) + \alpha_2(X^2 - 1) + \alpha_3(X - 1) \\ &= \alpha_1 X^3 + \alpha_2 X^2 + \alpha_3 X - (\alpha_1 + \alpha_2 + \alpha_3). \end{aligned}$$

  Comparing coefficients, it follows that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$ which shows that p_1, p_2 and p_3 are linearly independent. □

  If in addition we take $p_4 = X^3 - X^2$, then the system p_1, p_2, p_3 and p_4 is linearly dependent.

  Proof. As before, let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ such that $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 = 0$. This means that

$$\begin{aligned} 0 &= \alpha_1(X^3 - 1) + \alpha_2(X^2 - 1) + \alpha_3(X - 1) + \alpha_4(X^3 - X^2) \\ &= (\alpha_1 + \alpha_4)X^3 + (\alpha_2 - \alpha_4)X^2 + \alpha_3 X - (\alpha_1 + \alpha_2 + \alpha_3). \end{aligned}$$

  Comparing coefficients, this is equivalent to $\alpha_1 + \alpha_4 = 0$, $\alpha_2 - \alpha_4 = 0$, $\alpha_3 = 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$. This system of equations has infinitely many solutions. They are given by $\alpha_2 = \alpha_4 = -\alpha_1 \in \mathbb{R}$, $\alpha_3 = 0$ (verify this!). Therefore p_1, p_2, p_3 and p_4 are linearly dependent. □

  Exercise. Show that p_1, p_2, p_3 and p_5 are linearly independent if $p_5 = X^3 + X^2$.

- In P_2, we consider the vectors $p_1 = X^2 + 2X - 1$, $p_2 = 5X + 2$, $p_3 = 2X^2 - 11X - 8$. These vectors are linearly dependent.

  Proof. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$. This means that

$$\begin{aligned} 0 &= \alpha_1(X^2 + 2X - 1) + \alpha_2(5X + 2) + \alpha_3(2X^2 - 11X - 8) \\ &= (\alpha_1 + 2\alpha_3)X^2 + (2\alpha_1 + 5\alpha_2 - 11\alpha_3)X - \alpha_1 + 2\alpha_2 - 8\alpha_3). \end{aligned}$$

  Comparing coefficients, it follows that $\alpha_1 + 2\alpha_3 = 0$, $2\alpha_1 + 5\alpha_2 - 11\alpha_3 = 0$, $-\alpha_1 + 2\alpha_2 - 8\alpha_3 = 0$. We write this in matrix form and apply Gauß-Jordan:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 5 & -11 \\ -1 & 2 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & -15 \\ 0 & 2 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

  This shows that the system has non-trivial solutions (find them!) and therefore p_1, p_2 and p_3 are linearly dependent. □

- In $V = M(2\times 2)$ consider $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$. Then A, B, C are linearly dependent because $A - B - \frac{1}{5}C = 0$.
- In $V = M(2\times 3)$ consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$. Then A, B, C are linearly independent.

  Exercise. Prove this!

You should have understood

- what a linear combination is,
- the concept of linear independence,
- the concept of linear span and that it consists either of only the zero vector or of infinitely many vectors,
- geometrically the concept of linear independence in $\mathbb{R}^2$ and $\mathbb{R}^3$,
- that the coefficients in a linear combination are not necessarily unique,
- what the number of solutions of $A\vec{x} = \vec{0}$ says about the linear independence of the columns of A seen as vectors in $\mathbb{R}^n$,
- what the existence (or non-existence) of solutions of $A\vec{x} = \vec{b}$ for all $\vec{b} \in \mathbb{R}^m$ says about the span of the columns of A seen as vectors in $\mathbb{R}^n$,
- why a matrix $A \in M(n\times n)$ is invertible if and only if its columns are linearly independent,
- ...

You should now be able to

- verify if a given vector is a linear combination of a given set of vectors,
- verify if a given vector lies in the linear span of a given set of vectors,
- verify if a given set of vectors is a generator of a given vectors space,
- find a set of generators for a given vectors space,
- verify if a given set of vectors is a linearly independent,
- ...

5.4 Basis and dimension

In this section, we work with real vector spaces for definiteness sake. However, all the statements are also true for complex vector spaces. We only have to replace everywhere $\mathbb{R}$ by $\mathbb{C}$ and the word *real* by *complex*.

Definition 5.38. Let V be a vector space. A *basis* of V is a set of vectors $\{v_1, \dots, v_n\}$ in V which is linearly independent and generates V.

The following remark shows that a basis is a *minimal system of generators of* V and at the same time a *maximal system of linear independent vectors.*

Remark. Let $\{v_1, \dots, v_n\}$ be a basis of V.

(i) Let $w \in V$. Then $\{v_1, \dots, v_n, w\}$ in not a basis of V because this system of vectors is no longer linearly independent by Proposition 5.32 (v)

(ii) If we take away one of the vectors from $\{v_1, \dots, v_n\}$, then it is no longer a basis of V because the new system of vectors no longer generates V. For example, if we take away v_1, then $v_1 \notin \operatorname{span}\{v_2, \dots, v_n\}$ (otherwise $v_1, \dots, v_n$ would be linearly dependent), and therefore $\operatorname{span}\{v_2, \dots, v_n\} \neq V$.

Remark 5.39. Every basis of $\mathbb{R}^n$ has exactly n elements. To see this note that by Corollary 5.34, a basis can have at most n elements because otherwise it cannot be linearly independent. On the other hand, if it had less than n elements, then, by Remark 5.28, it cannot generate $\mathbb{R}^n$.

Examples 5.40.

- A basis of $\mathbb{R}^3$ is, for example, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. The vectors of this basis are the standard unit vectors. The basis is called the *standard basis* (or *canonical basis*) of $\mathbb{R}^3$.

  Other examples of bases of $\mathbb{R}^3$ are

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

  Exercise. Verify that the systems above are bases of $\mathbb{R}^3$.

  The following systems are **not** bases of $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

  Exercise. Verify that the systems above are not bases of $\mathbb{R}^3$.

- The *standard basis in* $\mathbb{R}^n$ (or *canonical basis in* $\mathbb{R}^n$) is $\{\vec{e}_1, \dots, \vec{e}_n\}$. Recall that the $\vec{e}_j$ are the standard unit vectors whose jth entry is 1 and all other entries are 0.

  Exercise. Verify that they form a basis of $\mathbb{R}^n$.

- The *standard basis in* P_n (or *canonical basis in* P_n) is $\{1, X, X^2, \dots, X^n\}$.

Exercise. Verify that they form a basis of P_n.

- Let $p_1 = X,\ p_2 = 2X^2 + 5X - 1, p_3 = 3X^2 + X + 2$. Then the system $\{p_1, p_2, p_3\}$ is a basis of P_2.

  Proof. We have to show that the system in linearly independent and that it generates the space P_2. Let $q = aX^2 + bX + c \in P_2$. We want to see if there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $q = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$. If we write this equation out, we find

$$\begin{aligned} aX^2 + bX + c &= \alpha_1 X + \alpha_2(2X^2 + 5X - 1) + \alpha_3(3X^2 + X + 2) \\ &= (2\alpha_2 + 3\alpha_3)X^2 + (\alpha_1 + 5\alpha_2 + \alpha_3)X - \alpha_2 + 2\alpha_3. \end{aligned}$$

  Comparing coefficients, we obtain the following system of linear equations for the α_j:

$$\left.\begin{array}{rrrl} & 2\alpha_2 & +3\alpha_3 & = a \\ \alpha_1 + & 5\alpha_2 & +\alpha_3 & = b \\ & -\alpha_2 & +2\alpha_3 & = c \end{array}\right\} \qquad \text{in matrix form:} \qquad \begin{pmatrix} 0 & 2 & 3 \\ 1 & 5 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

  Now we apply Gauß-Jordan to the augmented matrix:

$$\left(\begin{array}{rrr|r} 0 & 2 & 3 & a \\ 1 & 5 & 1 & b \\ 0 & -1 & 2 & c \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 5 & 1 & b \\ 0 & -1 & 2 & c \\ 0 & 2 & 3 & a \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 1 & 0 & 11 & b + 5c \\ 0 & 1 & -2 & c \\ 0 & 0 & 7 & a + 2c \end{array}\right).$$

  So we see that there is exactly one solution for any given q. The existence of such a solution shows that $\{p_1, p_2, p_3\}$ generates P_2. We also see that for any give $q \in P_2$ there is exactly one way to write it as a linear combination of p_1, p_2, p_3. If we take the special case $q = 0$, this shows that the system is linearly independent. In summary, $\{p_1, p_2, p_3\}$ is a basis of P_2. ☐

- Let $p_1 = X + 1,\ p_2 = X^2 + X, p_3 = X^3 + X^2 . p_4 = X^3 + X^2 + X + 1$. Then the system $\{p_1, p_2, p_3, p_4\}$ is **not** a basis of P_2.

Exercise. Show this!

- In the spaces $M(m \times n)$, the set of all matrices A_{ij} form a basis, where A_{ij} is the matrix with $a_{ij} = 1$ and all other entries equal to 0. For example, in $M(2 \times 3)$ we have the following basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\{A, B, C, D\}$ is a basis of $M(2 \times 2)$.

  Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary 2×2 matrix. Consider the equation $M = \alpha_1 A + \alpha_2 B + \alpha_3 C + \alpha_4 D$. This leads to

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_4 \\ \alpha_2 + \alpha_3 + \alpha_4 & \alpha_3 + \alpha_4 \end{pmatrix}. \end{aligned}$$

So we obtain the following set of equations for the α_j:

$$\left.\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= a \\ \alpha_4 &= b \\ \alpha_2 + \alpha_3 + \alpha_4 &= c \\ \alpha_3 + \alpha_4 &= d \end{aligned}\right\} \qquad \text{in matrix form:} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Now we apply Gauß-Jordan to the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 0 & 0 & 1 & b \\ 0 & 1 & 1 & 1 & c \\ 0 & 0 & 1 & 1 & d \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & c \\ 0 & 0 & 1 & 1 & d \\ 0 & 0 & 0 & 1 & b \end{array}\right) \longrightarrow \left(\begin{array}{cccc|r} 1 & 1 & 1 & 0 & a-b \\ 0 & 1 & 1 & 0 & c-b \\ 0 & 0 & 1 & 0 & d-b \\ 0 & 0 & 0 & 1 & b \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|r} 1 & 1 & 0 & 0 & a-d \\ 0 & 1 & 0 & 0 & c-d \\ 0 & 0 & 1 & 0 & d-b \\ 0 & 0 & 0 & 1 & b \end{array}\right) \longrightarrow \left(\begin{array}{cccc|r} 1 & 0 & 0 & 0 & a-c \\ 0 & 1 & 0 & 0 & c-d \\ 0 & 0 & 1 & 0 & d-b \\ 0 & 0 & 0 & 1 & b \end{array}\right).$$

We see that there is exactly one solution for any given $M \in M(2 \times 2)$. Existence of the solution shows that the matrices A, B, C, D generate $M(2 \times 2)$ and uniqueness shows that they are linearly independent if we choose $M = 0$. □

Next we show that every finitely generated vector space has a basis.

Theorem 5.41. *Let V be a vector space and assume that there are vectors $w_1, \dots, w_m \in V$ such that $V = \operatorname{span}\{w_1, \dots, w_m\}$. Then the set $\{w_1, \dots, w_m\}$ contains a basis of V. In particular, V has a finite basis and $\dim V \leq m$.*

Proof. Without restriction we may assume that all vectors w_j are different from $\mathbb{O}$. We start with the first vector. If $V = \operatorname{span}\{w_1\}$, then $\{w_1\}$ is a basis of V and $\dim V = 1$. Otherwise we set $V_1 := \operatorname{span}\{w_1\}$ and we note that $V_1 \neq V$. Now we check if $w_2 \in \operatorname{span}\{w_1\}$. If it is, we throw it out because in this case $\operatorname{span}\{w_1\} = \operatorname{span}\{w_1, w_2\}$ so we do not need w_2 to generate V. Next we check if $w_3 \in \operatorname{span}\{w_1\}$. If it is, we throw it out, etc. We proceed like this until we find a vector w_{i_2} in our list which does not belong to $\operatorname{span}\{w_1\}$. Such an i_2 must exist because otherwise we already had that $V_1 = V$. Then we set $V_2 := \operatorname{span}\{w_1, w_{i_2}\}$. If $V_2 = V$, then we are done. Otherwise, we proceed as before: We check if $w_{i_2+1} \in V_2$. If this is the case, then we can throw it out because $\operatorname{span}\{w_1, w_{i_2}\} = \operatorname{span}\{w_1, w_{i_2}, w_{i_2+1}\}$. Then we check w_{i_2+2}, etc., until we find a w_{i_3} such that $w_{i_3} \notin \operatorname{span}\{w_1, w_{i_2}\}$ and we set $V_3 := \operatorname{span}\{w_1, w_{i_2}, w_{i_3}\}$. If $V_3 = V$, then we are done. If not, then we repeat the process. Note that after at most m repetitions, this comes to an end. This shows that we can extract from the system of generators a basis $\{w_1, w_{i_2}, \dots, w_{i_k}\}$ of V. □

The following theorem complements the preceding one.

Theorem 5.42. *Let V be a finitely generated vector space. Then any system $w_1, \dots, w_m \in V$ of linearly independent vectors can be completed to a basis $\{w_1, \dots, w_m, v_{m+1}, \dots, v_n\}$ of V.*

Proof. Note that $\dim V < \infty$ by Theorem 5.41 and set $n = \dim V$. It follows that $n \geq m$ because we have m linearly independent vectors in V. If $m = n$, then $w_1, \dots, w_m$ is already a basis of V and we are done.

If $m < n$, then $\operatorname{span}\{w_1, \dots, w_m\} \neq V$ and we choose an arbitrary vector $v_{m+1} \notin \operatorname{span}\{w_1, \dots, w_m\}$ and we define $V_{m+1} := \operatorname{span}\{w_1, \dots, w_m, v_{m+1}\}$. Then $\dim V_{m+1}\} = m + 1$. If $m + 1 = n$, then necessarily $V_{m+1} = V$ and we are done. If $m + 1 < n$, then we choose an arbitrary vector $v_{m+2} \in V \setminus V_{m+1}$ and we let $V_{m+2} := \operatorname{span}\{w_1, \dots, w_m, v_{m+1}, v_{m+2}\}$. If $m+2 = n$, then necessarily $V_{m+2} = V$ and we are done. If not, we repeat the step before. Note that after $n - m$ steps we have found a basis $\{w_1, \dots, w_m, v_{m+1}, \dots, v_n\}$ of V. $\square$

In summary, the two preceding theorems say the following:

- If the set of vectors $v_1, \dots v_m$ generates the vector space V, then it is always possible to extract a subset which is a basis of V (we need to eliminate $m - n$ vectors).
- If we have a set of linearly independent vectors $v_1, \dots v_m$ in a finitely generated vector space V, then it is possible to find vectors $v_{m+1}, \dots, v_n$ such that $\{v_1, \dots, v_n\}$ is a basis of V (we need to add $\dim V - m$ vectors).

Corollary 5.43. *Let V be a vector space.*

- *If the vectors $v_1, \dots, v_k \in V$ are linearly independent, then $k \leq \dim V$.*
- *If the vectors $v_1, \dots, v_m \in V$ generate V, then $m \geq \dim V$.*

Example 5.44.

- Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M(2 \times 2)$ and suppose that we want to complete them to a basis of $M(2 \times 2)$ (it is clear that A and B are linearly independent, so this makes sense). Since $\dim(M(2 \times 2)) = 4$, we know that we need 2 more matrices. We take any matrix $C \notin \operatorname{span}\{A, B\}$, for example $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Finally we need a matrix $D \notin \operatorname{span}\{A, B, C\}$. We can take for example $D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then A, B, C, D is a basis of $M(2 \times 2)$.

  Check that $D \notin \operatorname{span}\{A, B, C\}$

  Find other matrices C' and D' such that $\{A, B, C', D'\}$ is a basis of $M(2 \times 2)$.

- Given the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $\vec{v}_6 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and we want to find a subset of them which form a basis of $\mathbb{R}^3$.

  Note that a priori it is not clear that this is possible because we do not know without further calculations that the given vectors generate $\mathbb{R}^3$. If they do not, then of course it is impossible to extract a basis from them.

  Let us start. First observe that we need 3 vectors for a basis since $\dim \mathbb{R}^3 = 3$. So we start with the first non-zero vector which is $\vec{v}_1$. We see that $\vec{v}_2 = 4\vec{v}_1$, so we discard it. We keep $\vec{v}_3$ since $\vec{v}_3 \notin \operatorname{span}\{\vec{v}_1\}$. Next, $\vec{v}_4 = \vec{v}_3 - \vec{v}_1$, so $\vec{v}_4 \in \operatorname{span}\{\vec{v}_1, \vec{v}_3\}$ and we discard it. A little calculation shows that $\vec{v}_5 \notin \operatorname{span}\{\vec{v}_1, \vec{v}_3\}$. Hence $\{\vec{v}_1, \vec{v}_3, \vec{v}_5\}$ is a basis of $\mathbb{R}^3$.

Remark 5.45. We will present a more systematic way to solve exercises of this type in Theorem 6.34 and Remark 6.35.

The next theorem is very important. It says that if V has a basis which consists of n vectors, then **every** basis consists of exactly n vectors.

Theorem 5.46. *Let V be a vector space and let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ be bases of V. Then $n = m$.*

Proof. Suppose that $m > n$. We will show that then the vectors $w_1, \dots, w_m$ cannot be linearly independent, hence they cannot be a basis of V. Since the vectors $v_1, \dots, v_n$ are a basis of V, every w_j can be written as a linear combination of them. Hence there exist numbers a_{ij} which

$$\begin{aligned} w_1 &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ w_2 &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ &\vdots \\ w_m &= a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n. \end{aligned} \tag{5.5}$$

Now we consider the equation

$$c_1w_1 + \cdots + c_mw_m = \mathbb{O}. \tag{5.6}$$

If the $w_1, \dots, w_m$ were linearly independent, then it should follow that all c_j are 0. We insert (5.5) into (5.6) and obtain

$$\begin{aligned} \mathbb{O} &= c_1(a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n) + c_2(a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n) \\ &\qquad + \cdots + c_m(a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n) \\ &= (c_1a_{11} + c_2a_{21} + \cdots + c_ma_{m1})v_1 + \cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_ma_{mn})v_n. \end{aligned}$$

Since the vectors $v_1, \dots, v_n$ are linearly independent, the expressions in the parentheses must be equal to zero. So we find

$$\begin{aligned} c_1a_{11} + c_2a_{12} + \cdots + c_ma_{1m} &= 0 \\ c_1a_{21} + c_2a_{22} + \cdots + c_ma_{2m} &= 0 \\ &\vdots \\ c_1a_{1n} + c_2a_{2n} + \cdots + c_ma_{mm} &= 0. \end{aligned} \tag{5.7}$$

This is a homogeneous system of n equations for the m unknowns $c_1, \dots, c_m$. Since $n < m$ it must have infinitely many solutions. So the system $\{w_1, \dots, w_m\}$ is not linearly independent and therefore it cannot be a basis of V. Therefore $m > n$ cannot be true and me must have that $n \geq m$.

If we assume that $n > m$, then the same argument as above, with the roles of the v_j and the w_j exchanged, leads to a contradiction and we must have $n \leq m$.

In summary we showed that both m$n \geq m$ and $n \leq m$ must be true. Therefore $m = n$. ∎

Definition 5.47.

- Let V be a finitely generated vector space. Then it has a basis by Theorem 5.41 and by Theorem 5.46 the number n of vectors needed for a basis does not depend on the particular chosen basis. This number is called the *dimension of V*. It is denoted by $\dim V$.

- If a vector space V is not finitely generated, then we set $\dim V = \infty$.

Theorem 5.48. *Let V be a vector space with basis $\{v_1, \dots, v_n\}$. Then every $x \in V$ can be written in unique way as linear combination of the vectors $v_1, \dots, v_n$.*

Proof. We have to show existence and uniqueness of numbers $c_1, \dots, c_n$ so that $w = c_1v_1 + \dots + c_nv_n$.
Existence is clear since the set $\{v_1, \dots, v_n\}$ is a set of generators of V (it is even a basis!).
Uniqueness can be shown as follows. Assume that there are numbers $c_1, \dots, c_n$ and $d_1, \dots, d_n$ such that $w = c_1v_1 + \cdots c_nv_n$ and $w = d_1v_1 + \cdots d_nv_n$. Then it follows that

$$\mathbb{O} = w - w = c_1v_1 + \cdots c_nv_n - (d_1v_1 + \cdots d_nv_n) = (c_1 - d_1)v_1 + \cdots (c_n - d_n)v_n.$$

Then all the coefficients $c_1 - d_1, \dots, c_n - d_n$ have to be zero because the vectors $v_1, \dots, v_n$ are linearly independent. Hence it follows that $c_1 = d_1, \dots, c_n = d_n$, which shows uniqueness. $\square$

If we have a vector space V and a subspace $W \subset V$, then we can ask ourselves what the relation between their dimensions is because W itself is a vector space.

Lemma 5.49. *Let V be a finitely generated vector space and let W be a subspace. Then W is finitely generated and $\dim W \le \dim V$.*

Proof. Let V be a finitely generated vector space with $\dim V = n < \infty$ and let W be a subspace of V. Assume that W is not finitely generated. Then we can construct an arbitrary large system of linear independent vectors in W as follows. Clearly, W cannot be the trivial space, so we can choose $w_1 \in W \setminus \{\mathbb{O}\}$ and we set $W_1 = \operatorname{span}\{w_1\}$. Then W_1 is a finitely generated subspace of W, therefore $W_1 \subsetneq W$ and we can choose $w_2 \in W \setminus W_1$. Clearly, the set $\{w_1, w_2\}$ is linearly independent. Let us set $W_2 = \operatorname{span}\{w_1, w_2\}$. Since W_2 is a finitely generated subspace of W, it follows that $W_2 \subsetneq W$ and we can choose $w_3 \in W \setminus W_2$. Then the vectors w_1, w_2, w_3 are linearly independent and we set $W_3 = \operatorname{span}\{w_1, w_2, w_3\}$. Continuing with this procedure we can construct subspaces $W_1 \subsetneq W_2 \subsetneq \cdots W$ with $\dim W_k = k$ for every k. In particular, we can find a system of $n + 1$ linear independent vectors in $W \subseteq V$ which contradicts the fact that any system of more than $n = \dim V$ vectors in V must be linearly dependent, see Corollary 5.43. This also shows that any system of more than n vectors in W must be linear dependent. Since a basis of W consists of linearly independent vectors, it follows that $\dim W \le n = \dim V$. $\square$

Theorem 5.50. *Let V be a finitely generated vector space and let $W \subseteq V$ be a subspace. Then the following is true:*

(i) $\dim W \le \dim V$.

(ii) $\dim W = \dim V$ *if and only if* $W = V$.

Proof. (i) follow immediately from Lemma 5.49.

(ii) If $V = W$, then clearly $\dim V = \dim W$. To show the converse, we assume that $\dim V = \dim W$ and we have to show that $V = W$. As before let $\{w_1, \dots, w_k\}$ be a basis of W. Then these vectors are linearly independent in W, and therefore also in V. Since $\dim W = \dim V$, we know that these vectors form a basis of V. Therefore $V = \operatorname{span}\{w_1, \dots, w_m\} = W$. $\square$

Remark 5.51. Note that (i) is true even when V is not finitely generated because $\dim W \leq \infty = \dim V$ whatever $\dim W$ may be. However (ii) is **not** true in general for infinite dimensional vector spaces. In Example 5.52 (f) and (g) we will show that $\dim P = \dim C(\mathbb{R})$ in spite of $P \neq C(\mathbb{R})$. (Recall that P is the set of all polynomials and that $C(\mathbb{R})$ is the set of all continuous functions. So we have $P \subsetneq C(\mathbb{R})$.)

Now we give a few examples of dimensions of spaces.

Examples 5.52. (a) $\dim \mathbb{R}^n = n$, $\dim \mathbb{C}^n = n$.

(b) $\dim M(m \times n) = mn$. This follows because the set of all $m \times n$ matrices A_{ij} which have a 1 in the ith row and jth column and all other entries are equal to zero form a basis of $M(m \times n)$ and there are exactly mn such matrices.

(c) Let $M_{sym}(n \times n)$ be the set of all symmetric $n \times n$ matrices. Then $\dim M_{sym}(n \times n) = \frac{n(n+1)}{2}$. To see this, let A_{ij} be the $n \times n$ matrix with $a_{ij} = a_{ji} = 1$ and all other entries equal to 0. Observe that $A_{ij} = A_{ji}$. It is not hard to see that the set of all A_{ij} with $i \leq j$ form a basis of $M_{sym}(n \times n)$. The dimension of $M_{sym}(n \times n)$ is the number of different matrices of this type. How many of them are there? If we fix $j = 1$, then only $i = 1$ is possible. If we fix $j = 2$, then $i = 1, 2$ is possible, etc. until for $j = n$ the allowed values for i are $1, 2, \dots, n$. In total we have $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ possibilities. For example, in the case $n = 2$, the matrices are

$$A_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the case $n = 3$, the matrices are

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Convince yourself that the A_{ij} form a basis of $M_{sym}(n \times n)$.

(d) Let $M_{asym}(n \times n)$ be the set of all antisymmetric $n \times n$ matrices. Then $\dim M_{asym}(n \times n) = \frac{n(n-1)}{2}$. To see this, for $i \neq j$ let A_{ij} be the $n \times n$ matrix with $a_{ij} = -a_{ji} = 1$ and all other entries equal to 0 form a basis of $M_{sym}(n \times n)$. It is not hard to see that the set of all A_{ij} with $i < j$ form a basis of $M_{asym}(n \times n)$. How many of these matrices are there? If we fix $j = 2$, then only $i = 1$ is possible. If we fix $j = 3$, then $i = 1, 2$ is possible, etc. until for $j = n$ the allowed values for i are $1, 2, \dots, n - 1$. In total we have $1 + 2 + \cdots + (n - 1) = \frac{n(n-1)}{2}$ possibilities. For example, in the case $n = 2$, the only matrix is

$$A_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the case $n = 3$, the matrices are

$$A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Convince yourself that the A_{ij} form a basis of $M_{\mathrm{asym}}(n \times n)$.

Remark. Observe that $\dim M_{\mathrm{sym}}(n \times n) + \dim M_{\mathrm{asym}}(n \times n) = n^2 = \dim M(n \times n)$. This is no coincidence. Note that every $n \times n$ matrix M can be written as

$$M = \tfrac{1}{2}(M + M^t) + \tfrac{1}{2}(M - M^t)$$

and that $\frac{1}{2}(M + M^t) \in M_{\mathrm{sym}}(n \times n)$ and $\frac{1}{2}(M - M^t) \in M_{\mathrm{asym}}(n \times n)$. Moreover it is easy to check that $M_{\mathrm{sym}}(n \times n) \cap M_{\mathrm{asym}}(n \times n) = \{0\}$. Therefore $M(n \times n)$ is the *direct sum* of $M_{\mathrm{sym}}(n \times n)$ and $M_{\mathrm{asym}}(n \times n)$. (For the definition of the *direct sum* of subspaces, see Definition 7.18.

(e) $\dim P_n = n + 1$ since $\{1, X, \dots, X^n\}$ is a basis of P_n and consists of $n + 1$ vectors.

(f) $\dim P = \infty$. Recall that P is the space of all polynomials.

Proof. We know that for every $n \in \mathbb{N}$, the space P_n is a subspace of P. Therefore for every $n \in \mathbb{N}$, we must have that $n + 1 = \dim P_n \leq \dim P$. This is possible only if $\dim P = \infty$. □

(g) $\dim C(\mathbb{R}) = \infty$. Recall that $C(\mathbb{R})$ is the space of all continuous functions.

Proof. Since P is a subspace of $C(\mathbb{R})$, it follows that $\dim P \leq \dim(C(\mathbb{R}))$, hence $\dim(C(\mathbb{R})) = \infty$. □

Now we use the concept of dimension to classify all subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$. We already know that for examples lines and planes which pass through the origin are subspaces of $\mathbb{R}^3$. Now we can show that there are no other proper subspaces.

Subspaces of $\mathbb{R}^2$. Let U be a subspace of $\mathbb{R}^2$. Then U must have a dimension. So we have the following cases:

- $\dim U = 0$. In this case $U = \{\vec{0}\}$ is the trivial subspace.
- $\dim U = 1$. Then U is of the form $U = \operatorname{span}\{\vec{v}_1\}$ with some vector $\vec{v}_1 \in \mathbb{R}^2 \setminus \{\vec{0}\}$. Therefore U is a line parallel to $\vec{v}_1$ passing through the origin.
- $\dim U = 2$. In this case $\dim U = \dim \mathbb{R}^2$. Hence it follows that $U = \mathbb{R}^2$ by Theorem 5.50 (ii).
- $\dim U \geq 3$ is not possible.

In conclusion, the only subspaces of $\mathbb{R}^2$ are $\{\vec{0}\}$, lines passing through the origin and $\mathbb{R}^2$ itself.

Subspaces of $\mathbb{R}^3$. Let U be a subspace of $\mathbb{R}^3$. Then U must have a dimension. So we have the following cases:

- $\dim U = 0$. In this case $U = \{\vec{0}\}$ is the trivial subspace.
- $\dim U = 1$. Then U is of the form $U = \operatorname{span}\{\vec{v}_1\}$ with some vector $\vec{v}_1 \in \mathbb{R}^3 \setminus \{\vec{0}\}$. Therefore U is a line parallel to $\vec{v}_1$ passing through the origin.
- $\dim U = 2$. Then U is of the form $U = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$ with linearly independent vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$. Hence U is a plane parallel to the vectors $\vec{v}_1$ and $\vec{v}_2$ which passes through the origin.
- $\dim U = 3$. In this case $\dim U = \dim \mathbb{R}^3$. Hence it follows that $U = \mathbb{R}^3$ by Theorem 5.50 (ii).
- $\dim U \geq 4$ is not possible.

In conclusion, the only subspaces of $\mathbb{R}^3$ are $\{\vec{0}\}$, lines passing through the origin, planes passing through the origin and $\mathbb{R}^3$ itself.

We conclude this section with the formal definition of lines and planes.

Definition 5.53. Let V be a vector space with $\dim V = n$ and let $W \subseteq V$ be a subspace. Then W is called a

- *line* if $\dim W = 1$,
- *plane* if $\dim W = 2$,
- *hyperplane* if $\dim W = n - 1$.

Note that in $\mathbb{R}^3$ the hyperplanes are exactly the planes.

You should have understood

- the concept of a basis of a finite dimensional vector space,
- that a given vector space has infinitely many bases, but the number of vectors in any basis of the space is the same,
- why and how the concept of dimension helps to classify all subspaces of given vector space,
- why a matrix $A \in M(n \times n)$ is invertible if and only if its columns are a basis of $\mathbb{R}^n$,
- ...

You should now be able to

- check if a system of vectors is a basis for a given vector space,
- find a basis for a given vector space,
- extend a system of linear independent vectors to a basis,
- find the dimension of a given vector space,
- ...

5.5 Summary

Let V be a vector space over $\mathbb{K}$ and let $v_1, \dots, v_k \in V$.

Linear combinations and linear independence

- A vector w is called a *linear combination* of the vectors $v_1, \dots, v_k$, if there exists scalars $\alpha_1, \dots, \alpha_k \in \mathbb{K}$ such that
$$w = \alpha_1 v_1 + \dots + \alpha_k v_k.$$

- The set of all linear combinations of the vectors $v_1, \dots, v_k$ is a subspace of V, called the *space generated by the vectors* $v_1, \dots, v_k$ or the *linear span of the vectors* $v_1, \dots, v_k$. Notation:
$$\begin{aligned}\operatorname{gen}\{v_1, \dots, v_k\} := \operatorname{span}\{v_1, \dots, v_k\} &:= \{w \in V : w \text{ is linear combination of } v_1, \dots, v_k\}\\ &= \{\alpha_1 v_1 + \dots + \alpha_k v_k \ : \ \alpha_1, \dots, \alpha_k \in \mathbb{K}\}.\end{aligned}$$

- The vectors $v_1, \dots, v_k$ are called *linearly independent* if the equation
$$\alpha_1 v_1 + \dots + \alpha_k v_k = \mathbb{O}$$
has only the trivial solution $\alpha_1 = \dots = \alpha_k = 0$.

Basis and dimension

- A system $v_1, \dots, v_m$ of vectors in in V is called a *basis of* V if it is linearly independent and $\operatorname{span}\{v_1, \dots, v_m\} = V$.

- A vector space V is called *finitely generated* if it has a finite basis. In this case, every basis of V has the same number of vectors. The number of vectors needed for a basis of a vector space V is called the *dimension of* V.

- If V is not finitely generated, we set $\dim V = \infty$.

- For $v_1, \dots, v_k \in V$, it follows that $\dim(\operatorname{span}\{v_1, \dots, v_k\}) \le k$ with equality if and only if the vectors $v_1, \dots, v_k$ are linearly independent.

- If V is finitely generated then every linearly independent system of vectors $v_1, \dots, v_k \in V$ can be extended to a basis of V.

- If $V = \operatorname{span}\{v_1, \dots, v_k\}$, then V has a basis consisting of a subsystem of the given vectors $v_1, \dots, v_k$.

- If U is a subspace of V, then $\dim U \le \dim V$.

- If V is finitely generated and U is a subspace of V, then $\dim U = \dim V$ if and only if $U = V$. This claim is false in general if $\dim V = \infty$.

Examples of the dimensions of some vector spaces:

- $\dim \mathbb{R}^n = n$, $\dim \mathbb{C}^n = n$,
- $\dim M(m \times n) = mn$,
- $\dim M_{\text{sym}}(n \times n) = \frac{n(n+1)}{2}$,
- $\dim M_{\text{asym}}(n \times n) = \frac{n(n-1)}{2}$,
- $\dim P_n = n + 1$,
- $\dim P = \infty$,
- $\dim C(\mathbb{R}) = \infty$.

Linear independence, generator property and bases in $\mathbb{R}^n$ and $\mathbb{C}^n$

Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ or $\mathbb{C}^n$ and let $A = (\vec{v}_1|\dots|\vec{v}_k) \in M(n \times k)$ be the matrix whose columns consist of the given vectors.

- $\text{gen}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$ if and only if the system $A\vec{x} = \vec{b}$ has at least one solution for every $\vec{b} \in \mathbb{R}^n$.
- The vectors $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent if and only if the system $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.
- The vectors $\vec{v}_1, \dots, \vec{v}_k$ are a basis of $\mathbb{R}^n$ if and only if $k = n$ and A is invertible.

5.6 Exercises

1. Sea X el conjunto de todas las funciones de $\mathbb{R}$ a $\mathbb{R}$. Demuestre que X con la suma y producto con números en $\mathbb{R}$ es un espacio vectorial.

   De los siguientes subconjuntos de X, diga si son subespacios de X.

   (a) Todas las funciones acotadas de $\mathbb{R}$ a $\mathbb{R}$.
   (b) Todas las funciones constantes.
   (c) Todas las funciones continuas.
   (d) Todas las funciones continuas con $f(3) = 0$.
   (e) Todas las funciones continuas con $f(3) = 4$.
   (f) Todas las funciones con $f(3) > 0$.
   (g) Todas las funciones pares.
   (h) Todas las funciones impares.
   (i) Todos los polinomios.
   (j) Todas las funciones nonegativas.
   (k) Todos los polinomios de grado ≥ 4.

2. Sean $A \in M(m \times n)$ y sea $\vec{a} \in \mathbb{R}^k$.
   (a) Demuestre que $U = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ es un subespacio de $\mathbb{R}^m$.
   (b) Demuestre que $W = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = 0\}$ es un subespacio de $\mathbb{R}^n$.
   (c) ¿Los conjuntos $R = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = (1, 1, \dots, 1)^t\}$ y $S = \{\vec{x} \in \mathbb{R}^n : A\vec{x} \neq 0\}$ son subespacios de $\mathbb{R}^n$?

3. Sean $A \in M(m \times n)$ y sea $\vec{a} \in \mathbb{R}^k$.
   (a) ¿El conjunto $T = \{\vec{x} \in \mathbb{R}^k : \langle \vec{x}, \vec{a} \rangle = 0\}$ es un subespacio de $\mathbb{R}^k$?
   (b) ¿Los conjuntos
   $$S_1 = \{\vec{x} \in \mathbb{R}^k : \|\vec{x}\| = 1\}, \quad B_1 = \{\vec{x} \in \mathbb{R}^k : \|\vec{x}\| \leq 1\}, \quad F = \{\vec{x} \in \mathbb{R}^k : \|\vec{x}\| \geq 1\}$$
   son subespacios de $\mathbb{R}^k$?

4. Considere el conjunto $\mathbb{R}^2$ con las siguientes operaciones:
$$\oplus : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_2 \\ x_2 + y_1 \end{pmatrix},$$
$$\odot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad \lambda \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$
¿Es $\mathbb{R}^2$ con esta suma y producto con escalares un espacio vectorial?

5. Considere el conjunto $\mathbb{R}^2$ con las siguientes operaciones:
$$\boxplus : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \boxplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix},$$
$$\boxdot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad \lambda \boxdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$
¿Es $\mathbb{R}^2$ con esta suma y producto con escalares un espacio vectorial?

6. (a) Sea $V = (-\frac{\pi}{2}, \frac{\pi}{2})$ y defina suma $\oplus : V \times V \to V$ y producto con escalar $\odot : \mathbb{R} \times V \to V$ por
   $$x \oplus y = \arctan(\tan(x) + \tan(y)), \qquad \lambda \odot x = \arctan(\lambda \tan(x))$$
   para todo $x, y \in V$, $\lambda \in \mathbb{R}$. Demuestre que $(V, \oplus, \odot)$ es un espacio vectorial sobre $\mathbb{R}$.

(b) Una generalización de la construcción en (a) es lo siguiente:
Sea V un conjunto y $f : \mathbb{R}^n \to V$ una función biyectiva. Entonces V es un espacio vectorial con suma y producto con escalar definido así:

$$x \oplus y = f(f^{-1}(x) + f^{-1}(y)), \qquad \lambda \odot x = f(\lambda f^{-1}(x))$$

para todo $x, y \in V,\ \lambda \in \mathbb{R}$.

7. Sea U un subespacio de $\mathbb{R}^n$. Demuestre que $\mathbb{R}^n \setminus U$ no es un subespacio de $\mathbb{R}^n$.

8. Sean $m, n \in \mathbb{N}$. Demuestre que $M(m \times n, \mathbb{R})$ con la suma y producto con números en $\mathbb{R}$ es un espacio vectorial.

De los siguientes subconjuntos de $M(n \times n)$, diga si son subespacios.

(a) Todas matrices con $a_{11} = 0$.
(b) Todas matrices con $a_{11} = 3$.
(c) Todas matrices con $a_{12} = \mu a_{11}$ para un $\mu \in \mathbb{R}$ fijo.
(d) Todas matrices cuya primera columna coincide con la última columna.

Para los siguientes numerales supongamos que $n = m$.

(e) Todas las matrices simétricas (es decir, todas las matrices A con $A^t = A$).
(f) Todas las matrices que no son simétricas.
(g) Todas las matrices antisimétricas (es decir, todas las matrices A con $A^t = -A$).
(h) Todas las matrices diagonales.
(i) Todas las matrices triangular superior.
(j) Todas las matrices triangular inferior.
(k) Todas las matrices invertibles.
(l) Todas las matrices no invertibles.
(m) Todas las matrices con $\det A = 1$.

9. Demuestre que

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \begin{array}{l} x_1 + x_2 - 2x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 + 7x_4 = 0 \end{array} \right\}$$

es un subespacio de $\mathbb{R}^4$.

10. Demuestre que

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \ : \ \begin{matrix} 3x_1 - x_2 - 2x_3 - x_4 = 3 \\ 4x_1 + x_2 + x_3 + 7x_4 = 5 \end{matrix} \right\}$$

es un subespacio afín de $\mathbb{R}^4$.

11. Considere los sistemas de ecuaciones lineales

$$(1) \quad \left\{ \begin{matrix} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{matrix} \right\}, \qquad\qquad (2) \quad \left\{ \begin{matrix} x + 2y + 3z = 3 \\ 4x + 5y + 6z = 9 \\ 7x + 8y + 9z = 15 \end{matrix} \right\}.$$

Sea U el conjunto de todas las soluciones de (1) y W el conjunto de todas las soluciones de (2). Note que se pueden ver como subconjuntos de $\mathbb{R}^3$.

(a) Demuestre que U es un subespacio de $\mathbb{R}^3$ y descríbalo geométricamente.

(b) Demuestre que W no es un subespacio de $\mathbb{R}^3$.

(c) Demuestre que W es un subespacio afín de $\mathbb{R}^3$ y descríbalo geométricamente.

12. (a) Sean $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \in \mathbb{R}^2$. Escriba $v = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ como combinación lineal de v_1 y v_2.

(b) ¿Es $v = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ combinación lineal de $v_1 = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$?

(c) ¿Es $A = \begin{pmatrix} 13 & -5 \\ 50 & 8 \end{pmatrix}$ combinación lineal de

$$A_1 = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}, \ A_3 = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}, \ A_4 = \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix} \ ?$$

13. (a) ¿Los vectores $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ son linealmente independientes en $\mathbb{R}^3$?

(b) ¿Los vectores $v_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ son linealmente independientes en $\mathbb{R}^3$?

(c) ¿Los vectores $p_1 = X^2 - X + 2$, $p_2 = X + 3$, $p_3 = X^2 - 1$ son linealmente independientes en P_2? Son linealmente independientes en P_n para $n \geq 3$?

(d) ¿Los vectores $A_1 = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 2 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 7 & 3 \\ 2 & -1 & 2 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & -1 & 0 \\ 5 & 2 & 8 \end{pmatrix}$ son linealmente independientes en $M(2 \times 3)$?

14. Sean $\vec{v}_1, \dots, \vec{v}_k, \vec{w} \in \mathbb{R}^n$. Suponga que $\vec{w} \neq \vec{0}$ y que $\vec{w} \in \mathbb{R}^n$ es ortogonal a todos los vectores $\vec{v}_j$. Demuestre que $\vec{w} \notin \operatorname{gen}\{\vec{v}_1, \dots, \vec{v}_m\}$. Se sigue que el sistema $\vec{w}, \vec{v}_1, \dots, \vec{v}_m$ es linealmente independiente?

15. Determine si $\operatorname{gen}\{a_1, a_2, a_3, a_4\} = \operatorname{gen}\{v_1, v_2, v_3\}$ para

$$a_1 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \ a_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \ a_3 = \begin{pmatrix} 1 \\ 2 \\ 13 \end{pmatrix}, \ a_4 = \begin{pmatrix} 2 \\ 1 \\ 11 \end{pmatrix}, \ v_1 = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}, \ v_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

16. (a) ¿Las siguientes matrices generan el espacio de todas las matrices simétricas 2×2?

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 13 & 0 \\ 0 & 5 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix},$$

Si no lo hacen, encuentre un $M \in M_{sym}(2 \times 2) \setminus \operatorname{span}\{A_1, A_2, A_3\}$.

(b) ¿Las siguientes matrices generan el espacio de todas las matrices simétricas 2×2?

$$B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 13 & 0 \\ 0 & 5 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix},$$

(c) ¿Las siguientes matrices generan el espacio de las matrices triangulares superiores 2×2?

$$C_1 = \begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 3 \\ 0 & 5 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 10 & -7 \\ 0 & 0 \end{pmatrix}.$$

Si no, encuentre una matriz M triangular superior que no pertence a $\operatorname{span}\{C_1, C_2, C_3\}$.

17. Sea $n \in \mathbb{N}$ y sea V el conjunto de las matrices simétricas $n \times n$ con la suma y producto con $\lambda \in \mathbb{R}$ usual.

(a) Demuestre que V es un espacio vectorial sobre $\mathbb{R}$.

(b) Encuentre matrices que generan V. ¿Cual es el número mínimo de matrices que se necesitan para generar V?

18. Determine si los siguientes conjuntos de vectores son bases del espacio vectorial indicado.

(a) $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$; $\mathbb{R}^2$.

(b) $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$; $M(2 \times 2)$.

(c) $p_1 = 1 + x,\ p_2 = x + x^2,\ p_3 = x^2 + x^3,\ p_4 = 1 + x + x^2 + x^3;\quad P_3$.

19. (a) Es F el plano dado por $F : 2x - 5y + 3z = 0$. Demuestre que F es subespacio de $\mathbb{R}^3$ y encuentre vectores $\vec{u}$ y $\vec{w} \in \mathbb{R}^3$ tal que $F = \text{gen}\{\vec{u}, \vec{w}\}$.

(b) Sean $v_1 = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix},\ v_2 = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{R}^3$. Sea E el plano $E = \text{gen}\{v_1, v_2\}$. Escriba E en la forma $E : ax + by + cz = d$.

(c) Encuentre un vector $w \in \mathbb{R}^3$, distinto de v_1 y v_2, tal que $\text{gen}\{v_1, v_2, w\} = E$.

(d) Encuentre un vector $v_3 \in \mathbb{R}^3$ tal que $\text{gen}\{v_1, v_2, v_3\} = \mathbb{R}^3$.

20. (a) Encuentre una base para el plano $E : x - 2y + 3z = 0$ in $\mathbb{R}^3$.

(b) Complete la base encontrado en (i) a una base de $\mathbb{R}^3$.

21. Sea $F := \{(x_1, x_2, x_3, x_4)^t : 2x_1 - x_2 + 4x_3 + x_4 = 0\}$.

(a) Demuestre que F es un subespacio de $\mathbb{R}^4$

(b) Encuentre una base para F y calcule $\dim F$.

(c) Complete la base encontrada en (ii) a una base de $\mathbb{R}^4$.

22. Sea $G := \{(x_1, x_2, x_3, x_4)^t : 2x_1 - x_2 + 4x_3 + x_4 = 0, x_1 - x_2 + x_3 + 2x_4 = 0\}$.

(a) Demuestre que G es un subespacio de $\mathbb{R}^4$

(b) Encuentre una base para G y calcule $\dim G$.

(c) Complete la base encontrada en (ii) a una base de $\mathbb{R}^4$.

23. Sean $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix},\ v_4 = \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix},\ v_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Determine si estos vectoren generan el espacio $\mathbb{R}^3$. Si lo hacen, escoja una base de $\mathbb{R}^3$ de los vectores dados.

24. Sean $C_1 = \begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix},\quad C_2 = \begin{pmatrix} 6 & 3 \\ 0 & 12 \end{pmatrix},\quad C_3 = \begin{pmatrix} 6 & -3 \\ 0 & 2 \end{pmatrix},\quad C_4 = \begin{pmatrix} 12 & -9 \\ 0 & -1 \end{pmatrix}$.

Determine si estas matrices generan el espacio de las matrices triangulares superiores 2×2. Si lo hacen, escoja una base de las matrices dadas.

25. Sean $p_1 = x^2+7, p_2 = x+1, p_3 = 3x^3+7x$. Determine si los polinomios p_1, p_2, p_3 son linealmente independientes. Si lo son, complételos a una base en P_3.

26. Para los siguientes conjuntos, determine si son espacios vectoriales. Si lo son, calcule su dimensión.

(a) $M_1 = \{A \in M(n \times n) : A \text{ es triangular superior}\}$.

(b) $M_2 = \{A \in M(n \times n) : A \text{ tiene zeros en la diagonal}\}$.

(c) $M_3 = \{A \in M(n \times n) : A^t = -A\}$.

(d) $M_4 = \{p \in P_5 : p(0) = 0\}$.

27. Para los siguientes sistemas de vectores en el espacio vectorial V, determine la dimensión del espacio vectorial generado por ellos y escoja un subsistema de ellos que es base del espacio vectorial generado por los vectores dados. Complete este subsistema a una base de V.

(a) $V = \mathbb{R}^3$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

(b) $V = P_4$, $p_1 = x^3 + x$, $p_2 = x^3 - x^2 + 3x$, $p_3 = x^2 + 2x - 5$, $p_4 = x^3 + 3x + 2$.

(c) $V = M(2 \times 2)$, $A = \begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 12 \\ -7 & 11 \end{pmatrix}$, $D = \begin{pmatrix} 9 & -12 \\ 10 & 1 \end{pmatrix}$.

28. Sea V un espacio vectorial. Falso o verdadero?

(a) Suponga $v_1, \dots, v_k, u, z \in V$ tal que z es combinación lineal de los $v_1, \dots, v_k$. Entonces que z es combinación lineal de $v_1, \dots, v_k, u$.

(b) Si u es combinación lineal de $v_1, \dots, v_k \in V$, entonces $v_1, \dots, v_k, u$ es un sistema de vectores linealmente dependientes.

(c) Si $v_1, \dots, v_k \in V$ es un sistema de vectores linealmente dependientes, entonces v_1 es combinación lineal de los $v_2, \dots, v_k$.

29. Sean V y W espacios vectoriales.

(a) Sea $U \subset V$ un subspacio y sean $u_1, \dots, u_k \in U$. Demuestre que $\text{gen}\{u_1, \dots u_k\} \subset U$.

(b) Sean $u_1, \dots, u_k, w_1, \dots, w_m \in V$. Demuestre que lo siguiente es equivalente:

- (i) $\text{gen}\{u_1, \dots, u_k\} = \text{gen}\{w_1, \dots, w_m\}$.
- (ii) Para todo $j = 1, \dots, k$ tenemos $u_j \in \text{gen}\{w_1, \dots, w_m\}$ y para todo $\ell = 1, \dots, m$ tenemos $w_\ell \in \text{gen}\{u_1, \dots, u_k\}$.
- (c) Sean $v_1, v_2, v_3, \dots, v_m \in V$ y sea $c \in \mathbb{R}$. Demuestre que $\text{gen}\{v_1, v_2, v_3, \dots, v_m\} = \text{gen}\{v_1 + cv_2, v_2, v_3, \dots, v_m\}$.

(d) Sean $v_1, \dots, v_k \in V$ y sea $A : V \to W$ una función lineal invertible. Demuestre que $\dim \text{gen}\{v_1, \dots, v_k\} = \dim \text{gen}\{Av_1, \dots, Av_k\}$. Es verdad si A no es invertible?

30. (a) ¿Es $\mathbb{C}^n$ un espacio vectorial sobre $\mathbb{R}$?

(b) ¿Es $\mathbb{C}^n$ un espacio vectorial sobre $\mathbb{Q}$?

(c) ¿Es $\mathbb{R}^n$ un espacio vectorial sobre $\mathbb{C}$?

(d) ¿Es $\mathbb{R}^n$ un espacio vectorial sobre $\mathbb{Q}$?

(e) ¿Es $\mathbb{Q}^n$ un espacio vectorial sobre $\mathbb{R}$?

(f) ¿Es $\mathbb{Q}^n$ un espacio vectorial sobre $\mathbb{C}$?

Chapter 6

Linear transformations and change of bases

In the first section of this chapter we will define linear maps between vector spaces and discuss their properties. These are functions which "behave well" with respect to the vector space structure. For example, $m \times n$ matrices can be viewed as linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. We will prove the so-called *dimension formula* for linear maps. In Section 6.2 we will study the special case of matrices. One of the main results will be the dimension formula (6.4). In Section 6.4 we will see that, after choice of a basis, every linear map between finite dimensional vector spaces can be represented as a matrix. This will allow us to carry over results on matrices to the case of linear transformations.

As in previous chapters, we work with vector spaces over $\mathbb{R}$ or $\mathbb{C}$. Recall that $\mathbb{K}$ always stands for either $\mathbb{R}$ or $\mathbb{C}$.

6.1 Linear maps

Definition 6.1. Let U, V be vector spaces over the same field $\mathbb{K}$. A function $T : U \to V$ is called a *linear map* if for all $x, y \in U$ and $\lambda \in \mathbb{K}$ the following is true:

$$T(x + y) = Tx + Ty, \quad T(\lambda x) = \lambda Tx. \tag{6.1}$$

Other words for *linear map* are *linear function*, *linear transformation* or *linear operator*.

Remark. Note that very often one writes Tx instead of $T(x)$ when T is a linear function.

Remark 6.2. (i) Clearly, (6.1) is equivalent to

$$T(x + \lambda y) = Tx + \lambda Ty \qquad \text{for all } x, y \in U \text{ and } \lambda \in \mathbb{K}. \tag{6.1'}$$

(ii) It follows immediately from the definition that

$$T(\lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda_1 T v_1 + \cdots + \lambda_k T v_k$$

for all $v_1, \dots, v_k \in V$ and $\lambda_1, \dots, \lambda_k \in \mathbb{K}$.

(iii) The condition (6.1) says that **a linear map respects the vector space structures of its domain and its target space**.

Exercise 6.3. Let U, V be vector spaces over $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let us denote the set of all linear maps from U to V by $\mathcal{L}(U, V)$. Show that $\mathcal{L}(U, V)$ is a vector spaces over $\mathbb{K}$. That means you have to show that the sum of two linear maps is a linear map and that the a scalar multiple of linear map is a linear map.

Exercise 6.4. Let U, V, W be vector spaces over $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

- Suppose that $T : U \to V$ and $S : V \to W$ are linear functions. Show that their composition $ST : U \to W$ is a linear function too.
- Suppose that $T : U \to V$ is a linear invertible linear function so that we can define its inverse function $T^{-1} : \mathrm{Im}(T) \to U$. Show that it is a linear function too.

Examples 6.5 (Linear maps). (a) Every matrix $A \in M(m \times n)$ can be identified with a linear map $\mathbb{R}^n \to \mathbb{R}^m$.

(b) Differentiation is a linear map, for example:

(i) Let $C(\mathbb{R})$ be the space of all continuous functions and $C^1(\mathbb{R})$ the space of all continuously differentiable functions. Then

$$T : C^1(\mathbb{R}) \to C(\mathbb{R}), \quad Tf = f'$$

is a linear map.

Proof. First of all note that $f' \in C(\mathbb{R})$ if $f \in C^1(\mathbb{R})$, so the map T is well-defined. Now we want to see that it is linear. So we take $f, g \in C^1(\mathbb{R})$ and $\lambda \in \mathbb{R}$. We find

$$T(\lambda f + g) = (\lambda f + g)' = (\lambda f)' + g' = \lambda f' + g' = \lambda Tf + Tg. \qquad \square$$

(ii) The following maps are linear, too. Note that their action is the same as the one of T above, but we changed the vector spaces where it acts on.

$$R : P_n \to P_{n-1}, \quad Rf = f', \qquad\qquad S : P_n \to P_n, \quad Sf = f'.$$

(c) Integration is a linear map. For example:

$$I : C([0,1]) \to C([0,1]), \quad f \mapsto If \quad \text{where } (If)(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Proof. Clearly I is well-defined since the integral of a continuous function is again continuous. In order to show that I is linear, we fix $f, g \in C(\mathbb{R})$ and $\lambda \in \mathbb{R}$. We find for every $x \in \mathbb{R}$:

$$\begin{aligned}(I(\lambda f + g))(x) &= \int_0^x (\lambda f + g)(t)\,\mathrm{d}t = \int_0^x \lambda f(t) + g(t)\,\mathrm{d}t = \lambda \int_0^t f(t)\,\mathrm{d}t + \int_0^x g(t)\,\mathrm{d}t \\ &= \lambda (If)(x) + (Ig)(x).\end{aligned}$$

Since this is true for every x, it follows that $I(\lambda f + g) = \lambda(If) + (Ig)$. $\square$

(d) As an example for a linear map from $M(n \times n)$ to itself, we consider

$$T : M(n \times n) \to M(n \times n), \quad T(A) = A + A^t.$$

Proof that T is a linear map. Let $A, B \in M(n \times n)$ and let $c \in \mathbb{R}$. Then

$$\begin{aligned} T(A + cB) &= (A + cB) + (A + cB)^t = A + cB + A^t + (cB)^t = A + cB + A^t + cB^t \\ &= A + A^t + c(B + B^t) = T(A) + cT(B). \end{aligned}$$ □

The next lemma shows that a linear map always maps the zero vector to the zero vector.

Lemma 6.6. *If T is a linear map, then $T\mathbb{O} = \mathbb{O}$.*

Proof. $\mathbb{O} = T\mathbb{O} - T\mathbb{O} = T(\mathbb{O} - \mathbb{O}) = T\mathbb{O}$. □

Definition 6.7. Let $T : U \to V$ be a linear map.

(i) T is called *injective* (or *one-to-one*) if

$$x, y \in U, \; x \neq y \quad \Longrightarrow \quad Tx \neq Ty.$$

(ii) T is called *surjective* if for all $v \in V$ there exists at least one $x \in U$ such that $Tx = v$.

(iii) T is called *bijective* if it is injective and surjective.

(iv) The *kernel of T* (or *null space of T*) is

$$\ker(T) := \{x \in U : Tx = 0\}.$$

Sometimes the notations $N(T)$ or N_T instead of $\ker(T)$ are used.

(v) The *image of T* (or *range of T*) is

$$\operatorname{Im}(T) := \{v \in V : y = Tx \text{ for some } y \in U\}.$$

Sometimes the notations $\operatorname{Rg}(T)$ or $\mathrm{R}(T)$ or $T(U)$ instead of $\operatorname{Im}(T)$ are used.

Remark 6.8. (i) Observe that $\ker(T)$ is a subset of U, $\operatorname{Im}(T)$ is a subset of V. In Proposition 6.11 we will show that they are even subspaces.

(ii) Clearly, T is injective if and only if for all $x, y \in U$ the following is true:

$$Tx = Ty \quad \Longrightarrow \quad x = y.$$

(iii) If T is a linear injective map, then its inverse $T^{-1} : \operatorname{Im}(T) \to U$ exists and is linear too.

The following lemma is very useful.

Lemma 6.9. *Let $T : U \to V$ be a linear map.*

(i) *T is injective if and only if* $\ker(T) = \{\mathbb{O}\}$.

(ii) *T is surjective if and only if* $\operatorname{Im}(T) = V$.

Proof. (i) From Lemma 6.6, we know that $\mathbb{O} \in \ker(T)$. Assume that T is injective, then $\ker(T)$ cannot contain any other element, hence $\ker(T) = \{\mathbb{O}\}$.
Now assume that $\ker(T) = \{\mathbb{O}\}$ and let $x, y \in U$ with $Tx = Ty$. By Remark 6.8 it is sufficient to show that $x = y$. By assumption, $\mathbb{O} = Tx - Ty = T(x-y)$, hence $x - y \in \ker(T) = \{\mathbb{O}\}$. Therefore $x - y = \mathbb{O}$, which means that $x = y$.
(ii) follows directly from the definitions of surjectivity and the image of a linear map. $\square$

Examples 6.10 (Kernels and ranges of the linear maps from Examples 6.5).

(a) We will discuss the case of matrices at the beginning of Section 6.2.

(b) If $T : C^1(\mathbb{R}) \to C(\mathbb{R}), \quad Tf = f'$, then it is easy to see that the kernel of T consists exactly of the constant functions. Moreover T is surjective because every continuous functions is the derivative of another function because for every $f \in C(\mathbb{R})$ we can set $g(x) = \int_0^x f(t)\,\mathrm{d}t$. Then $g \in C^1(\mathbb{R})$ and $Tg = g' = f$ which shows that $\operatorname{Im}(T) = C(\mathbb{R})$.

(c) For the integration operator in Example 6.5((c)) we have that $\ker(I) = \{0\}$ and $\operatorname{Im}(I) = C^1(\mathbb{R})$. In other words, I is injective but not surjective.

Proof. First we prove the claim about the range of I. Suppose that $g \in \operatorname{Im}(I)$. Then g is of the form $g(x) = \int_0^x f(t)\,\mathrm{d}t$ for some $f \in C(\mathbb{R})$. By the fundamental theorem of calculus, it follows that $g \in C^1(\mathbb{R})$, so we proved $\operatorname{Im}(I) \subseteq C^1(\mathbb{R})$. To show the other inclusion, let $g \in C^1(\mathbb{R})$. Then g is differentiable and $g' \in C(\mathbb{R})$ and, again by the fundamental theorem of calculus, we have that $g(x) = \int_0^x g'(t)\,\mathrm{d}t$, so $g \in \operatorname{Im}(I)$ and it follows that $C^1(\mathbb{R}) \subseteq \operatorname{Im}(I)$.

Now assume that $Ig = 0$. If we differentiate, we find that $0 = (Ig)'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x g(t)\,\mathrm{d}t = g(x)$ for all $x \in \mathbb{R}$, therefore $g \equiv 0$, hence $\ker(I) = \{0\}$. $\square$

(d) Let $T : M(n \times n) \to M(n \times n), \quad T(A) = A + A^t$. Then $\ker T = M_{\mathrm{asym}}(n \times n)$ (= the space of all antisymmetric $n \times n$ matrices) and $\operatorname{Im} T = M_{\mathrm{sym}}(n \times n)$ (= the space of all symmetric $n \times n$ matrices).

Proof. First we prove the claim about the range of T. Clearly, $\operatorname{Im}(T) \subseteq M_{\mathrm{sym}}(n \times n)$ because for every $A \in M(n \times n)$ we have that $T(A)$ is symmetric because $(T(A))^t = (A + A^t)^t = A^t + (A^t)^t = A^t + A = T(A)$. To prove $M_{\mathrm{sym}}(n \times n) \subseteq \operatorname{Im}(T)$ we take some $B \in M_{\mathrm{sym}}(n \times n)$. Then $T(\frac{1}{2}B) = \frac{1}{2}B + (\frac{1}{2}B)^t = \frac{1}{2}B + \frac{1}{2}B = B$ where we used that B is symmetric. In summary we showed that $\operatorname{Im}(T) = M_{\mathrm{sym}}(n \times n)$.

The claim on the kernel of T follows from

$$A \in \ker T \iff T(A) = 0 \iff A + A^t = 0 \iff A = -A^t \iff A \in M_{\mathrm{asym}}(n \times n). \quad \square$$

Proposition 6.11. *Let $T : U \to V$ be a linear map. Then*

(i) $\ker(T)$ *is a subspace of* U.

(ii) $\operatorname{Im}(T)$ *is a subspace of* V.

Proof. (i) By Lemma 6.6, $\mathbb{O} \in \ker(T)$. Let $x, y \in \ker(T)$ and $\lambda \in \mathbb{K}$. Then $x + \lambda y \in \ker(T)$ because

$$T(x + \lambda y) = Tx + \lambda Ty = \mathbb{O} + \lambda 0 = \mathbb{O}.$$

Hence $\ker(T)$ is a subspace of U by Proposition 5.10.

(ii) C;early, $\mathbb{O} \in \operatorname{Im}(T)$. Let $v, w \in \operatorname{Im}(T)$ and $\lambda \in \mathbb{K}$. Then there exist Let $x, y \in U$ such that $Tx = v$ and $Ty = w$. Then $v + \lambda w = Tx + \lambda Ty = T(x + \lambda y) \in \operatorname{Im}(T)$. hence $v + \lambda w \in \operatorname{Im}(T)$. Therefore $\operatorname{Im}(T)$ is a subspace of V by Proposition 5.10. $\square$

Since we now know that $\ker(T)$ and $\operatorname{Im}(T)$ are subspaces, the following definition makes sense.

Definition 6.12. Let $T : U \to V$ be a linear map. We define

$$\dim(\ker(T)) = \textit{nullity of } T, \qquad \dim(\operatorname{Im}(T)) = \textit{rank of } T.$$

Sometimes the notations $\nu(T) = \dim(\ker(T))$ and $\rho(T) = \dim(\operatorname{Im}(T))$ are used.

Example. Let $T : P_4 \to P_4$ be defined by $Tp = p'$. Then $\operatorname{Im}(T) = \{q \in P_3 : \deg q \leq 2\}$ and $\ker(T) = \{q \in P_3 : \deg q = 0\}$. In particular $\dim(\operatorname{Im}(T)) = 3$ and $\dim(\ker(T)) = 1$.

Proof.
- First we show the claim about the image of T. We know that differentiation lowers the degree of a polynomial by 1. Hence $\operatorname{Im}(T) \subseteq \{q \in P_3 : \deg q \leq 2\}$. On the other hand, we know that every polynomial of degree ≤ 2 is the derivative of a polynomial of degree ≤ 3. So the claim follows.
- First we show the claim about the kernel of T. Recall that $\ker(T_) = \{p \in P_3 : Tp = 0\}$. So the kernel of T are exactly those polynomials whose first derivative is 0. These are exactly the constant polynomials, i.e., the polynomials of degree 0. $\square$

Lemma 6.13. *Let* $T : U \to V$ *be a linear map between two vector spaces* U, V *and let* $\{u_1, \dots, u_k\}$ *be a basis of* U. *Then* $\operatorname{Im} T = \operatorname{span}\{Tu_1, \dots, Tu_k\}$.

Proof. Clearly, $Tu_1, \dots, Tu_k \in \operatorname{Im} T$. Since the image of T is a vector space, all linear combinations of these vectors must belong to $\operatorname{Im} T$ too which shows $\operatorname{span}\{Tu_1, \dots, Tu_k\} \subseteq \operatorname{Im} T$. To show the other inclusion, let $y \in \operatorname{Im} T$. Then there is an $x \in U$ such that $y = Tx$. Let us express x as linear combination of the vectors of the basis: $x = \alpha_1 u_1 + \dots \alpha_k u_k$. Then we obtain

$$y = Tx = T(\alpha_1 u_1 + \dots \alpha_k u_k) = \alpha_1 T u_1 + \dots \alpha_k T u_k \in \operatorname{span}\{Tu_1, \dots, Tu_k\}.$$

Since y was arbitrary in $\operatorname{Im} T$, we conclude that $\operatorname{Im} T \subseteq \operatorname{span}\{Tu_1, \dots, Tu_k\}$. So in summary we proved the claim. $\square$

Proposition 6.14. *Let* U, V *be* $\mathbb{K}$*-vector spaces,* $T : U \to V$ *a linear map. Let* $x_1, \dots, x_k \in U$ *and set* $y_1 := Tx_1, \dots, y_k := Tx_k$. *Then the following is true.*

(i) *If the* $x_1, \dots, x_k$ *are linearly dependent, then* $y_1, \dots, y_k$ *are linearly dependent too.*

(ii) *If the* $y_1, \dots, y_k$ *are linearly independent, then* $x_1, \dots, x_k$ *are linearly independent too.*

(iii) *Suppose additionally that* T *invertible. Then* $x_1, \dots, x_k$ *are linearly independent if and only if* $y_1, \dots, y_k$ *are linearly independent.*

In general the implication "If $x_1, \dots, x_k$ are linearly independent, then $y_1, \dots, y_k$ are linearly independent." is *false*. Can you give an example?

Proof of Proposition 6.14. (i) Assume that the vectors $x_1, \dots, x_k$ are linearly dependent. Then there exist $\lambda_1, \dots, \lambda_k \in \mathbb{K}$ such that $\lambda_1 x_1 + \dots + \lambda_k x_k = 0$ and at least one $\lambda_j \neq 0$. But then

$$\begin{aligned} \mathbb{O} = T\mathbb{O} &= T(\lambda_1 x_1 + \dots + \lambda_k x_k) = \lambda_1 T x_1 + \dots + \lambda_k T x_k \\ &= \lambda_1 y_1 + \dots + \lambda_k y_k, \end{aligned}$$

hence the vectors $y_1, \dots, y_k$ are linearly dependent.

(ii) follows directly from (i).

(iii) Suppose that the vectors $y_1, \dots, y_k$ are linearly independent. Then so are the $x_1, \dots, x_k$ by (i). Now suppose that $x_1, \dots, x_k$ are linearly independent. Note that T is invertible, so T^{-1} exists. Therefore we can apply (i) to T^{-1} in order to conclude that the system $y_1, \dots, y_k$ is linearly independent. (Note that $x_j = T^{-1} y_j$.) □

Exercise 6.15. Assume that $T : U \to V$ is an injective linear map and suppose that $\{u_1, \dots, u_\ell\}$ is a set of are linearly independent vectors in U. Show that $\{Tu_1, \dots, Tu_\ell\}$ is a set of are linearly independent vectors in V.

The following lemma is **very** useful and it is used in the proof of Theorem 6.18.

Proposition 6.16. *Let* U, V *be* $\mathbb{K}$*-vector spaces with* $\dim U = k < \infty$.

(i) *If* $T : U \to V$ *is linear transformation, then* $\dim U \geq \dim \operatorname{Im}(T)$.

(ii) *If* $T : U \to V$ *is an injective linear transformation, then* $\dim U = \dim \operatorname{Im}(T)$.

(iii) *If* $T : U \to V$ *is a bijective linear transformation, then* $\dim U = \dim V$.

Proof. Let $u_1, \dots, u_k$ be a basis of U.

(i) From Lemma 6.13 we know that $\operatorname{Im} T = \operatorname{span}\{Tu_1, \dots, Tu_k\}$. Therefore $\dim \operatorname{Im} T \geq k = \dim U$ by Theorem 5.41.

(ii) Assume that T is injective. We will show that $Tu_1, \dots, Tu_k$ are linearly independent. Let $\alpha_1, \dots, \alpha_k \in \mathbb{K}$ such that $\alpha_1 Tu_1 + \dots + \alpha_k Tu_k = \mathbb{O}$. Then

$$\mathbb{O} = \alpha_1 Tu_1 + \dots + \alpha_k Tu_k = T(\alpha_1 u_1 + \dots + \alpha_k u_k).$$

Since T is injective, it follows that $\alpha_1 u_1 + \dots + \alpha_k u_k = \mathbb{O}$, hence $\alpha_1 = \dots = \alpha_k = 0$ which shows that the vectors $Tu_1, \dots, Tu_k$ are indeed linearly independent. Therefore they are a basis of $\operatorname{span}\{Tu_1, \dots, Tu_k\} = \operatorname{Im} T$ and we conclude that $\dim \operatorname{Im} T = k = \dim U$.

(iii) Since T is bijective, it is surjective and injective. Surjectivity means that $\operatorname{Im} T = V$ and injectivity of T implies that $\dim \operatorname{Im} T = \dim U$ by (ii). In conclusion,

$$\dim U = \dim \operatorname{Im} T = \dim V. \qquad \square$$

The previous theorem tells us for example that there is no injective linear map from $\mathbb{R}^5$ to $\mathbb{R}^3$; or that there is no surjective linear map from $\mathbb{R}^3$ to $M(2 \times 2)$.

Remark 6.17. Proposition 6.16 is true also for $\dim U = \infty$. In this case, (i) clearly holds whatever $\dim \operatorname{Im}(T)$ may be. To prove (ii) we need to show that $\dim \operatorname{Im}(T) = \infty$ if T is injective. Note that for every $n \in \mathbb{N}$ we can find a subspace U_n of U with $\dim U_n = n$ and we define T_n to be the restriction of T to U_n, that is, $T_n : U_n \to V$. Since the restriction of an injective map is injective, it follows from (ii) that $\dim \operatorname{Im}(T_n) = n$. On the other hand, $\operatorname{Im}(T_n)$ is a subspace of V, therefore $\dim V \geq \dim \operatorname{Im}(T_n) = n$ by Theorem 5.50 and Remark 5.51. Since this is true for any $n \in \mathbb{N}$, it follows that $\dim V = \infty$. The proof of (iii) is the same as in the finite dimensional case.

Theorem 6.18. *Let U, V be $\mathbb{K}$-vector spaces and $T : U \to V$ a linear map. Moreover, let $E : U \to U$, $F : V \to V$ be linear bijective maps. Then the following is true:*

(i) $\operatorname{Im}(T) = \operatorname{Im}(TE)$, *in particular* $\dim(\operatorname{Im}(T)) = \dim(\operatorname{Im}(TE))$.

(ii) $\ker(TE) = E^{-1}(\ker(T))$ *and* $\dim(\ker(T)) = \dim(\ker(TE))$.

(iii) $\ker(T) = \ker(FT)$, *in particular* $\dim(\ker(T)) = \dim(\ker(FT))$.

(iv) $\operatorname{Im}(FT) = F(\operatorname{Im}(T))$ *and* $\dim(\operatorname{Im}(T)) = \dim(\operatorname{Im}(FT))$.

In summary we have

$$\boxed{\begin{array}{ll} \ker(FT) = \ker(T), & \ker(TE) = E^{-1}(\ker(T)), \\ \operatorname{Im}(FT) = F(\operatorname{Im}(T)), & \operatorname{Im}(TE) = \operatorname{Im}(T). \end{array}} \tag{6.2}$$

and

$$\boxed{\begin{array}{l} \dim \ker(T) = \dim \ker(FT) = \dim \ker(TE) = \dim \ker(FTE), \\ \dim \operatorname{Im}(T) = \dim \operatorname{Im}(FT) = \dim \operatorname{Im}(TE) = \dim \operatorname{Im}(FTE). \end{array}} \tag{6.3}$$

Proof. (i) Let $v \in V$. If $v \in \operatorname{Im}(T)$, then there exists $x \in U$ such that $Tx = v$. Set $y = E^{-1}x$. Then $v = Tx = TEE^{-1}x = TEy \in \operatorname{Im}(TE)$. On the other hand, if $v \in \operatorname{Im}(TE)$, then there exists $y \in U$ such that $TEy = v$. Set $x = E$. Then $v = TEy = Tx \in \operatorname{Im}(T)$.

(ii) To show $\ker(TE) = E^{-1}\ker(T)$ observe that

$$\ker(TE) = \{x \in U : Ex \in \ker(T)\} = \{E^{-1}u : u \in \ker(T)\} = E^{-1}(\ker(T)).$$

It follows that

$$E^{-1} : \ker T \to \ker(TE)$$

is a linear bijection and therefore $\dim T = \dim \ker(TE)$ by Proposition 6.16(iii) (or Remark 6.17 in the infinite dimensional case) with E^{-1} as T, $\ker(T)$ as U and $\ker(TE)$ as V.

(iii) Let $x \in U$. Then $x \in \ker(FT)$ if and only if $FTx = \mathbb{O}$. Since F is injective, we know that $\ker(F) = \{\mathbb{O}\}$, hence it follows that $Tx = \mathbb{O}$. But this is equivalent to $x \in \ker(T)$.

(iv) To show $\operatorname{Im}(FT) = F \operatorname{Im}(T)$ observe that

$$\operatorname{Im}(FT) = \{y \in V : y = FTx \text{ for some } x \in U\} = \{Fv : v \in \operatorname{Im}(T)\} = F(\operatorname{Im}(T)).$$

It follows that

$$F : \operatorname{Im} T \to \operatorname{Im}(FT)$$

is a linear bijection and therefore $\dim T = \dim \operatorname{Im}(FT)$ by Proposition 6.16(iii) (or Remark 6.17 in the infinite dimensional case) with F as T, $\operatorname{Im}(T)$ as U and $\operatorname{Im}(FT)$ as V. □

Remark 6.19. In general, $\ker(T) = \ker(TE)$ and $\ker(T) = \ker(FT)$ is false. Take for example $U = V = \mathbb{R}^2$, $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E = F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then clearly the hypotheses of the theorem are satisfied and

$$\ker(T) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \qquad \operatorname{Im}(T) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},$$

but

$$\ker(TE) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \qquad \operatorname{Im}(FT) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

Draw a picture to visualise the example above, taking into account that T represents the projection onto the x-axis and E and F are rotation by $45°$ and a "stretching" by the factor $\sqrt{2}$.

We end this section with one of the main theorems of linear algebra. In the next section we will re-prove it for the special case when T is given a matrix in Theorem 6.33. The theorem below can be considered a coordinate free version of Theorem 6.33.

Theorem 6.20. *Let U, V be vector spaces with $\dim U = n < \infty$ and let $T : U \to V$ be a linear map. Then*

$$\dim(\ker(T)) + \dim(\operatorname{Im}(T)) = n. \tag{6.4}$$

Proof. Let $k = \dim(\ker(T))$ and let $\{u_1, \dots, u_k\}$ be a basis of $\ker(T)$. We complete it to a basis $\{u_1, \dots, u_k, w_{k+1}, \dots, w_n\}$ of U and we set $W := \operatorname{span}\{w_{k+1}, \dots, w_n\}$. Note that by construction $\ker(T) \cap W = \{\mathbb{O}\}$. (Prove this!) Let us consider $\widetilde{T} = T|_W$ the restriction of T to W.
It follows that $\widetilde{T}$ *is injective* because if $\widetilde{T}x = \mathbb{O}$ for some $x \in W$ then also $Tx = \widetilde{T}x = \mathbb{O}$, hence $x \in \ker(T) \cap W = \{\mathbb{O}\}$. It follows from Proposition 6.16(ii) that

$$\dim \operatorname{Im} \widetilde{T} = \dim W = n - k. \tag{6.5}$$

To complete the proof, it suffices to show that $\operatorname{Im} \widetilde{T} = \operatorname{Im} T$. Recall that by Lemma 6.13, we have that the range of a linear map is generated by the images of a basis of the initial vector space. Therefore we find that

$$\begin{aligned} \operatorname{Im} T &= \operatorname{span}\{Tu_1, \dots, Tu_k, Tw_{k+1}, \dots, Tw_n\} = \operatorname{span}\{Tw_{k+1}, \dots, Tw_n\} \\ &= \operatorname{span}\{\widetilde{T}w_{k+1}, \dots, \widetilde{T}w_n\} \\ &= \operatorname{Im} \widetilde{T} \end{aligned}$$

where in the second step we used that $Tu_1 = \cdots = Tu_k = \mathbb{O}$ and therefore they do not contribute to the linear span and in the third step we used that $Tw_j = \widetilde{T}w_j$ for $j = k+1, \dots, n$. So we showed that $\operatorname{Im}\widetilde{T} = \operatorname{Im} T$, in particular their dimensions are equal and the claim follows from (6.5) because, recalling that $k = \dim\ker(T)$,

$$n = \dim \operatorname{Im}\widetilde{T} + k = \dim\operatorname{Im} T + \dim\ker T. \qquad \square$$

Note that an alternative way to prove the theorem above is to first prove Theorem 6.33 for matrices and then use the results on representations of linear maps in Section 6.4 to conclude (6.4).

You should now have understood

- what a linear map is and why they are the natural maps to consider on vector spaces,
- what injectivity, surjectivity and bijectivity means,
- what the kernel and image of a linear map is,
- why the dimension formula (6.4) is true,
- . . . ,

You should now be able to

- give examples of linear maps,
- check if a given function is a linear maps,
- find bases and the dimension of kernels and ranges of a given linear map,
- . . .

6.2 Matrices as linear maps

In this section, we work mostly with real vector spaces for definiteness sake. However, all the statements are also true for complex vector spaces. We only have to replace everywhere $\mathbb{R}$ by $\mathbb{C}$ and the word *real* by *complex*.

Let $A \in M(m\times n)$. We already know that we can view A as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Hence $\ker(A)$ and $\operatorname{Im}(A)$ and the terms *injectivity* and *surjectivity* are defined.

Strictly speaking, we should distinguish between a matrix and the linear map induced by it. So we should write $T_A : \mathbb{R}^n \to R^m$ for the map $x \mapsto Ax$. The reason is that if we view A directly as a linear map then this implies that we tacitly have already chosen a basis in $\mathbb{R}^n$ and $\mathbb{R}^m$, see Section 6.4 for more on that. However, we will usually abuse notation and write A instead of T_A.

If we view a matrix A as a linear map and at the same time as a linear system of equations, then we obtain the following.

Remark 6.21. Let $A \in M(m\times n)$ and denote the columns of A by $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^m$. Then the following is true.

(i) $\ker(A) =$ all solutions $\vec{x}$ of the homogeneous system $A\vec{x} = \vec{0}$.

(ii) $\operatorname{Im}(A) =$ all vectors $\vec{b}$ such that the system $A\vec{x} = \vec{b}$ has a solution
$= \operatorname{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$.

Consequently,

(iii) A is injective $\iff \ker(A) = \{\vec{0}\}$
$\iff$ the homogenous system $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

(iv) A is surjective $\iff \operatorname{Im}(A) = \mathbb{R}^m$
$\iff$ for every $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} = \vec{b}$ has at least one solution.

Proof. All claims should be clear except maybe the second equality in (ii). This follows from

$$\begin{aligned}\operatorname{Im} A = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\} &= \left\{ (\vec{a}_1|\ldots|\vec{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \right\} \\ &= \{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n) : x_1, \ldots, x_n \in \mathbb{R}\} \\ &= \operatorname{span}\{\vec{a}_1, \ldots, \vec{a}_n\},\end{aligned}$$

see also Remark 3.18. $\square$

To practice a bit, we prove the following two remarks in two ways.

Remark 6.22. Let $A \in M(m \times n)$. If $m > n$, then M cannot be surjective.

Proof with Gauß-Jordan. Let A' be the row reduced echelon form of A. Then there must be an invertible matrix E such that $A = EA'$ and A' the last row of A' must be zero because it can have at most n pivots. But then $(A'|\vec{e}_m)$ is inconsistent, which means that $(A|E^{-1}\vec{e}_m)$ is inconsistent. Hence $E^{-1}\vec{e} \notin \operatorname{Im} A$ so A cannot be surjective. (Basically we say that clearly A' is not surjective because we can easily find a right side to that $A'\vec{x}' = \vec{b}'$ is inconsistent. Just pick any vector $\vec{b}'$ whose last coordinate is different from 0. The easiest such vector is $\vec{e}_m$. Now do the Gauß-Jordan process backwards on this vector in order to obtain a right hand side $\vec{b}$ such that $A\vec{x} = \vec{b}$ is inconsistent.) $\square$

Proof using the concept of dimension. We already saw that $\operatorname{Im} A$ is the linear span of its columns. Therefore $\dim \operatorname{Im} A \leq \#$columns of $A = n < m = \dim \mathbb{R}^m$, therefore $\operatorname{Im} A \subsetneq \mathbb{R}^m$. $\square$

Remark 6.23. Let $A \in M(m \times n)$. If $m < n$, then M cannot be injective.

Proof with Gauß-Jordan. Let A' be the row reduced echelon form of A. Then A' can have at most m pivots. Since A' has more columns than pivots, the homogeneous system $A\vec{x} = \vec{0}$ has infinitely solutions, but then also $\ker A$ contains infinitely many vectors, in particular A cannot be injective. $\square$

Proof using the concept of dimension. We already saw that $\operatorname{Im} A$ is the linear span of its n columns in $\mathbb{R}^m$. Since $n > m$ it follows that the column vectors are linearly dependent in $\mathbb{R}^m$, hence $A\vec{x} = \vec{0}$ has a non-trivial solution. Therefore $\ker A$ is not trivial and it follows that A is not injective. $\square$

Note that the remarks do **not** imply that A is surjective if $m \le n$ or that A is injective if $n \le m$. Find examples!

From Theorem 3.43 we obtain the following very important theorem for the special case $m = n$.

Theorem 6.24. *Let $A \in M(n \times n)$ be a square matrix. Then the following is equivalent.*

(i) *A is invertible.*

(ii) *A is injective, that is,* $\ker A = \{\vec{0}\}$.

(iii) *A is surjective, that is,* $\operatorname{Im} A = \mathbb{R}^n$.

In particular, A is injective if and only if A is surjective if and only if A is bijective.

Definition 6.25. Let $A \in M(m \times n)$ and let $\vec{c}_1, \dots, \vec{c}_n$ be the columns of A and $\vec{r}_1, \dots, \vec{r}_m$ be the rows of A. We define

(i) $C_A := \operatorname{span}\{\vec{c}_1, \dots, \vec{c}_m\} =:$ *column space of* $A \subseteq \mathbb{R}^m$,

(ii) $R_A := \operatorname{span}\{\vec{r}_1, \dots, \vec{r}_n\} =:$ *row space of* $A \subseteq \mathbb{R}^n$,

The next proposition follows immediately from the definition above and from Remark 6.21(ii).

Proposition 6.26. *For $A \in M(m \times n)$ it follows that*

(i) $R_A = C_{A^t}$ *and* $C_A = R_{A^t}$,

(ii) $C_A = \operatorname{Im}(A)$ *and* $R_A = \operatorname{Im}(A^t)$.

The next proposition follows directly from the general theory in Section 6.1. We will give another proof at the end of this section.

Proposition 6.27. *Let $A \in M(m \times n)$, $E \in M(n \times n)$, $F \in M(m \times m)$ and assume that E and F are invertible. Then*

(i) $C_A = C_{AE}$.

(ii) $R_A = R_{FA}$.

Proof. (i) Note that $C_A = \operatorname{Im}(A) = \operatorname{Im}(AE) = C_{AE}$, where in the first and third equality we used Proposition 6.26, and in the second equality we used Theorem 6.18.

(ii) Recall that, if F is invertible, then F^t is invertible too. With Proposition 6.26(i) and what we already proved in (i), we obtain $R_{FA} = C_{(FA)^t} = C_{A^t F^t} = C_{A^t} = R_A$. □

This proposition implies immediately the following proposition.

Proposition 6.28. *Let $A, B \in M(m \times n)$.*

(i) *If A and B are row equivalent, then*

$$\dim(\ker(A)) = \dim(\ker(B)), \quad \dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(B)), \quad \operatorname{Im}(A^t) = \operatorname{Im}(B^t), \quad R_A = R_B.$$

(ii) *If A and B are column equivalent, then*

$$\dim(\ker(A)) = \dim(\ker(B)), \quad \dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(B)), \quad \operatorname{Im}(A) = \operatorname{Im}(B), \quad C_A = C_B.$$

Proof. We will only prove (i). The claim (ii) can be proved similar (or can be deduced easily from (i) by applying (i) to the transposed matrices). That A and B are row equivalent means that we can transform B into A by row transformations. Since row transformations can be represented by multiplication by elementary matrices from the left, there are elementary matrices $F_1, \dots, F_k \in M(m \times m)$ such that $A = F_1 \dots F_k B$. Note that all F_j are invertible. Let $F := F_1 \dots F_k$. Then F is invertible and $A = FB$. Hence all the claims in (i) follow from Theorem 6.18 and Proposition 6.27. □

The proposition above is very useful to calculate the kernel of a matrix A: Let A' be the reduced row-echelon form of A. Then the proposition can be applied to A and A', and we find that $\ker(A) = \ker(A')$.

In fact, we know this since the first chapter of this course, but back then we did not have fancy words like "kernel" at our disposal. It says nothing else than that the solutions of a homogenous system do not change if we apply row transformations which is exactly what we are doing in the Gauß-Jordan elimination.

In Examples 6.36 and 6.37 we will calculate the kernel and range of a matrix. Now we will prove two technical lemmas.

Lemma 6.29. *Let $A \in M(m \times n)$. Then there exist elementary matrices $E_1, \dots, E_k \in M(n \times n)$ and $F_1, \dots, F_\ell \in M(m \times m)$ such that*

$$F_1 \cdots F_\ell A E_1 \cdots E_k = A''$$

where A'' is of the form

$$A'' = \begin{array}{c} \overbrace{\qquad\qquad}^{r}\;\overbrace{\qquad}^{n-r} \\ \left(\begin{array}{ccc:c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ \hdashline & 0 & & 0 \end{array}\right) \end{array} \begin{array}{l} \\ \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} r \\ \left.\vphantom{\begin{array}{c}0\end{array}}\right\} m-r \end{array} \tag{6.6}$$

Proof. Let A' be the reduced row-echelon form of A. Then there exist $F_1, \dots, F_\ell \in M(m \times m)$ such that $F_1 \cdots F_\ell A = A'$ and A' is of the form

$$A' = \begin{pmatrix} 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots \\ & & & & 1 & * & \cdots & * & 0 & * & \cdots \\ & & & & & & & & 1 & * & \cdots \\ & & & & & & & & & & \\ & & & & & & & & & & \end{pmatrix}. \tag{6.7}$$

Now clearly we can find "allowed" column transformations such that A' is transformed into the form A''. If we observe that applying column transformations is equivalent to multiplying A' from the right by elementary matrices, then we can find elementary matrices $E_1, \dots, E_k$ such that $A'E_1 \dots E_k$ if of the form (6.6). ☐

Lemma 6.30. *Let A'' be as in (6.6). Then*

(i) $\dim(\ker(A)) = m - r =$ *number of zero rows of A'',*

(ii) $\dim(\operatorname{Im}(A)) = r =$ *number of pivots A'',*

(iii) $\dim(C_{A''}) = \dim(R_{A''}) = r$.

Proof. All assertions are clear if we note that

$$\ker(A'') = \operatorname{span}\{\vec{e}_{r+1}, \dots, \vec{e}_n\} \quad \text{and} \quad \operatorname{Im}(A'') = \operatorname{span}\{\vec{e}_1, \dots, \vec{e}_r\},$$

where the $\vec{e}_j$ are the standard unit vectors (that is, their jth component is 1 and all other components are 0). ☐

Proposition 6.31. *Let $A \in M(m \times n)$ and let A' be its reduced row-echelon form. Then*

$$\dim(\operatorname{Im}(A)) = \textit{ number of pivots of } A'.$$

Proof. Let $F_1, \dots, F_\ell, E_1, \dots, E_k$ and A'' be as in (6.29) and set $F := F_1 \cdots F_\ell$ and $E := E_1 \cdots E_k$. It follows that $A' = FA$ and $A'' = FAE$. Clearly, the number of pivots of A' and A'' coincide. Therefore, with the help of Theorem 6.18 we obtain

$$\begin{aligned} \dim(\operatorname{Im}(A)) &= \dim(\operatorname{Im}(FAE)) \\ &= \text{ number of pivots of } A'' \\ &= \text{ number of pivots of } A'. \end{aligned}$$

☐

Proposition 6.32. *Let $A \in M(m \times n)$. Then*

$$\dim(\operatorname{Im}(A)) = \dim C_A = \dim R_A.$$

That means: (rank of row space) = (rank of column space).

Proof. Since $C_A = \operatorname{Im}(A)$ by Proposition 6.26, the first equality is clear.

Now let $F_1, \dots, F_\ell$, $E_1, \dots, E_k$ and A', A'' be as in Lemma 6.29 and set $F := F_1 \cdots F_\ell$ and $E := E_1 \cdots E_k$. Then

$$\begin{aligned}\dim(R_A) &= \dim(R_{FAE}) = \dim(R_{A''}) = r = \dim(C_{A''}) = \dim(C_{FAE}) \\ &= \dim(C_A).\end{aligned}$$ □

As an immediate consequence we obtain the following theorem which is a special case of Theorem 6.20, see also Theorem 6.46.

Theorem 6.33. *Let $A \in M(m \times n)$. Then*

$$\dim(\ker(A)) + \dim(\operatorname{Im}(A)) = n. \tag{6.8}$$

Proof. With the notation a above, we obtain

$$\begin{aligned}\dim(\ker(A)) &= \dim(\ker(A'')) = n - r,\\ \dim(\operatorname{Im}(A)) &= \dim(\operatorname{Im}(A'')) = r\end{aligned}$$

and the desired formula follows. □

For the calculation of a basis of $\operatorname{Im}(A)$, the following theorem is useful.

Theorem 6.34. *Let $A \in M(m \times n)$ and let A' be its reduced row-echelon form with columns $\vec{c}_1, \dots, \vec{c}_n$ and $\vec{c}_1{}', \dots, \vec{c}_n{}'$ respectively. Assume that the pivot columns of A' are the columns $j_1 < \dots < j_k$. Then $\dim(\operatorname{Im}(A)) = k$ and a basis of $\operatorname{Im}(A)$ is given by the columns $\vec{c}_{j_1}, \dots, \vec{c}_{j_k}$ of A.*

Proof. Let E be an invertible matrix such that $A = EA'$. By assumption on the pivot columns of A', we know that $\dim(\operatorname{Im}(A')) = k$ and that a basis of $\operatorname{Im}(A')$ is given by the columns $\vec{c}_{j_1}{}', \dots, \vec{c}_{j_k}{}'$. By Theorem 6.18 it follows that $\dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(A')) = k$. Now observe that by definition of E we have that $E\vec{c}_\ell{}' = \vec{c}_\ell$ for every $\ell = 1, \dots, n$; in particular this is true for the pivot columns of A'. Moreover, since E in invertible and the vectors $\vec{c}_{j_1}{}', \dots, \vec{c}_{j_k}{}'$ are linearly independent, it follows from Theorem 6.14 that the vectors $\vec{c}_{j_1}, \dots, \vec{c}_{j_k}$ are linearly independent. Clearly they belong to $\operatorname{Im}(A)$, so we have $\operatorname{span}\{\vec{c}_{j_1}, \dots, \vec{c}_{j_k}\} \subseteq \operatorname{Im}(A)$. Since both spaces have the same dimension, they must be equal. □

Remark 6.35. The theorem above can be used to determine a basis of a subspace given in the form $U = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^m$ as follows: Define the matrix $A = (\vec{v}_1 | \dots | \vec{v}_k)$. Then clearly $U = \operatorname{Im} A$ and we can apply Theorem 6.34 to find a basis of U.

Example 6.36. Find $\ker(A)$, $\operatorname{Im}(A)$, $\dim(\ker(A))$, $\dim(\operatorname{Im}(A))$ and R_A for

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 3 & 2 & 13 & 1 \\ 0 & 2 & 4 & -1 \\ 4 & 5 & 22 & 1 \end{pmatrix}.$$

Solution. First, let us row-reduce the matrix A:

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 3 & 2 & 13 & 1 \\ 0 & 2 & 4 & -1 \\ 4 & 5 & 22 & 1 \end{pmatrix} \xrightarrow[]{\substack{Q_{21}(-1) \\ Q_{41}(-4)}} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 2 & 4 & -1 \\ 0 & 1 & 2 & -3 \end{pmatrix} \xrightarrow[]{\substack{Q_{32}(2) \\ Q_{42}(1)}} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

$$\xrightarrow[]{\substack{S_2(-1) \\ Q_{43}(-1)}} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[]{\substack{S_4(1/5) \\ Q_{12}(-1)}} \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[]{\substack{Q_{14}(1) \\ Q_{24}(-2)}} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: A'.$$

Now it follows immediately that $\dim R_A = \dim C_A = 3$ and

$$\begin{aligned} \dim(\operatorname{Im}(A)) &= \#\text{non-zero rows of } A' = 3, \\ \dim(\ker(A)) &= 4 - \dim(\operatorname{Im}(A)) = 1 \end{aligned}$$

(or: $\dim(\operatorname{Im}(A)) = \#$pivot columns $A' = 3$, or: $\dim(\operatorname{Im}(A)) = \dim(R_A) = 3$ or: $\dim(\ker(A)) = \#$non-pivot columns $A' = 1$).

Kernel of A: We know that $\ker(A) = \ker(A')$ by Theorem 6.18 or Proposition 6.28. From the explicit form of A', it is clear that $A\vec{x} = 0$ if and only if $x_4 = 0$, x_3 arbitrary, $x_2 = -2x_3$ and $x_1 = -3x_3$. Therefore

$$\ker(A) = \ker(A') = \left\{ \begin{pmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{pmatrix} : x_3 \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Image of A: The pivot columns of A' are the columns 1, 2 and 4. Therefore, by Theorem 6.34 a basis of $\operatorname{Im}(A)$ are the columns 1, 2 and 4 of A:

$$\operatorname{Im}(A) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Example 6.37. Find a basis of $\operatorname{span}\{p_1, p_2, p_3, p_4\} \subseteq P_3$ and its dimension for

$$\begin{aligned} p_1 &= x^3 - x^2 + 2x + 2, & p_2 &= x^3 + 2x^2 + 8x + 13, \\ p_3 &= 3x^3 - 6x^2 - 5, & p_3 &= 5x^3 + 4x^2 + 26x - 9. \end{aligned}$$

Solution. First we identify P_3 with $\mathbb{R}^4$ by $ax^3 + bx^2 + cx + d \;\widehat{=}\; (a, b, c, d)^t$. The polynomials p_1, p_2, p_3, p_4 correspond to the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 8 \\ 13 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 3 \\ -6 \\ 0 \\ -5 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 5 \\ 4 \\ 26 \\ -9 \end{pmatrix}.$$

Now we use Remark 6.35 to find a basis of $\operatorname{span}\{v_1, v_2, v_3, v_4\}$. To this end we consider the A whose columns are the vectors $\vec{v}_1, \dots, \vec{v}_4$:

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 \\ -1 & 2 & -6 & 4 \\ 2 & 8 & 0 & 26 \\ 2 & 13 & -5 & -9 \end{pmatrix}.$$

Clearly, $\operatorname{span}\{v_1, v_2, v_3, v_4\} = \operatorname{Im}(A)$, so it suffices to find a basis of $\operatorname{Im}(A)$. Applying row transformation to A, we obtain

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 \\ -1 & 2 & -6 & 4 \\ 2 & 8 & 0 & 26 \\ 2 & 13 & -5 & -9 \end{pmatrix} \longrightarrow \quad \cdots \quad \longrightarrow \begin{pmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A'.$$

The pivot columns of A' are the first and the second column, hence by Theorem 6.34, a basis of $\operatorname{Im}(A)$ are its first and second columns, i.e. the vectors $\vec{v}_1$ and $\vec{v}_2$.
It follows that $\{p_1, p_2\}$ is a basis of $\operatorname{span}\{p_1, p_2, p_3, p_4\} \subseteq P_3$, hence $\dim(\operatorname{span}\{p_1, p_2, p_3, p_4\}) = 2$.

Remark 6.38. Let us use the abbreviation $\pi = \operatorname{span}\{p_1, p_2, p_3, p_4\}$. The calculation above actually shows that any two vectors of p_1, p_2, p_3, p_4 form a basis of π. To see this, observe that clearly any two of them are linearly independent, hence the dimension of their generated space is 2. On the other hand, this generated space is a subspace of π which has the same dimension 2. Therefore they must be equal.

Remark 6.39. If we wanted to complete p_1, p_2 to a basis of P_3, we have (at least) the two following options:

(i) In order to find $q_3, q_4 \in P_3$ such that p_1, p_2, q_3, q_4 forms a basis of P_3 we can use the reduction process that was employed to find A'. Assume that E is an invertible matrix such that $A = EA'$. Such an E can be found by keeping track of the row operations that transform A into A'. Let $\vec{e}_j$ be the standard unit vectors of $\mathbb{R}^4$. Then we already know that $\vec{v}_1 = E\vec{e}_1$ and $\vec{v}_2 = E\vec{e}_2$. If we set $\vec{w}_3 = E\vec{e}_3$ and $\vec{w}_4 = E\vec{e}_4$, then $\vec{v}_1, \vec{v}_2, \vec{w}_3, \vec{w}_4$ form a basis of $\mathbb{R}^4$. This is because $\vec{e}_1, \dots, \vec{e}_4$ are linearly independent and E is injective. Hence $E\vec{e}_1, \dots, E\vec{e}_4$ are linearly independent too (by Proposition 6.14).

If we already have some knowledge of orthogonal complements as discussed in Chapter 7, then we know that any basis of the orthogonal complement of $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$ completes them to a basis of $\mathbb{R}^4$ which we then only have to translate back to vectors in P_3. In order to two linearly independent vectors which are orthogonal to $\vec{v}_1$ an $\vec{v}_2$ we have to find linearly independent solutions of the homogenous system of two equations for four unknowns

$$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 &= 0, \\ x_1 + 2x_2 - 6x_3 + 4x_4 &= 0 \end{aligned}$$

or, in matrix notation, $P\vec{x} = 0$ where P is the 2×4 matrix whose rows are p_1 and p_2. Since clearly $\operatorname{Im}(P) \subseteq \mathbb{R}^2$, it follows that $\dim(\operatorname{Im}(P)) \leq 2$ and therefore $\dim(\ker(P)) \geq 4 - 2 = 2$.

The following theorem is sometimes useful, cf. Lemma 7.26. For the definition of the orthogonal complement see Definition 7.22.

Theorem 6.40. *Let* $A \in M(m \times n)$. *Then* $\ker(A) = (R_A)^\perp$.

Proof. Observe that $R_A = C_{A^t} = \operatorname{Im}(A^t)$. So we have to show that $\ker(A) = (\operatorname{Im}(A^t))^\perp$. Recall that $\langle Ax\,, y\rangle = \langle x\,, A^t y\rangle$. Therefore

$$\begin{aligned} x \in \ker(A) &\iff Ax = 0 \iff Ax \perp \mathbb{R}^m \\ &\iff \langle Ax\,, y\rangle = 0 \text{ for all } y \in \mathbb{R}^m \\ &\iff \langle x\,, A^t y\rangle = 0 \text{ for all } y \in \mathbb{R}^m \iff x \in (\operatorname{Im}(A))^t. \end{aligned}$$ □

Alternative proof of Theorem 6.40. Let $\vec r_1, \dots, \vec r_n$ be the rows of A. Since $R_A = \operatorname{span}\{\vec r_1, \dots, \vec r_n\}$, it suffices to show that $\vec x \in \ker(A)$ if and only if $\vec x \perp \vec r_j$ for all $j = 1, \dots, m$.
By definition $\vec x \in \ker(A)$ if and only if

$$\vec 0 = A\vec x = \left(\begin{array}{c} \vec r_1 \\ \hline \vdots \\ \hline \vec r_m \end{array}\right) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \langle \vec r_1\,, \vec x\rangle \\ \vdots \\ \langle \vec r_m\,, \vec x\rangle \end{pmatrix}$$

This is the case if and only if $\langle \vec r_j\,, \vec x\rangle$ for all $j = 1, \dots, m$, that is, if and only if $\vec x \perp \vec r_j$ for all $j = 1, \dots, m$. ($\langle \cdot\,, \cdot\rangle$ denotes the inner product on $\mathbb{R}^n$.) □

You should now have understood

- what the relation between the solutions of a homogeneous system and the kernel of the associated coefficient matrix is,
- what the relation between the admissible right hand sides of a system of linear equations and the range of the associated coefficient matrix is,
- why the dimension formula (6.8) holds and why it is only a special case of (6.4),
- why the Gauß-Jordan process works,
- ...,

You should now be able to

- calculate a basis of the kernel of a matrix and its dimension,
- calculate a basis of the range of a matrix and its dimension,
- ...

6.3 Change of bases

In this section, we work mostly with real vector spaces for definiteness sake. However, all the statements are also true for complex vector spaces. We only have to replace everywhere $\mathbb{R}$ by $\mathbb{C}$ and the word *real* by *complex*.

Usually we represent vectors in $\mathbb{R}^n$ as column of numbers, for example

$$\vec{v} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \qquad \text{or more generally,} \qquad \vec{w} = \begin{pmatrix} x \\ y \\ z. \end{pmatrix}, \tag{6.9}$$

Such columns of numbers are usually interpreted as the Cartesian coordinates of the tip of the vector if its initial point is in the origin. So for example, we can visualise $\vec{v}$ as the vector which we obtain when we move 3 units along the x-axis, 2 units along the y-axis and -1 unit along the z-axis.
If we set $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ the unit vectors which are parallel to the x-, y- and z-axis, respectively, then we can write $\vec{v}$ as a weighted sum of them:

$$\vec{v} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 3\vec{e}_1 + 2\vec{e}_2 - \vec{e}_3. \tag{6.10}$$

So the column of numbers which we use to describe $\vec{v}$ in (6.9) can be seen as a convenient way to abbreviate the sum in (6.10).

Sometimes however, it may make more sense to describe a certain vector not by its Cartesian coordinates. For instance, think of an infinitely large chess field (this is $\mathbb{R}^2$). Then the rock is moving a along the Cartesian axis while the bishop moves a along the diagonals, that is along $\vec{b}_1 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, $\vec{b}_2 = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ and the knight moves in directions parallel to $\vec{k}_1 = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, $\vec{k}_2 = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$. We suppose that in our imaginary chess game the rock, the bishop and the knight may move in arbitrary multiples of their directions. Suppose all three of them are situated in the origin of the field and we want to move them to the field $(3,5)$. For the rock, this is very easy. It only has to move 3 steps to the right and then 5 steps up. He would denote his movement as $\vec{v}_{\mathcal{R}} = \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right)_{\mathcal{R}}$. The bishop cannot do this. He can move only along the diagonals. So what does he have to do? He has to move 4 steps in direction of $\vec{b}_1$ and 1 step indirection $\vec{b}_2$. So he would denote his movement with respect to his bishop coordinate system as $\vec{v}_{\mathcal{B}} = \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right)_{\mathcal{B}}$. Finally the knight has to move $\frac{1}{3}$ steps in direction $\vec{k}_1$ and $\frac{7}{3}$ steps in direction $\vec{k}_2$ to reach the point $(3,5)$. So he would denote his movement with respect to his knight coordinate system as $\vec{v}_{\mathcal{K}} = \left(\begin{smallmatrix} 1/3 \\ 7/3 \end{smallmatrix}\right)_{\mathcal{K}}$. See Figure 6.1.

Exercise. Check that $\vec{v}_{\mathcal{B}} = \left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right)_{\mathcal{B}} = 4\vec{b}_1 + 2\vec{b}_2 = \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right)$ and that $\vec{v}_{\mathcal{K}} = \left(\begin{smallmatrix} 1/3 \\ 7/3 \end{smallmatrix}\right)_{\mathcal{K}} = 1/3\vec{k}_1 + 7/3\vec{k}_2 = \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right)$.

Although the three vectors $\vec{v}$, $\vec{v}_B$ and $\vec{v}_{\mathcal{K}}$ look very differently but they describe the same vector only from three different perspectives (the rock, the bishop and the knight perspective). We have to remember that they have to be interpreted as linear combinations of the vectors that describe their movements.

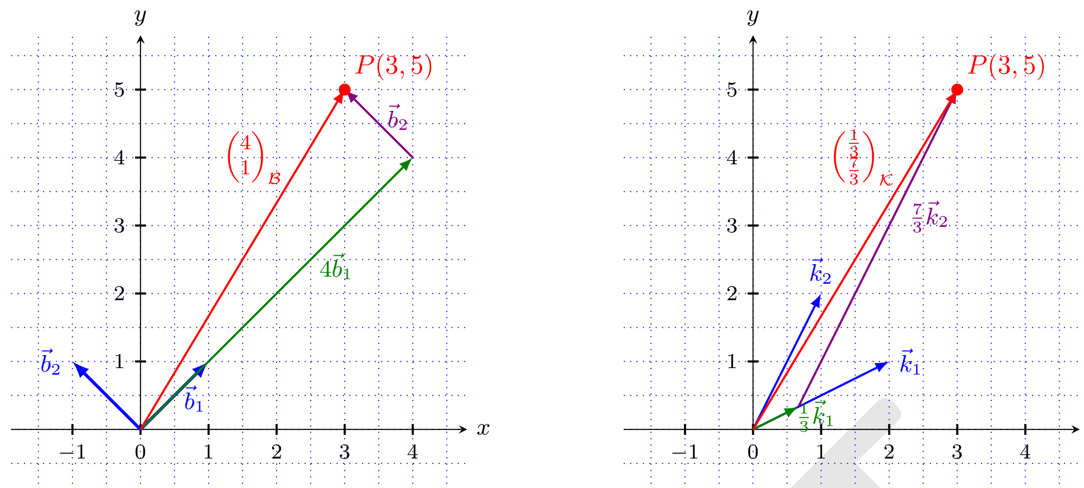


FIGURE 6.1

The pictures shows the point $(3, 5)$ in "bishop" and "knight" coordinates. The vectors for the bishop are $\vec{b}_1 = \binom{1}{1}$, $\vec{b}_2 = \binom{-1}{1}$ and $\vec{x}_B = \binom{3}{1}$. The vectors for the knight are $\vec{k}_1 = \binom{2}{1}$, $\vec{k}_2 = \binom{1}{2}_B$ and $\vec{x}_{\mathcal{K}} = \begin{pmatrix}\frac{1}{3}\\\frac{7}{3}\end{pmatrix}_{\mathcal{K}}$.

What we just did was to perform a change of bases in $\mathbb{R}^2$: Instead of describing a point in the plane in Cartesian coordinates, we used "bishop"- and "knight"-coordinates.

We can also go in the other direction and transform from "bishop"- or "knight"-coordinates to Cartesian coordinates. Assume that we know that the bishop moves 3 steps in his direction $\vec{b}_1$ and -2 steps in his direction $\vec{b}_2$, where does he end up? In his coordinate system, he is displaced by the vector $\vec{u} = \binom{3}{-2}_{\mathcal{B}}$. In Cartesian coordinates this vector is

$$\vec{u} = \begin{pmatrix}3\\-2\end{pmatrix}_{\mathcal{B}} = 3\vec{b}_1 - 2\vec{b}_2 = \begin{pmatrix}3\\3\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}.$$

If we move the knight 3 steps in his direction $\vec{k}_1$ and -2 step in his direction $\vec{k}_2$, that is, we move him along $\vec{w} = \binom{3}{-2}_{\mathcal{K}}$ according to his coordinate system, then in Cartesian coordinates this vector is

$$\vec{w} = \begin{pmatrix}3\\-2\end{pmatrix}_{\mathcal{K}} = 3\vec{b}_1 - 2\vec{b}_2 = \begin{pmatrix}6\\3\end{pmatrix} + \begin{pmatrix}-2\\-4\end{pmatrix} = \begin{pmatrix}4\\-1\end{pmatrix}.$$

Can the bishop and the knight reach every point in the plane? If so, in how many ways? The answer is yes, and they can do so in exactly one way. The reason is that for the bishop and for the knight, their set of direction vectors each form a basis of $\mathbb{R}^2$ (verify this!).

Let us make precise the concept of change of basis. Assume we are given an ordered basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ of $\mathbb{R}^n$. If we write

$$\vec{x} = \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}_{\mathcal{B}} \tag{6.11}$$

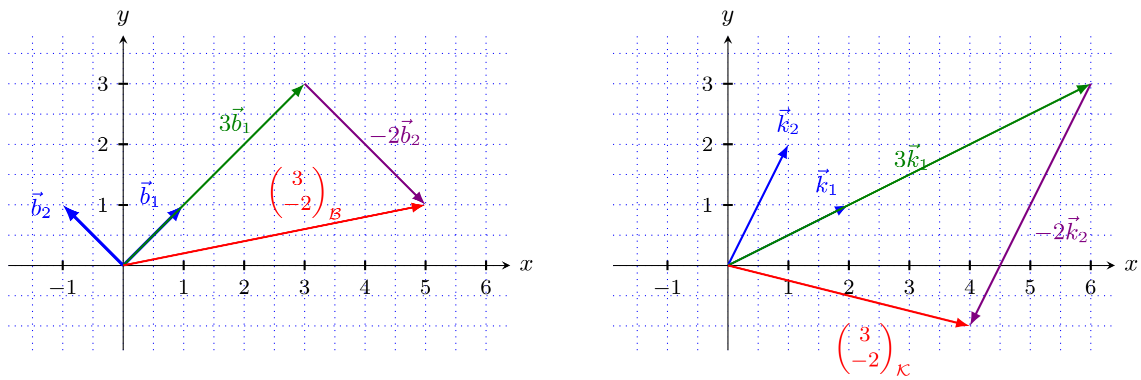


FIGURE 6.2: The pictures shows the vectors $\left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)_{\mathcal{B}}$ and $\left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)_{\mathcal{K}}$.

then we interprete it as a vector which is expressed with respect to the basis $\mathcal{B}$ and

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n. \tag{6.12}$$

If there is no index attached to the column vector, then we interprete it as a vector with respect to the canonical basis $\vec{e}_1, \dots, \vec{e}_n$ of $\mathbb{R}^n$. Now we want to find a way to calculate the Cartesian coordinates (that is, those with respect to the canonical basis) if we are given a vector in $\mathcal{B}$-coordinates and the other way around.
It will turn out that the following matrix will be very useful:

$$A_{\mathcal{B}\to can} = (\vec{v}_1|\dots|\vec{v}_n) = \text{matrix whose columns are the vectors of the basis } \mathcal{B}.$$

We will explain the index "$\mathcal{B} \to can$" in a moment.

Transition from representation with respect to a given basis to Cartesian coordinates.

Suppose we are given a vector as in (6.12). How do we obtain its Cartesian coordinates?

This is quite straightforward. We only need to remember what the notation $(\vdots)_{\mathcal{B}}$ means. We will denote by $\vec{x}_{\mathcal{B}}$ the representation of the vector with respect to the basis $\mathcal{B}$ and by $\vec{x}$ its representation with respect to the standard basis of $\mathbb{R}^n$.

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n = (\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A_{\mathcal{B}\to can}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A_{\mathcal{B}\to can}\vec{x}_{\mathcal{B}},$$

that is

$$\vec{x} = A_{\mathcal{B}\to can}\vec{x}_{\mathcal{B}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{can}. \tag{6.13}$$

The last vector (the one with the $y_1, \dots, y_n$ in it) describes the same vector as $\vec{x}_{\mathcal{B}}$, but it does so with respect to the standard basis of $\mathbb{R}^n$). The matrix $A_{\mathcal{B}\to can}$ is called the *transition matrix from the basis $\mathcal{B}$ to the canonical basis* (which explains the subscript "$\mathcal{B} \to can$"). The matrix is also called the *change-of-coordinates matrix*

Transition from Cartesian coordinates to representation with respect to a given basis.

Suppose we are given a vector $\vec{x}$ in Cartesian coordinates. How do we calculate its coordinates $\vec{x}_{\mathcal{B}}$ with respect to the basis $\mathcal{B}$?

We only need to remember the relation between $\vec{x}$ and $\vec{x}_{\mathcal{B}}$ which according to (6.13) is

$$\vec{x} = A_{\mathcal{B}\to can}\vec{x}_{\mathcal{B}}.$$

In this case, we know the entries of the vector $\vec{x}_{\mathcal{B}}$. So we only need to invert the matrix $A_{\mathcal{B}\to can}$ in order to obtain the entries of $\vec{x}_{\mathcal{B}}$:

$$\vec{x}_{\mathcal{B}} = A_{\mathcal{B}\to can}^{-1}\vec{x}.$$

This requires of course to know that $A_{\mathcal{B}\to can}$ invertible. But this is guaranteed by Theorem 5.36 since we know that its columns are linearly independent. So it follows that the transitions matrix from the canonical basis to the basis $\mathcal{B}$ is given by

$$A_{can\to\mathcal{B}} = A_{\mathcal{B}\to can}^{-1}.$$

Note that we could do this also "by hand": We are given $\vec{x} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{can}$ and we want to find the entries $x_1, \dots, x_n$ of the vector $\vec{x}_{\mathcal{B}}$ which describes the same vector. That is, we need numbers $x_1, \dots, x_n$ such that

$$\vec{x} = x_1\vec{b}_1 + \cdots + \vec{b}_n x_n.$$

If we know the vectors $\vec{b}_1, \dots, \vec{b}_n$, then we can write this as an $n \times n$ system of linear equations and then solve it for $x_1, \dots, x_n$ which of course in reality is the same as applying the inverse of the matrix $A_{\mathcal{B}\to can}$ to the vector $\vec{x} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{can}$.

Now assume that we have two ordered bases $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ of $\mathbb{R}^n$ and we are given a vector $\vec{x}_{\mathcal{B}}$ with respect to the basis $\mathcal{B}$. How can we calculate its representation $\vec{x}_{\mathcal{C}}$ with respect to the basis $\mathcal{C}$? The easiest way is to use the canonical basis of $\mathbb{R}^n$ as an auxiliary basis. So we first calculate the given vector $\vec{x}_{\mathcal{B}}$ with respect to the canonical basis, we call this vector $\vec{x}$. Then we go from $\vec{x}$ to $\vec{x}_{\mathcal{C}}$. According to the formulas above, this is

$$\vec{x}_{\mathcal{C}} = \vec{A}_{can\to\mathcal{C}}\vec{x} = A_{can\to\mathcal{C}}A_{\mathcal{B}\to can}\vec{x}_{\mathcal{B}}$$

Hence the transition matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ is

$$A_{\mathcal{B}\to\mathcal{C}} = A_{can\to\mathcal{C}}A_{\mathcal{B}\to can}.$$

Example 6.41. Let us go back to our example of our imaginary chess board. We have the "bishop basis" $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, $\vec{b}_2 = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ and the "knight basis" $\mathcal{K} = \{\vec{k}_1, \vec{k}_2\}$ $\vec{k}_1 = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, $\vec{k}_2 = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$. Then the transition matrices to the canonical basis are

$$A_{\mathcal{B}\to can} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A_{\mathcal{K}\to can} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

their inverses are

$$A_{can\to\mathcal{B}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad A_{can\to\mathcal{K}} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the transition matrices from $\mathcal{C}$ to $\mathcal{K}$ and from $\mathcal{K}$ to $\mathcal{C}$ are

$$A_{\mathcal{B}\to\mathcal{K}} = \frac{1}{3}\begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix}, \quad A_{\mathcal{K}\to\mathcal{C}} = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}.$$

- Given a vector $\vec{x} = \left(\begin{smallmatrix} 2 \\ 7 \end{smallmatrix}\right)_{\mathcal{B}}$ in bishop coordinates, how does it look like in knight coordinates?

  Solution. $(\vec{x})_{\mathcal{K}} = A_{\mathcal{B}\to\mathcal{K}}\vec{x}_{\mathcal{B}} = \frac{1}{3}\left(\begin{smallmatrix} 3 & -3 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 \\ 7 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -5 \\ 3 \end{smallmatrix}\right)_{\mathcal{K}}$. ◇

- Given a vector $\vec{y} = \left(\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right)_{\mathcal{K}}$ in knight coordinates, how does it look like in bishop coordinates?

  Solution. $(\vec{y})_{\mathcal{B}} = A_{\mathcal{K}\to\mathcal{B}}\vec{y}_{\mathcal{K}} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 3 \\ -1 & 3 \end{smallmatrix}\right)\left(\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 \\ -1 \end{smallmatrix}\right)_{\mathcal{B}}$. ◇

- Given a vector $\vec{z} = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$ in standard coordinates, how does it look like in bishop coordinates?

  Solution. $(\vec{z})_{\mathcal{B}} = A_{can\to\mathcal{B}}\vec{z} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)_{\mathcal{B}}$. ◇

Example 6.42. Recall the example on page 96 where we had a shop that sold different types of packages of food. Package type A contains 1 peach and 3 mangos and package type B contains 2 peaches and 1 mango. We asked two types of questions:

Question 1. If we by a packages of type A and b packages of type B, how many peaches and mangos will we get? We could rephrase this question so that it becomes more similar to Question 2: How many peaches and mangos do we need in order to fill a packages of type A and b packages of type B?

Question 2. How many packages of type A and of type B do we have to buy in order to get p peaches and m mangos?

Recall that we had the relation

$$M\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m \\ p \end{pmatrix}, \quad M^{-1}\begin{pmatrix} m \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{where} \quad M = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad M^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}. \tag{6.14}$$

We can view these problems in two different coordinate systems. We have the "fruit basis" $\mathcal{F} = \{\vec{p}, \vec{m}\}$ and the "package basis" $\mathcal{P} = \{\vec{A}, \vec{B}\}$ where

$$\vec{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \vec{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

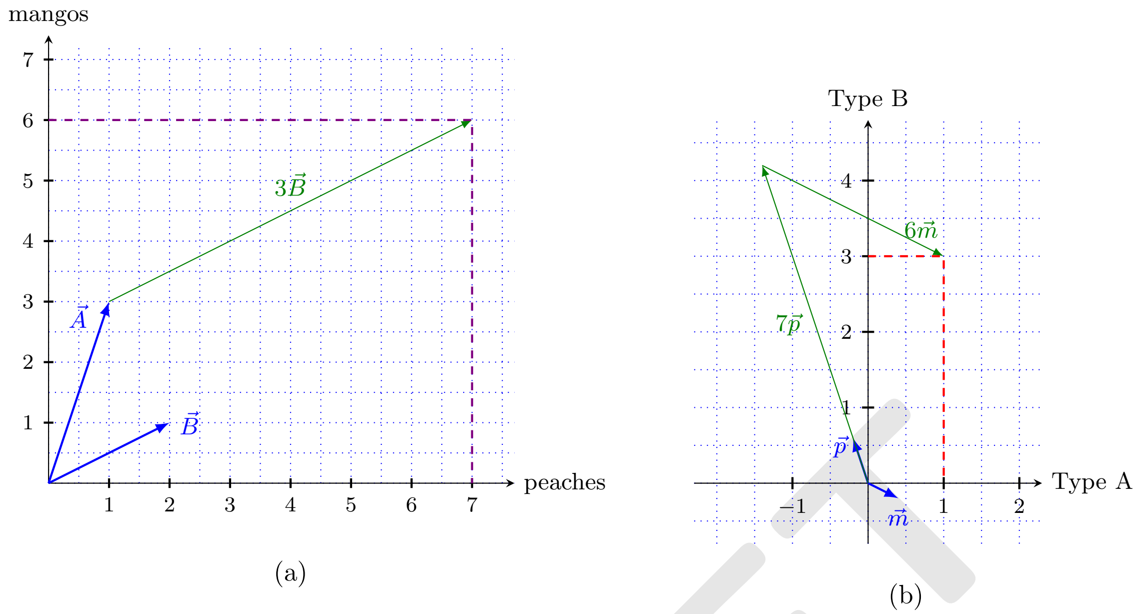


FIGURE 6.3: How many peaches and mangos do we need to obtain 1 package of type A and 3 packages of type B? Answer: 7 peaches and 6 mangos. Figure (a) describes the situation in the "fruit plane" while Figure (b) describes the same situation in the "packages plane". In both figures we see that $\vec{A} + 3\vec{B} = 7\vec{p} + 6\vec{m}$.

Note that $\vec{A} = \vec{m} + 3\vec{p}$, $\vec{B} = 2\vec{m} + \vec{p}$, and that $\vec{m} = \frac{1}{5}(-\vec{A} + 3\vec{B})$ and $\vec{p} = \frac{1}{5}(2\vec{A} - \vec{B})$ (that means for example that one mango is three fifth of a package B minus one fifth of a package A).

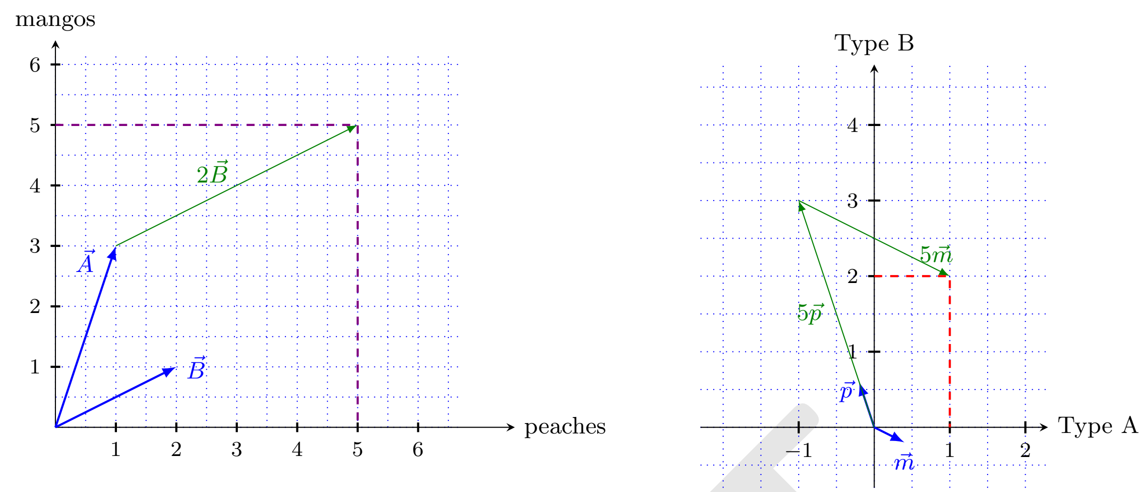
An example for the first question is: How many peaches and mangos do we need to obtain 1 package of type A and 3 packages of type B? Clearly, we need 7 peaches and 6 mangos. So the point that we want to reach is in "package coordinates" $\begin{pmatrix}1\\3\end{pmatrix}_{\mathcal{P}}$ and in "fruit coordinates" $\begin{pmatrix}7\\6\end{pmatrix}_{\mathcal{F}}$. This is sketched in Figure 6.3.

An example for the second question is: How many packages of type A and of type B do we have to buy in order to obtain 5 peaches and 5 mangos? Using (6.14) we find that we need 1 package of type A and 3 packages of type B. So the point that we want to reach is in "package coordinates" $\begin{pmatrix}1\\2\end{pmatrix}_{\mathcal{P}}$ and in "fruit coordinates" $\begin{pmatrix}5\\5\end{pmatrix}_{\mathcal{F}}$. This is sketched in Figure 6.4.

In the rest of this section we will apply these ideas to introduce coordinates in abstract (finitely generated) vector spaces V with respect to a given a basis. This allows us to identify in a certain sense V with $\mathbb{R}^n$ or $\mathbb{C}^n$ for an appropriate n.

Assume we are given a real vector space V with an ordered basis $\mathcal{B} = \{v_1, \dots, v_n\}$. Given a vector $w \in V$, we know that there are uniquely determined real numbers $\alpha_1, \dots, \alpha_n$ such that

$$w = \alpha_1 v_1 + \dots + \alpha_n v_n.$$



(a) (b)

FIGURE 6.4: How many packages of type A and of type B do we need to get 5 peaches and 5 mangos? Answer: 1 package of type A and 2 packages of type B. Figure (a) describes the situation in the "fruit plane" while Figure (b) describes the same situation in the "packages plane". In both figures we see that $\vec{A} + 2\vec{B} = 5\vec{p} + 5\vec{m}$.

So, if we are given w, we can find the numbers $\alpha_1, \dots, \alpha_n$. On the other hand, if we are given the numbers $\alpha_1, \dots, \alpha_n$, we can easily reconstruct the vector w (just replace in the right hand side of the above equation). Therefore it makes sense to write

$$w = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}_{\mathcal{B}}$$

where again the index $\mathcal{B}$ reminds us that the column of numbers has to be understood as the coefficients with respect to the basis $\mathcal{B}$. In this way, we identify V with $\mathbb{R}^n$ since every column vector gives a vector w in V and every vector w gives one column vector in $\mathbb{R}^n$. Note that if we start with some w in V, calculate its coordinates with respect to a given basis and then go back to V, we get back our original vector w.

Example 6.43. In P_2, consider the bases $\mathcal{B} = \{p_1, p_2, p_3\}$, $\mathcal{C} = \{q_1, q_2, q_3\}$, $\mathcal{D} = \{r_1, r_2, r_3\}$ where

$$p_1 = 1,\ p_2 = X,\ p_3 = X^2, \qquad q_1 = X^2,\ q_2 = X,\ q_3 = 1, \qquad r_1 = X^2 + 2X,\ r_2 = 5X + 2,\ r_3 = 1.$$

We want to write the polynomial $\pi(X) = aX^2 + bX + c$ with respect to the given basis.

- Basis $\mathcal{B}$: Clearly, $\pi = cp_1 + bp_2 + ap_3$, therefore $\pi, = \begin{pmatrix} c \\ b \\ a \end{pmatrix}_{\mathcal{B}}$.

- Basis $\mathcal{C}$: Clearly, $\pi = aq_1 + bq_2 + cq_3$, therefore $\pi = \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{\mathcal{C}}$.
- Basis $\mathcal{D}$: This requires some calculations. Recall that we need numbers $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\pi = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{\mathcal{D}} = \alpha r_1 + r_2\beta + r_3\gamma.$$

This leads to the following equation

$$aX^2 + bX + c = \alpha(X^2 + 2X) + \beta(5X + 2) + \gamma = \alpha X^2 + (2\alpha + 5\beta)X + 2\beta + \gamma.$$

Comparing coefficients we obtain

$$\left.\begin{aligned} \alpha \qquad\quad &= a \\ 2\alpha + 5\beta \qquad &= b \\ 2\beta + \gamma &= c. \end{aligned}\right\} \qquad \text{in matrix form:} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{6.15}$$

Note that the columns of the matrix appearing on the right hand side are exactly the vector representations of r_1, r_2, r_3 with respect to the basis $\mathcal{C}$ and the column vector $\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$ is exactly the vector representation of t with respect to the basis $\mathcal{C}$! The solution of the system is

$$\alpha = a, \quad \beta = -\tfrac{1}{5}a + \tfrac{1}{5}b, \quad \gamma = \tfrac{4}{5}a - \tfrac{2}{5}b + c,$$

therefore

$$\pi = \begin{pmatrix} a \\ -\frac{1}{5}a + \frac{1}{5}b \\ \frac{4}{5}a - \frac{2}{5}b + c \end{pmatrix}_{\mathcal{D}}.$$

We could have found the solution also by doing a detour through $\mathbb{R}^3$ as follows: We identify the vectors q_1, q_2, q_3 with the canonical basis vectors $\vec{e}_1$, $\vec{e}_2, \vec{e}_3$ of $\mathbb{R}^3$. Then the vectors r_1, r_2, r_3 and π correspond to

$$\vec{r}_1' = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{r}_2' = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad \vec{r}_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\pi}' = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Let $R = \{\vec{r}_1', \vec{r}_2', \vec{r}_3'\}$. In order to find the coordinates of $\vec{\pi}'$ with respect to the basis $\vec{r}_1'$, $\vec{r}_2'$, $\vec{r}_3'$, we note that

$$\vec{\pi}' = A_{R\to can}\vec{\pi}'_R$$

where $A_{R\to can}$ is the transition matrix from the basis R to the canonical basis of $\mathbb{R}$ whose columns consist of the vectors $\vec{r}_1'$, $\vec{r}_2'$, $\vec{r}_3'$. So we see that this is exactly the same equation as the one in (6.15).

We give an example in a space of matrices.

Example 6.44. Consider the matrices

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}.$$

(i) Show that $\mathcal{B} = \{R, S, T\}$ is a basis of $M_{sym}(2\times 2)$ (the space of all symmetric 2×2 matrices).

(ii) Write Z in terms of the basis $\mathcal{B}$.

Solution. (i) Clearly, $R, S, T \in M_{sym}(2\times 2)$. Since we already now that $\dim M_{sym}(2\times 2) = 3$, it suffices to show that R, S, T are linearly independent. So let us consider the equation

$$0 = \alpha R + \beta S + \gamma T = \begin{pmatrix} \alpha+\beta & \alpha+\gamma \\ \alpha+\gamma & \alpha+3\beta \end{pmatrix}.$$

We obtain the system of equations

$$\left.\begin{aligned} \alpha + \ \beta \qquad &= 0 \\ \alpha + \qquad + \gamma &= 0 \\ \alpha + 3\beta + \qquad &= 0 \end{aligned}\right\} \qquad \text{in matrix form:} \quad \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.16}$$

Doing some calculations, if follows that $\alpha = \beta = \gamma = 0$. Hence we showed that R, S, T are linearly independent and therefore they are a basis of $M_{sym}(2\times 2)$.

(ii) In order to write Z in terms of the basis $\mathcal{B}$, we need to find $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$Z = \alpha R + \beta S + \gamma T = \begin{pmatrix} \alpha+\beta & \alpha+\gamma \\ \alpha+\gamma & \alpha+3\beta \end{pmatrix}.$$

We obtain the system of equations

$$\left.\begin{aligned} \alpha + \ \beta \qquad &= 2 \\ \alpha + \qquad + \gamma &= 3 \\ \alpha + 3\beta + \qquad &= 0 \end{aligned}\right\} \qquad \text{in matrix form:} \quad \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}. \tag{6.17}$$

Therefore

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 0 & -1 \\ -1 & 0 & 1 \\ -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix},$$

therefore $Z = 3R - S = \left(\begin{smallmatrix} 3 \\ -1 \\ 0 \end{smallmatrix}\right)_{\mathcal{B}}$. ◇

Now we give an alternative solution (which is essentially the same as the above) doing a detour through $\mathbb{R}^3$. Let $\mathcal{C} = \{A_1, A_2, A_3\}$ where $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is clearly a basis of $M_{sym}(2\times 2)$. We identify it with the standard basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ of $\mathbb{R}^3$. Then the vectors R, S, T in this basis look like

$$R' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad S' = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \qquad T' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad Z' = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

(i) In order to show that R, S, T are linearly independent, we only have to show that the vectors R', S' and T' are linearly independent in $\mathbb{R}$. To this end, we consider the matrix A whose columns are these vectors. Note that this is the same matrix that appeared in (6.17). It is easy to show that this matrix is invertible (we already calculated its inverse!). Therefore the vectors R', S', T' are linearly independent in $\mathbb{R}^3$, hence R, S, T are linearly independent in $M_{sym}(2 \times 2)$.

(ii) Now in order to find the representation of Z in terms of the basis $\mathcal{B}$, we only need to find the representation of Z' in terms of the basis $\mathcal{B}' = \{R', S', T'\}$. This is done as follows:

$$Z'_{\mathcal{B}'} = A_{can \to \mathcal{B}'} Z' = A^{-1} Z' = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

You should now have understood

- the geometric meaning of a change of bases in R^n,
- how an abstract finite dimensional vector space can be represented as $\mathbb{R}^n$ or $\mathbb{C}^n$ and that the representation depends on the chosen basis of V,
- how the vector representation changes if the chosen basis is reordered,
- ...,

You should now be able to

- perform a change of basis in $\mathbb{R}^n$ and $\mathbb{C}^n$ given a basis,
- represent vectors in a finite dimensional vector space V as column vectors after the choice of a basis,
- ...

6.4 Linear maps and their matrix representations

Let U, V be $\mathbb{K}$-vector spaces and let $T : U \to V$ be a linear map. Recall that T satisfies

$$T(\lambda_1 x_1 + \cdots + \lambda_k x_k) = \lambda_1 T(x_1) + \cdots + \lambda_k T(x_k)$$

for all $x_1, \dots, x_k \in U$ and $\lambda_1, \dots, \lambda_k \in \mathbb{K}$. This shows that in order to know T, it is in reality enough to know how T acts on a basis of U. Suppose that we are given a basis $\mathcal{B} = \{u_1, \dots, u_n\} \in U$ and take an arbitrary vector $w \in U$. Then there exist uniquely determined $\lambda_1, \dots, \lambda_k \in \mathbb{K}$ such that $w = \lambda_1 u_1 + \cdots + \lambda_n u_n$. Hence

$$Tw = T(\lambda_1 u_1 + \cdots + \lambda_n u_n) = \lambda_1 T u_1 + \cdots + \lambda_n T u_n. \tag{6.18}$$

So Tw is a linear combination of the vectors $Tu_1, \dots, Tu_n \in V$, and the coefficients are exactly the $\lambda_1, \dots, \lambda_n$.

Suppose we are given a basis $\mathcal{C} = \{v_1, \ldots, v_k\}$ of V. Then we know that for every $j = 1, \ldots, n$, the vector Tu_j is a linear combination of the basis vectors $v_1, \ldots, v_m$ of V. Therefore there exist uniquely determined numbers $a_{ij} \in K$ $(i = 1, \ldots, m,\ j = 1, \ldots n)$ such that $Tu_j = a_{j1}v_1 + \cdots + a_{jm}v_m$, that is

$$\begin{aligned} Tu_1 &= a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m, \\ Tu_2 &= a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m, \\ &\ \vdots \qquad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ Tu_n &= a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m. \end{aligned} \tag{6.19}$$

Let us define the matrix A_T and the vector $\vec{\lambda}$ by

$$A_T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in M(m \times n), \qquad \vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{R}^n.$$

Recall that A_T represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Now let us come back to the calculation of Tw and its connection with the matrix A_T. From (6.18) and (6.19) we obtain

$$\begin{aligned} Tw &= \lambda_1 Tu_1 + \lambda_2 Tu_2 + \cdots + \lambda_n Tu_n \\ &= \quad \lambda_1(a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m) \\ &\quad + \lambda_2(a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m) \\ &\quad + \qquad\qquad \cdots \\ &\quad + \lambda_n(a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m) \\ &= \quad (a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots + a_{1n}\lambda_n)v_1 \\ &\quad + (a_{21}\lambda_1 + a_{22}\lambda_2 + \cdots + a_{2n}\lambda_n)v_2 \\ &\quad + \cdots \\ &\quad + (a_{m1}\lambda_1 + a_{m2}\lambda_2 + \cdots + a_{mn}\lambda_n)v_m. \end{aligned}$$

The calculation shows that for every k the coefficient of v_k is the kth component of the vector $A_T\vec{\lambda}$! Now we can go one step further. Recall that the choice of the basis $\mathcal{B}$ of U and the basis $\mathcal{C}$ of V lets us write w and Tw as a column vectors:

$$w = \vec{w}_{\mathcal{B}} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_1 \end{pmatrix}_{\mathcal{B}}, \qquad Tw = \begin{pmatrix} a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots + a_{1n}\lambda_n \\ a_{21}\lambda_1 + a_{22}\lambda_2 + \cdots + a_{2n}\lambda_n \\ \vdots \\ a_{m1}\lambda_1 + a_{m2}\lambda_2 + \cdots + a_{mn}\lambda_n \end{pmatrix}_{\mathcal{C}}.$$

This shows that

$$(Tw)_{\mathcal{C}} = A_T \vec{w}_{\mathcal{B}}.$$

For now hopefully obvious reasons, the matrix A_T is called the *matrix representation of T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.*

So every linear transformation $T : U \to V$ can be represented as a matrix $A_T \in M(m \times n)$. On the other hand, every a matrix $A(m \times n)$ induces a linear transformation $T_A : U \to V$.

Very important remark. This identification of $m\times n$-matrices with linear maps $U \to V$ depends on the choice of the basis! See Example 6.47.

Let us summarise what we have found so far.

Theorem 6.45. *Let U, V be finite dimensional vector spaces and let $\mathcal{B} = \{u_1, \dots, u_n\}$ be an ordered basis of U and let $\mathcal{C} = \{v_1, \dots, v_n\}$ be an ordered basis of V. Then the following is true:*

(i) Every linear map $T : U \to V$ can be represented as a matrix $A_T \in M(m \times n)$ such that

$$(Tw)_{\mathcal{C}} = A_T \vec{w}_B$$

where $(Tw)_{\mathcal{C}}$ is the representation of $Tw \in V$ with respect to the basis $\mathcal{C}$ and $\vec{w}_{\mathcal{B}}$ is the representation of $w \in U$ with respect to the basis $\mathcal{B}$. The entries a_{ij} of A_T can be calculated as in (6.19).

(ii) Every matrix $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in M(mt \times n$ induces a linear transformation $T : U \to V$ defined by

$$T(u_j) = a_{1j}v_1 + \dots a_{mj}v_m, \qquad j = 1, \dots, n.$$

(iii) $T = T_{A_T}$ and $A = A_{T_A}$. , That means: If we start with a linear map $T : U \to V$, calculate its matrix representation A_T and then the linear map $T_{A_T} : U \to V$ induced by A_T, then we get back our original map T. If on the other hand we start with a matrix $A \in M(m\times n)$, calculate the linear map $T_A : U \to V$ induced by A and then calculate its matrix representation A_{T_A}, then we get back our original matrix A.

Proof. We already showed (i) and (ii) in the text before the theorem. To see (iii), let us start with a linear transformation $T : U \to V$ and let $A_T = (a_{ij})$ be the matrix representation of T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$. For T_{A_T}, the linear map induced by A_T, it follows that

$$TA_T u_j = a_{1j}v_1 + \dots a_{mj}v_m = Tu_j, \qquad j = 1, \dots, n$$

Since this is true for all basis vectors and both T and T_{A_T} are linear, they must be equal.

If on the other hand we are given a matrix $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in M(mt \times n$ then we have that the linear transformation T_A induced by A acts on the basis vectors $u_1, \dots, u_n$ as follows:

$$T_A u_j = TA_T u_j = a_{1j}v_1 + \dots a_{mj}v_m.$$

But then, by definition of the matrix representation A_{T_A} of T_A, it follows that $A_{T_A} = A$. $\square$

Let us see this "identifications" of matrices with linear transformations a bit more formally. By choosing a basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ in U and thereby identifying U with $\mathbb{R}^n$, we are in reality defining a linear bijection

$$\Psi : U \to \mathbb{R}^n, \qquad \Psi(\lambda u_1 + \cdots + \lambda_n u_n) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

Recall that we denoted the vector on the right hand side by $\vec{w}_{\mathcal{B}}$.
The same happens if we choose a basis $\mathcal{C} = \{v_1, \ldots, v_m\}$ of V. We obtain a linear bijection

$$\Phi : V \to \mathbb{R}^m, \qquad \Phi(\mu v_1 + \cdots + \mu_m v_m) = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}.$$

With these linear maps, we find that

$$A_T = \Phi \circ T \circ \Psi^{-1} \qquad \text{and} \qquad T_A = \Phi^{-1} \circ A \circ \Psi.$$

The maps Ψ and Φ "translate" the spaces U and V to $\mathbb{R}^n$ and $\mathbb{R}^m$ where the chosen bases serve as "dictionary". Thereby they "translate" linear maps $U : U \to V$ to matrices $A \in M(m \times n)$ and vice versa. In a diagram this looks likes this:

$$\begin{array}{ccc} U & \xrightarrow{\quad T \quad} & V \\ {\scriptstyle\Psi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ \mathbb{R}^n & \xrightarrow{\quad A_T \quad} & \mathbb{R}^m \end{array}$$

So in order to go from U to V, we can take the detour through $\mathbb{R}^n$ and $\mathbb{R}^m$. The diagram above is called *commutative diagram.* That means that it does not matter which path we take to go from on corner of the diagram to another one as long as we move in the directions of the arrows. Note that in this case we are even allowed to go in the opposite directions of the arrows representing Ψ and Φ because they are bijections.

What is the use of a matrix representation of a linear map? Sometimes calculations are easier in the world of matrices. For example, we know how to calculate the range and the kernel of a matrix. Therefore:

- If we want to calculate $\operatorname{Im} T$, we only need to calculate $\operatorname{Im} A_T$ and then use Φ to "translate back" to the range of T. In formula: $\operatorname{Im} T = \operatorname{Im}(\Phi A_T) = \Phi(\operatorname{Im} A_T)$.

- If we want to calculate $\ker T$, we only need to calculate $\ker A_T$ and then use Ψ to "translate back" to the kernel of T. In formula: $\ker T = \ker(A_T \Psi) = \Psi^{-1}(\ker A_T)$.

- If $\dim U = \dim V$, i.e., if $n = m$, then T is invertible if and only if A_T is invertible. This is the case if and only if $\det A_T \neq 0$.

Let us summarise. From Theorem 6.24 we obtain again the following very important theorem, see Theorem 6.20 and Proposition 6.16.

Theorem 6.46. *Let U, V be vector spaces and let $T : U \to V$ be a linear transformation. Then*

$$\dim U = \dim(\ker T) + \dim(\operatorname{Im} T). \tag{6.20}$$

If $\dim U = \dim V$, then the following is equivalent:

(i) *T is invertible.*

(ii) *T is injective, that is, $\ker T = \{\mathbb{O}\}$.*

(iii) *T is surjective, that is, $\operatorname{Im} T = V$.*

Note that if T is bijective, then we must have that $\dim U = \dim V$.

Let us see some examples.

Example 6.47. We consider the operator of differentiation

$$T : P_3 \to P_3, \quad Tp = p'.$$

Note that in this case the vector spaces U and V are both equal to P_3.

(i) Represent T with respect to the basis $\mathcal{B} = \{p_1, p_2, p_3, p_4\}$ and find its kernel where $p_1 = 1, p_2 = X, p_3 = X^2, p_4 = X^3$.

Solution. We only need to evaluate T in the elements of the basis and then write the result again as linear combination of the basis. Since in this case, the bases are "easy", the calculations are fairly simple:

$$Tp_1 = 0, \quad Tp_2 = 1 = p_1, \quad Tp_3 = 2X = 2p_2, \quad Tp_4 = 3X^2 = 3p_3.$$

Therefore the matrix representation of T is

$$A_T^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The kernel of A_T is clearly $\operatorname{span}\{\vec{e}_1\}$, hence $\ker T = \operatorname{span}\{p_1\} = \operatorname{span}\{1\}$. $\diamond$

(ii) Represent T with respect to the basis $\mathcal{C} = \{q_1, q_2, q_3, q_4\}$ and find its kernel where $q_1 = X^3, q_2 = X^2, q_3 = X, q_4 = 1$.

Solution. Again we only need to evaluate T in the elements of the basis and then write the result as linear combination of the basis.

$$Tq_1 = 3X^2 = 3q_2, \quad Tq_2 = 2X = 2q_3, \quad Tq_3 = X = q_4, \quad Tq_4 = 0.$$

Therefore the matrix representation of T is

$$A_T^{\mathcal{C}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The kernel of A_T is clearly $\operatorname{span}\{\vec{e}_4\}$, hence $\ker T = \operatorname{span}\{q_4\} = \operatorname{span}\{1\}$. $\diamond$

(iii) Represent T with respect to the basis $\mathcal{B}$ in the domain of T (in the “left” P_3) and the basis $\mathcal{C}$ in the target space (in the “right” P_3).

Solution. We calculate

$$Tp_1 = 0, \quad Tp_2 = 1 = q_4, \quad Tp_3 = 2X = 2q_3, \quad Tp_4 = 3X^2 = 3q_2.$$

Therefore the matrix representation of T is

$$A_T^{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The kernel of A_T is clearly $\operatorname{span}\{\vec{e}_1\}$, hence $\ker T = \operatorname{span}\{p_1\} = \operatorname{span}\{1\}$. $\diamond$

(iv) Represent T with respect to the basis $\mathcal{D} = \{r_1, r_2, r_3, r_4\}$ and find its kernel where

$$r_1 = X^3 + X, \quad r_2 = 2X^2 + X^2 + 2X, \quad r_3 = 3X^3 + X^2 + 4X + 1, \quad r_4 = 4X^3 + X^2 + 4X + 1.$$

Solution 1. Again we only need to evaluate T in the elements of the basis and then write the result again as linear combination of the basis. This time the calculations are a bit more tedious.

$$\begin{aligned} Tr_1 &= 3X^2 + 1 &&= -8r_1 + 2r_2 + r_4, \\ Tr_2 &= 6X^2 + 2X + 2 &&= -14r_1 + 4r_2 + r_3, \\ Tr_3 &= 9X^2 + 2X + 4 &&= -24r_1 + 5r_2 + 2r_3 + 2r_4, \\ Tr_4 &= 12X^2 + 2X + 4 &&= 30r_1 + 8r_2 + 2r_3 + 2r_4. \end{aligned}$$

Therefore the matrix representation of T is

$$A_T^{\mathcal{D}} = \begin{pmatrix} -8 & -14 & -24 & -30 \\ 2 & 4 & 5 & 8 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix}.$$

In order to calculate the kernel of A_T, we apply the Gauß-Jordan process and obtain

$$A_T^{\mathcal{D}} = \begin{pmatrix} -8 & -14 & -24 & -30 \\ 2 & 4 & 5 & 8 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix} \longrightarrow \quad \cdots \longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The kernel of A_T is clearly $\operatorname{span}\{-2\vec{e}_1 - \vec{e}_2 + \vec{e}_4\}$, hence $\ker T = \operatorname{span}\{-2r_1 - r_2 + r_4\} = \operatorname{span}\{1\}$. $\diamond$

Solution 2. We already have the matrix representation $A_T^{\mathcal{C}}$ and we can use it to calculate $A_T^{\mathcal{D}}$. To this end define the vectors

$$\vec{\rho}_1 = \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}, \ \vec{\rho}_2 = \begin{pmatrix} 2\\1\\2\\0 \end{pmatrix}, \ \vec{\rho}_3 = \begin{pmatrix} 3\\1\\4\\1 \end{pmatrix}, \ \vec{\rho}_4 = \begin{pmatrix} 4\\1\\4\\1 \end{pmatrix}.$$

Note that these vectors are the representations of our basis vectors $r_1, \dots, r_4$ in the basis $\mathcal{C}$. The change-of-bases matrix from $\mathcal{C}$ to $\mathcal{D}$ and its inverse are, in coordinates,

$$S_{\mathcal{D}\to\mathcal{C}} = \begin{pmatrix} 1&2&3&4\\ 0&1&1&1\\ 1&2&4&4\\ 0&0&1&1 \end{pmatrix}, \qquad S_{\mathcal{C}\to\mathcal{D}} = S_{\mathcal{D}\to\mathcal{C}}^{-1} = \begin{pmatrix} 0&-2&1&-2\\ 0&1&0&-1\\ -1&0&1&0\\ 1&0&-1&1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} A_T^{\mathcal{D}} &= S_{\mathcal{C}\to\mathcal{D}} A_T^{\mathcal{C}} S_{\mathcal{D}\to\mathcal{C}} \\ &= \begin{pmatrix} 0&-2&1&-2\\ 0&1&0&-1\\ -1&0&1&0\\ 1&0&-1&1 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\ 3&0&0&0\\ 0&2&0&0\\ 0&0&1&0 \end{pmatrix} \begin{pmatrix} 1&2&3&4\\ 0&1&1&1\\ 1&2&4&4\\ 0&0&1&1 \end{pmatrix} = \begin{pmatrix} -8&-14&-24&-30\\ 2&4&5&8\\ 0&2&2&2\\ 1&0&2&2 \end{pmatrix}. \end{aligned}$$

Let us see how this looks in diagrams. We define the two bijections of P_3 with $\mathbb{R}^4$ which are given by choosing the bases $\mathcal{C}$ and $\mathcal{D}$ by $\Psi_{\mathcal{C}}$ and $\Psi_{\mathcal{D}}$

$$\begin{aligned} &\Psi_{\mathcal{C}} : P_3 \to \mathbb{R}^4, && \Psi_{\mathcal{C}}(q_1) = \vec{e}_1,\ \Psi_{\mathcal{C}}(q_2) = \vec{e}_2,\ \Psi_{\mathcal{C}}(q_3) = \vec{e}_3,\ \Psi_{\mathcal{C}}(q_4) = \vec{e}_4, \\ &\Psi_{\mathcal{D}} : P_3 \to \mathbb{R}^4, && \Psi_{\mathcal{D}}(r_1) = \vec{e}_1,\ \Psi_{\mathcal{D}}(r_2) = \vec{e}_2,\ \Psi_{\mathcal{D}}(r_3) = \vec{e}_3,\ \Psi_{\mathcal{D}}(r_4) = \vec{e}_4. \end{aligned}$$

Then we have the following diagrams:

$$\begin{tikzcd} P_3 \arrow[r, "T"] \arrow[d, "\Psi_{\mathcal{C}}"'] & P_3 \arrow[d, "\Psi_{\mathcal{C}}"] \\ \mathbb{R}^4 \arrow[r, "A_T^{\mathcal{C}}"] & \mathbb{R}^4 \end{tikzcd} \qquad \begin{tikzcd} P_3 \arrow[r, "T"] \arrow[d, "\Psi_{\mathcal{D}}"'] & P_3 \arrow[d, "\Psi_{\mathcal{D}}"] \\ \mathbb{R}^4 \arrow[r, dashed, "A_T^{\mathcal{D}}"] & \mathbb{R}^4 \end{tikzcd}$$

We already know everything in the diagram on the left and we want to calculate $A_T^{\mathcal{D}}$ in the diagram on the right. We can put the diagrams together as follows:

$$\begin{tikzcd} P_3 \arrow[rrr, "T"] \arrow[d, "\Psi_{\mathcal{D}}"'] \arrow[dr, "\Psi_{\mathcal{C}}"] & & & P_3 \arrow[d, "\Psi_{\mathcal{D}}"] \arrow[dl, "\Psi_{\mathcal{C}}"'] \\ \mathbb{R}^4 \arrow[r, "S_{\mathcal{D}\to\mathcal{C}}"] \arrow[rrr, dashed, bend right, "A_T^{\mathcal{D}}"'] & \mathbb{R}^4 \arrow[r, "A_T^{\mathcal{C}}"] & \mathbb{R}^4 \arrow[r, "S_{\mathcal{C}\to\mathcal{D}}"] & \mathbb{R}^4 \end{tikzcd}$$

We can also see that the change-of-basis maps $S_{\mathcal{D}\to\mathcal{C}}$ and $S_{\mathcal{C}\to\mathcal{D}}$ are

$$S_{\mathcal{D}\to\mathcal{C}} = \Psi_{\mathcal{C}} \circ \Psi_{\mathcal{D}}^{-1}, \qquad S_{\mathcal{C}\to\mathcal{D}} = \Psi_{\mathcal{D}} \circ \Psi_{\mathcal{C}}^{-1}.$$

For $A_T^{\mathcal{D}}$ we obtain

$$A_T^{\mathcal{D}} = \Psi_{\mathcal{D}} \circ T \circ \Psi_{\mathcal{D}}^{-1} = S_{\mathcal{D}\to\mathcal{C}} \circ A_T^{\mathcal{C}} \circ S_{\mathcal{C}\to\mathcal{D}}.$$

Another way to draw the diagram above is

$$\begin{tikzcd}
P_3 \arrow[rrr, "T"] \arrow[dr, "\Psi_{\mathcal{C}}"] \arrow[dd, "\Psi_{\mathcal{D}}"'] & & & P_3 \arrow[dl, "\Psi_{\mathcal{C}}"'] \arrow[dd, "\Psi_{\mathcal{D}}"] \\
& \mathbb{R}^4 \arrow[r, "A_T^{\mathcal{C}}"] & \mathbb{R}^4 \arrow[dr, "S_{\mathcal{C}\to\mathcal{D}}"] & \\
\mathbb{R}^4 \arrow[ur, "S_{\mathcal{D}\to\mathcal{C}}"] \arrow[rrr, dashed, "A_T^{\mathcal{D}}"] & & & \mathbb{R}^4
\end{tikzcd}$$

$\diamond$

Note that the matrices $A_T^{\mathcal{B}}$, $A_T^{\mathcal{C}}$, $A_T^{\mathcal{D}}$ and $A_T^{\mathcal{B},\mathcal{C}}$ all look different but they describe the same linear transformation. The reason why they look different is that in each case we used different bases to describe them.

Example 6.48. The next example is not very applied but it serves to practice a bit more. We consider the operator given

$$T : M(2\times 2) \to P_2, \quad T\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = (a+c)X^2 + (a-b)X + a - b + d.$$

Show that T is a linear transformation and represent T with respect to the bases $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ of $M(2\times 2)$ and $\mathcal{C} = \{p_1, p_2, p_3\}$ of P_2 where

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$p_1 = 1, \quad p_2 = X, \quad p_3 = X^2.$$

Find based for $\ker T$ and $\operatorname{Im} T$ and their dimensions.

Solution. First we verify that T is indeed a linear map. To this end, we take matrices $A_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}
T(\lambda A_1 + A_2) &= T\left(\lambda \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = T\left(\lambda \begin{pmatrix} \lambda a_1 + a_2 & \lambda b_1 + b_2 \\ \lambda c_1 + c_2 & \lambda d_1 + d_2 \end{pmatrix}\right) \\
&= (\lambda a_1 + a_2 + \lambda c_1 + c_2)X^2 + (\lambda a_1 + a_2 - \lambda b_1 - b_2)X + \lambda a_1 + a_2 - (\lambda b_1 + b_2) + \lambda d_1 + d_2 \\
&= \lambda(a_1 + c_1)X^2 + (a_1 - b_1)X + a_1 - b_1 + d_1) \\
&\quad + \left[(a_2 + c_2)X^2 + (a_2 - b_2)X + a_2 - b_2 + d_2)\right] \\
&= \lambda T(A_1) + T(A_2).
\end{aligned}$$

This shows that T is a linear transformation.
Now we calculate its matrix representation with respect to the given bases.

$$\begin{aligned} TB_1 &= X^2 + X + 1 = p_1 + p_2 + p_3, \\ TB_2 &= -X = -p_2, \\ TB_3 &= X^2 = p_3, \\ TB_4 &= 1 = p_1. \end{aligned}$$

Therefore the matrix representation of T is

$$A_T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

In order to determine the kernel and range of A_T, we apply the Gauß-Jordan process:

$$A_T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

So the range of A_T is $\mathbb{R}^3$ and its kernel is $\ker A_T = \operatorname{span}\{\vec{e}_1 + \vec{e}_2 - \vec{e}_3 - \vec{e}_3\}$. Therefore $\operatorname{Im} T = P_2$ and $\ker T = \operatorname{span}\{B_1 + B_2 - B_3 - B_4\}$. For their dimensions we find $\dim(\operatorname{Im} T) = 3$ and $\dim(\ker T) = 1$. $\diamond$

Example 6.49 (Reflection in $\mathbb{R}^2$). In $\mathbb{R}^2$, consider the line $L : 3x - 2y = 0$. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ which takes a vector in $\mathbb{R}^2$ and reflects it on the line L, see Figure 6.5. Find the matrix representation of R with respect to the standard basis of $\mathbb{R}^2$.
Observation. Note that L is the line which passes through the origin and is parallel to the vector $\vec{v} = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$.

Solution 1 (use coordinates adapted to the problem). Clearly, there are two directions which are special in this problem: the direction parallel and the direction orthogonal to the line. So a basis which is adapted to the exercise, is $\mathcal{B} = \{\vec{v}, \vec{w}\}$ where $\vec{v} = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ and $\vec{w} = \left(\begin{smallmatrix} -3 \\ 2 \end{smallmatrix}\right)$. Clearly, $R\vec{v} = \vec{v}$ and $R\vec{w} = -\vec{w}$. Therefore the matrix representation of R with respect to the basis $\mathcal{B}$ is

$$A_R^{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to obtain the representation A_R with respect to the standard basis, we only need to perform a change of basis. Recall that change-of-bases matrices are given by

$$S_{\mathcal{B}\to can} = (\vec{v}|\vec{w}) = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \qquad S_{can\to\mathcal{B}} = S_{\mathcal{B}\to can}^{-1} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}.$$

Therefore

$$A_R = S_{\mathcal{B}\to can} A_R^{\mathcal{B}} S_{can\to\mathcal{B}} = \frac{1}{13}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}.$$ $\diamond$

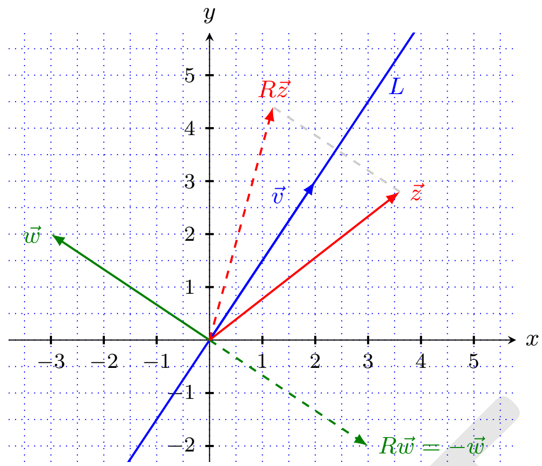


FIGURE 6.5: The pictures shows the reflection R on the line L. The vector $\vec{v}$ is parallel to L, hence $R\vec{v} = \vec{v}$. The vector $\vec{w}$ is perpendicular to L, hence $R\vec{w} = -\vec{w}$.

Solution 2 (reduce the problem to a known reflection). The problem would be easy if we were asked to calculate the matrix representation of the reflection on the x-axis. This would simply be $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Now we can proceed as follows: First we rotate $\mathbb{R}^2$ about the origin such that the line L is parallel to the x-axis, then we reflect on the x-axis and then we rotate back. The result is the same as reflecting on L. Assume that Rot is the rotation matrix. Then

$$A_T = \mathrm{Rot}^{-1} \circ A_0 \circ \mathrm{Rot}. \tag{6.21}$$

How can we calculate Rot? We know that $\mathrm{Rot}\vec{v} = \vec{e}_1$ and that $\mathrm{Rot}\vec{w} = \vec{e}_2$. It follows that $\mathrm{Rot}^{-1} = (\vec{v}|\vec{w}) = \left(\begin{smallmatrix} 2 & 3 \\ -3 & 2 \end{smallmatrix}\right)$. Note that up to a numerical factor, this is $S_{\mathcal{B}\to can}$. We can calculate easily that $\mathrm{Rot} = (\mathrm{Rot}^{-1})^{-1} = \frac{1}{13}\left(\begin{smallmatrix} 2 & -3 \\ 3 & 2 \end{smallmatrix}\right)$. If we insert this in (6.21), we find again $A_R = \left(\begin{smallmatrix} -5 & 12 \\ 12 & 5 \end{smallmatrix}\right)$. $\diamond$

Solution 3 (straight forward calculation). We can form a system of linear equations in order to find A_T. We write $A_R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with unknown numbers a, b, c, d. Again, we use that we know that $A_T\vec{v} = \vec{v}$ and $A_T\vec{w} = -\vec{w}$. This gives the following equations:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \vec{v} = A_T\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2a+3b \\ 2c+3d \end{pmatrix},$$
$$\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \vec{w} = -A_T\vec{w} = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3a-2b \\ 3c-2d \end{pmatrix}$$

which gives the system

$$2a+3b = 2, \quad 2c+3d = 3, \quad 3a-2b = -3, \quad 3c-2d = 2,$$

Its unique solution is $a = -\frac{5}{13}$, $b = c = \frac{12}{13}$, $d = \frac{5}{13}$, hence $A_R = \left(\begin{smallmatrix} -5 & 12 \\ 12 & 5 \end{smallmatrix}\right)$. $\diamond$

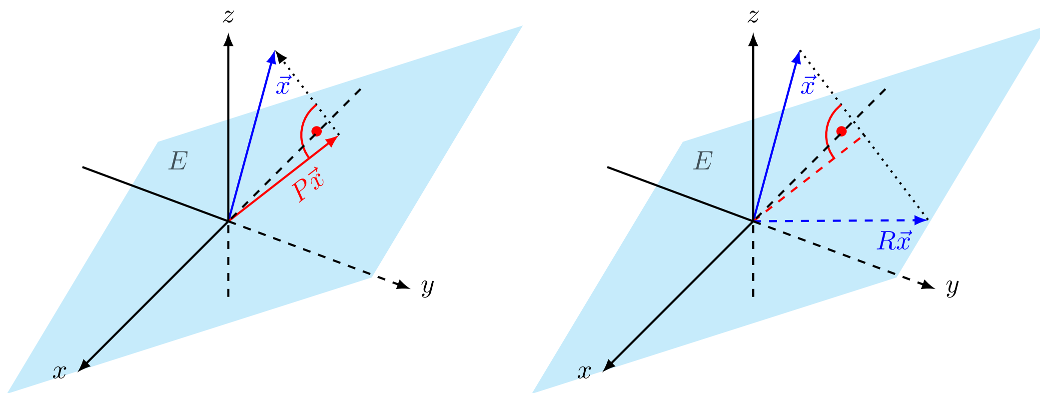


FIGURE 6.6: The figure shows the plane $E: x-2y+3z=0$ and for the vector $\vec{x}$ it shows its orthogonal projection $P\vec{x}$ onto E and its reflection $R\vec{x}$ about E, see Example 6.50.

Example 6.50 (Reflection and orthogonal projection in $\mathbb{R}^3$). In $\mathbb{R}^3$, consider the plane $E: x-2y+3z=0$. Let $R: \mathbb{R}^3 \to \mathbb{R}^3$ which takes a vector in $\mathbb{R}^3$ and reflects it on the plane E and let $P: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto E. Find the matrix representation of R with respect to the standard basis of $\mathbb{R}^E$.

Observation. Note that E is the plane which passes through the origin and is orthogonal to the vector $\vec{n} = \left(\begin{smallmatrix}1\\-2\\3\end{smallmatrix}\right)$. Moreover, if we set $\vec{v} = \left(\begin{smallmatrix}2\\1\\0\end{smallmatrix}\right)$ and $\vec{w} = \left(\begin{smallmatrix}0\\3\\2\end{smallmatrix}\right)$, then it is easy to see that $\{\vec{v}, \vec{w}\}$ is a basis of E.

Solution 1 (use coordinates adapted to the problem). Clearly, a basis which is adapted to the exercise is $\mathcal{B} = \{\vec{n},\ \vec{v},\ \vec{w}\}$ because for these vectors we have $R\vec{v} = \vec{v}$, $R\vec{w} = \vec{w}$, $R\vec{n} = -\vec{n}$, and $P\vec{v} = \vec{v}$, $P\vec{w} = \vec{w}$, $P\vec{n} = \vec{0}$. Therefore the matrix representation of R with respect to the basis $\mathcal{B}$ is

$$A_R^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the one of P is

$$A_R^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In order to obtain the representations A_R and A_P with respect to the standard basis, we only need to perform a change of basis. Recall that change-of-bases matrices are given by

$$S_{\mathcal{B}\to can} = (\vec{v}|\vec{w}|\vec{n}) = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -2 \\ 0 & 2 & 3 \end{pmatrix}, \qquad S_{can\to\mathcal{B}} = S_{\mathcal{B}\to can}^{-1} = \frac{1}{28}\begin{pmatrix} 13 & 2 & -3 \\ -3 & 6 & 5 \\ 2 & -4 & 6 \end{pmatrix}.$$

Therefore

$$A_R = S_{\mathcal{B}\to can} A_R^{\mathcal{B}} S_{can\to\mathcal{B}} = \frac{1}{28}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -2 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 13 & 2 & -3 \\ -3 & 6 & 5 \\ 2 & -4 & 6 \end{pmatrix}$$
$$= \frac{1}{7}\begin{pmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ -3 & 6 & -2 \end{pmatrix}$$

and

$$A_P = S_{\mathcal{B}\to can} A_P^{\mathcal{B}} S_{can\to\mathcal{B}} = \frac{1}{28}\begin{pmatrix} 2 & 0 & 2 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 13 & 2 & -3 \\ -3 & 6 & 5 \\ 2 & -4 & 6 \end{pmatrix}$$
$$= \frac{1}{14}\begin{pmatrix} 13 & 2 & -3 \\ 2 & 10 & 6 \\ -3 & 6 & 5 \end{pmatrix}$$

◇

Solution 2 (reduce the problem to a known reflection). The problem would be easy if we were asked to calculate the matrix representation of the reflection on the xy-plane. This would simply be $A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Now we can proceed as follows: First we rotate $\mathbb{R}^3$ about the origin such that the plane E is parallel to the xy-axis, then we reflect on the xy-plane and then we rotate back. The result is the same as reflecting on the plane E. We leave the details to the reader. An analogous procedure works for the orthogonal projection. ◇

Solution 3 (straight forward calculation). Lastly, we can form a system of linear equations in order to find A_R. We write $A_R = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ with unknowns a_{ij}. Again, we use that we know that $A_R\vec{v} = \vec{v}$, $A_R\vec{w} = \vec{w}$ and $A_R\vec{n} = -\vec{n}$. This gives a system of 9 linear equations for the nine unknowns a_{ij} which can be solved. ◇

Remark 6.51. Yet another solution is the following. Let Q be the orthogonal projection onto $\vec{n}$. We already know how to calculate its representing matrix:

$$Q\vec{x} = \frac{\langle \vec{x}\,, \vec{n}\rangle}{\|\vec{n}\|^2}\vec{n} = \frac{x-2y+3z}{14}\vec{n} = \frac{1}{14}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Hence $A_Q = \frac{1}{14}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix}$. Geometrically, it is clear that $P = \mathrm{id} - Q$ and $R = \mathrm{id} - 2Q$. Hence it follows that

$$A_P = \mathrm{id} - A_Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 13 & 2 & -3 \\ 2 & 10 & 6 \\ -3 & 6 & 5 \end{pmatrix}$$

and

$$A_R = \mathrm{id} - 2A_Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{7}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ -3 & 6 & -2 \end{pmatrix}.$$

Change of bases as matrix representation of the identity

Finally let us observe that a change-of-bases matrix is nothing else than the identity matrix written with respect to different bases. To see this let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\mathcal{C} = \{\vec{w}_1, \dots, \vec{v}_w\}$ be bases of $\mathbb{R}^n$. We define the the linear bijections $\Psi_{\mathcal{B}}$ and $\Psi_{\mathcal{C}}$ as follows:

$$\begin{aligned} \Psi_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^n, &\qquad \Psi_{\mathcal{B}}(\vec{e}_1) = \vec{v}_1, \ \dots, \ \Psi_{\mathcal{B}}(\vec{e}_n) = \vec{v}_n, \\ \Psi_{\mathcal{C}} : \mathbb{R}^n \to \mathbb{R}^n, &\qquad \Psi_{\mathcal{C}}(\vec{e}_1) = \vec{w}_1, \ \dots, \ \Psi_{\mathcal{C}}(\vec{e}_n) = \vec{w}_n, \end{aligned}$$

Moreover we define the change-of-bases matrices

$$S_{\mathcal{B}\to can} = (\vec{v}_1|\cdots|\vec{v}_n), \qquad S_{\mathcal{C}\to can} = (\vec{w}_1|\cdots|\vec{w}_n).$$

Note that these matrices are exactly the matrix representations of $\Psi_{\mathcal{B}}$ and $\Psi_{\mathcal{C}}$. Now let us consider the diagram

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\mathrm{id}"] \arrow[d, "\Psi_{\mathcal{B}}^{-1}"'] & \mathbb{R}^n \arrow[d, "\Psi_{\mathcal{C}}^{-1}"] \\ \mathbb{R}^n \arrow[r, "A_{\mathrm{id}}"] & \mathbb{R}^n \end{tikzcd}$$

Therefore

$$A_{\mathrm{id}} = \Psi_{\mathcal{C}}^{-1} \circ \mathrm{id} \circ \Psi_{\mathcal{B}} = \Psi_{\mathcal{C}} \circ \Psi_{\mathcal{B}}^{-1} = S_{\mathcal{C}\to can}^{-1} \circ S_{\mathcal{B}\to can} = S_{can\to\mathcal{C}} \circ S_{\mathcal{B}\to can} = S_{\mathcal{B}\to\mathcal{C}\to can}.$$

You should now have understood

- why every linear map between finite dimensional vector spaces can be written as a matrix and why the matrix depends on the chosen bases,
- how the matrix representation changes if the chosen bases changes
- in particular, how the matrix representation changes if the chosen bases are reordered,
- …,

You should now be able to

- represent a linear map between finite dimensional vector spaces as a matrix,
- use the matrix representation of a linear map to calculate its kernel and range,
- interpret a matrix as a linear map between finite dimensional vector spaces,
- …

6.5 Summary

Linear maps

A function $T : U \to V$ between two $\mathbb{K}$-vector spaces U and V is called *linear map* (or *linear function* or *linear transformation*) if it satisfies

$$T(u_1 + \lambda u_2) = T(u_1) + \lambda T(u_2) \qquad \text{for all} \ \ u_1, u_2 \in U \ \ \text{and} \ \ \lambda \in \mathbb{K}.$$

The set of all linear maps from U to V is denoted by $\mathcal{L}(U, V)$.

- The composition of linear maps is a linear map.
- If a linear map is invertible, then its inverse is a linear map.
- If U, V are $\mathbb{K}$-vector spaces then $\mathcal{L}(U, V)$ is a $\mathbb{K}$-vector space. This means: If $S, T \in \mathcal{L}(U, V)$ and $\lambda \in \mathbb{K}$, then $S + \lambda T \in \mathcal{L}(U, V)$.

For a linear map $T : U \to V$ we define the following sets

$$3 \ker T = \{u \in U : Tu = \mathbb{O}\} \subseteq U,$$
$$\operatorname{Im} T = \{Tu : u \in U\} \subseteq V.$$

$\ker T$ is called *kernel of* T or *null space of* T. It is a subspace of U. $\operatorname{Im} T$ is called *image of* T or *range of* T. It is a subspace of V.
The linear map T is called *injective* if $Tu_1 = Tu_2$ implies $u_1 = u + 2$ for all $u_1, u_2 \in U$. The linear map T is called *surjective* if for every $v \in V$ exist some $u \in U$ such that $Tu = v$. The linear map T is called *bijective* if it is injective and surjective.

Let $T : U \to V$ be a linear map.

- The following are equivalent:
  - (i) T is injective.
  - (ii) $Tu = \mathbb{O}$ implies that $u = \mathbb{O}$.
  - (iii) $\ker T = \{\mathbb{O}\}$.
- The following are equivalent:
  - (i) T is surjective.
  - (ii) $\operatorname{Im} T = V$.
- If T is bijective, then necessarily $\dim U = \dim V$. In other words: if $\dim U \neq \dim v$, then there exists no bijection between them.

Let U, V be $\mathbb{K}$-vector spaces and $T : U \to V$ a linear map. Moreover, let $E : U \to U$, $F : V \to V$ be linear bijective maps. Then

$$\boxed{\begin{aligned} \ker(FT) &= \ker(T), & \ker(TE) &= E^{-1}(\ker(T)), \\ \operatorname{Im}(FT) &= F(\operatorname{Im}(T)), & \operatorname{Im}(TE) &= \operatorname{Im}(T). \end{aligned}}$$

and

$$\boxed{\begin{aligned} \dim\ker(T) &= \dim\ker(FT) = \dim\ker(TE) = \dim\ker(FTE), \\ \dim\operatorname{Im}(T) &= \dim\operatorname{Im}(FT) = \dim\operatorname{Im}(TE) = \dim\operatorname{Im}(FTE). \end{aligned}}$$

If $\dim U = n < \infty$ then

$$\dim(\ker(T)) + \dim(\operatorname{Im}(T)) = n.$$

Linear maps and matrices

Every matrix $A \in M_{\mathbb{K}}(m \times n)$ represents a linear map from $\mathbb{K}^n$ to $\mathbb{K}^m$ by

$$T_A : \mathbb{K}^n \to \mathbb{K}^m, \quad \vec{x} \mapsto A\vec{x}.$$

Very often we write A instead of T_A.
On the other hand, every linear map $T : U \to V$ between finite dimensional vector spaces U and V has a matrix representation. Let $\mathcal{B} = \{u_1, \dots, u_n\}$ be a basis of U and $\mathcal{C} = \{v_1, \dots, v_m\}$ be a basis of V. Assume that $Tu_j = a_{1j}v_1 + \dots + a_{mj}v_m$. Then the matrix representation of T with respect to the basis $\mathcal{B}$ and $\mathcal{C}$ is $A_T = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in M(m \times n)$. Note that the matrix representation of T depends on the chosen bases in U and V.
If we define the functions Ψ and Φ as

$$\Psi : U \to \mathbb{K}^n, \quad \Psi(\alpha_1 u_1 + \dots \alpha_n u_n) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \qquad \Phi : V \to \mathbb{K}^m, \quad \Phi(\beta_1 v_1 + \dots \beta_m v_m) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix},$$

then these functions are linear and $\Phi \circ A_T \circ \Psi = T$ and $\Psi^{-1} \circ T \circ \Phi^{-1} = A_T$. In a diagram this is

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[d, "\Psi"'] & V \arrow[d, "\Phi"] \\ \mathbb{R}^n \arrow[r, "A_T"] & \mathbb{R}^m \end{tikzcd}$$

Matrices

Let $A \in M(m \times n)$.

- The *column space* C_A of A is the linear span of its column vectors. It is equal to $\operatorname{Im} A$.
- The *row space* R_A of A is the linear span of its row vectors. It is equal to the orthogonal complement of $\ker A$.
- $\dim R_A = \dim C_A = \dim(\operatorname{Im} A) =$ number of columns with pivots in echelon form of A.

Kernel and image of A:

- $\dim(\ker A) =$ number of free variables $=$ number of columns without pivots in any row echelon form of A.

  $\ker A$ is equal to the solution set of $A\vec{x} = \vec{0}$ which can be determined for instance with the Gauß or Gauß-Jordan elimination.
- $\dim(\operatorname{Im} A) = \dim C_A =$ number of columns with pivots in any row echelon form of A.

  $\operatorname{Im}(A)$ be be found by either of the following two methods:

  (i) **columns** reduction of A. The remaining columns are a basis of $\operatorname{Im} A$.

  (ii) **row** reduction of A. The columns of the original matrix A which correspond to the columns of the row reduced echelon form of A are a basis of $\operatorname{Im} A$.

6.6 Exercises

1. Determine si las siguientes funciones son lineales. Si lo son, calcule el kernel y la dimensión del kernel.

   (a) $A : \mathbb{R}^3 \to M(2x2),\ A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y & x-z \\ x+y-3z & z \end{pmatrix}$,

   (b) $B : \mathbb{R}^3 \to M(2x2),\ A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2xy & x-z \\ x+y-3z & z \end{pmatrix}$,

   (c) $C : M(2\times 2) \to M(2\times 2), \quad C(M) = M + M^t$

   (d) $D : P_3 \to P_4, \quad Dp = p' + xp$,

   (e) $T : P_3 \to M(2\times 3), \quad T(ax^3 + bx^2 + cx + d) = \begin{pmatrix} a+b & b+c & c+d \\ 0 & a+d & 0 \end{pmatrix}$,

   (f) $T : P_3 \to M(2\times 3), \quad T(ax^3 + bx^2 + cx + d) = \begin{pmatrix} a+b & b+c & c+d \\ 0 & a+d & 3 \end{pmatrix}$.

2. Sean U, V espacios vectoriales sobre $\mathbb{K}$ (con $\mathbb{K} = \mathbb{R}$ o $\mathbb{K} = \mathbb{C}$) y sea $T : U \to V$ una función lineal invertible. Entonces podemos considerar su función inversa $T^{-1} : \mathrm{Im}(T) \to U$. Demuestre que es una función lineal.

3. Sean U, V, W espacios vectoriales sobre $\mathbb{K}$ (con $\mathbb{K} = \mathbb{R}$ o $\mathbb{K} = \mathbb{C}$) y sean $T : U \to V$, $S : V \to W$ funciones lineales. Demuestre que la composición $ST : U \to W$ también es una función lineal.

4. Sean U, V espacios vectoriales sobre $\mathbb{K}$ (con $\mathbb{K} = \mathbb{R}$ o $\mathbb{K} = \mathbb{C}$). Con $\mathcal{L}(U, V)$ denotamos el conjunto de todas las transformaciones lineales de U a V. Demuestre que $\mathcal{L}(U, V)$ es un espacio vectorial sobre $\mathbb{K}$. ¿Qué se puede decir sobre $\dim \mathcal{L}(U, V)$?

5. Sean U, V espacios vectoriales sobre $\mathbb{K}$ (con $\mathbb{K} = \mathbb{R}$ o $\mathbb{K} = \mathbb{C}$). Sabemos de Ejercicio 4 que $\mathcal{L}(U, V)$ es un espacio vectorial. Fije un vector $v_0 \in V$. Demuestre que la siguiente función es una función lineal:

$$\Phi_{v_0} : \mathcal{L}(U, V) \to U, \quad \Phi_{v_0}(T) >= T(v_0).$$

6. Sean

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Demuestre que E y F son invertibles. Describa como actuan geométricamente en $\mathbb{R}^2$.

(b) Calcule $\operatorname{Im}(A)$, $\ker(A)$ y sus dimensiones. Dibuja $\operatorname{Im}(A)$ y $\ker(A)$, diga qué objetos geométricas son.

(c) Calcule $\operatorname{Im}(A)$, $\operatorname{Im}(FA)$, $\operatorname{Im}(AE)$ y sus dimensiones. Dibújalos y diga cual es la relación entre ellos.

(d) Calcule $\ker(A)$, $\ker(FA)$, $\ker(AE)$ y sus dimensiones. Dibújalos y diga cual es la relación entre ellos.

7. De los siguientes matrices, calcule kernel, imagen y las dimensiones correspondientes.

$$A = \begin{pmatrix} 1 & 4 & 7 & 2 \\ 2 & 5 & 8 & 4 \\ 3 & 6 & 9 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 3 & 2 & 13 & 1 \\ 0 & 2 & 7 & -1 \\ 4 & 5 & 25 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 9 \end{pmatrix}.$$

8. Sea $A \in M(m \times n)$. Demuestre:

   (i) A inyectiva $\Longrightarrow m \geq n$.

   (ii) A sobreyectiva $\Longrightarrow n \geq m$.

   Demuestre que la implicación "$\Longleftarrow$" en (i) and (ii) en general es falsa.

9. Sea $A \in M(m \times n)$ y suponga que A es invertible. Demuestre que $m = n$.

10. Sean $m, n \in \mathbb{N}$ y $A \in M(m \times n)$.

    (a) ¿Cuáles son las dimensiones posibles de $\ker A$ y $\operatorname{Im} A$?

    (b) Para cada $j = 0, 1, 2, 3$ encuentre una matriz $A_j \in M(2 \times 3)$ con $\dim(\ker A_j) = j$, es decir: encuentre matrices A_0, A_1, A_2, A_3 con $\dim(\ker A_0) = 0,\ \dim(\ker A_1) = 1, \dots$. Si tal matriz no existe, explique por qué no existe.

11. (a) Encuentre por lo menos dos diferentes funciones lineales biyectivas de $M(2 \times 2)$ a P_3.

    (b) Existe una función lineal biyectiva $S : M(2 \times 2) \to P_k$ para $k \in \mathbb{N},\ k \neq 3$?

12. Sean V y W espacios vectoriales.

    (a) Sea $U \subset V$ un subspacio y sean $u_1, \dots, u_k \in U$. Demuestre que $\operatorname{gen}\{u_1, \dots u_k\} \subset U$.

    (b) Sean $u_1, \dots, u_k, w_1, \dots, w_m \in V$. Demuestre que lo siguiente es equivalente:

      (i) $\operatorname{gen}\{u_1, \dots, u_k\} = \operatorname{gen}\{w_1, \dots, w_m\}$.

      (ii) Para todo $j = 1, \dots, k$ tenemos $u_j \in \operatorname{gen}\{w_1, \dots, w_m\}$ y para todo $\ell = 1, \dots, m$ tenemos $w_\ell \in \operatorname{gen}\{u_1, \dots, u_k\}$.

      (iii) Sean $v_1, v_2, v_3, \dots, v_m \in V$ y sea $c \in \mathbb{R}$. Demuestre que $\operatorname{gen}\{v_1, v_2, v_3, \dots, v_m\} = \operatorname{gen}\{v_1 + cv_2, v_2, v_3, \dots, v_m\}$.

    (c) Sean $v_1, \dots, v_k \in V$ y sea $A : V \to W$ una función lineal invertible. Demuestre que $\dim \operatorname{gen}\{v_1, \dots, v_k\} = \dim \operatorname{gen}\{Av_1, \dots, Av_k\}$. Es verdad si A no es invertible?

13. (a) Sean $\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},\ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ y sea $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Demuestre que $\mathcal{B}$ es una base de $\mathbb{R}^3$ y escriba los vectores $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ en términos de la base $\mathcal{B}$.

14. Sean $R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},\ S = \begin{pmatrix} 3 & 2 \\ 0 & 7 \end{pmatrix},\ T = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$. Demuestre que $\mathcal{B} = \{R, S, T\}$ es una base del espacio de las matrices triangulares superiores y exprese las matrices

$$K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

en términos de la base $\mathcal{B}$.

15. Sean $\vec{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{b}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ y sean $\mathcal{A} = \{\vec{a}_1, \vec{a}_2\}$, $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$.

    (a) Demuestre qu $\mathcal{A}$ y $\mathcal{B}$ son bases de $\mathbb{R}^2$.

    (b) Sea $(\vec{x})_{\mathcal{A}} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$. Encuentre $(\vec{x})_{\mathcal{B}}$ y $\vec{x}$ (en la representación estandar).

    (c) Sea $(\vec{y})_{\mathcal{B}} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Encuentre $(\vec{y})_{\mathcal{A}}$ y $\vec{y}$ (en la representación estandar).

16. Sea $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ una base de $\mathbb{R}^2$ y sean $\vec{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ (dados en coordenadas cartesianas).

    (a) Si se sabe que $\vec{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}_{\mathcal{B}}$, $\vec{x}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{\mathcal{B}}$, es posible calcular $\vec{b}_1$ y $\vec{b}_2$? Si sí, calcúlelos. Si no, explique por qué no es posible.

    (b) Si se sabe que $\vec{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}_{\mathcal{B}}$, $\vec{x}_3 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}_{\mathcal{B}}$, es posible calcular $\vec{b}_1$ y $\vec{b}_2$? Si sí, calcúlelos. Si no, explique por qué no es posible.

    (c) ¿Existen $\vec{b}_1$ y $\vec{b}_2$ tal que $\vec{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}_{\mathcal{B}}$, $\vec{x}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}_{\mathcal{B}}$? Si sí, calcúlelos. Si no, explique por qué no es posible.

    (d) ¿Existen $\vec{b}_1$ y $\vec{b}_2$ tal que $\vec{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}_{\mathcal{B}}$, $\vec{x}_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}_{\mathcal{B}}$? Si sí, calcúlelos. Si no, explique por qué no es posible.

17. (a) Demuestre que la siguente función es lineal:

$$\Phi : M(2 \times 2) \to M(2 \times 2), \qquad \Phi(A) = A^t$$

    (b) Sea $\mathcal{B} = \{E_1, E_2, E_3, E_4\}$ la base estandar[1] de $M(2 \times 2)$. Encuentre la matriz que representa a Φ con respecto a esta base.

    (c) Sean $R = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ y sea $\mathcal{C} = \{R, S, T, U\}$. Demuestre que $\mathcal{C}$ es una base de $M(2 \times 2)$ y escriba Φ como matriz con respecto a esta base.

18. (a) Demuestre que $T : P_3 \to P_3$, $Tp = p'$ es una función lineal.

    (b) Determine $\ker(T)$, $\operatorname{Im}(T)$, $\dim(\ker(T))$, $\dim(\operatorname{Im}(T))$.

    (c) Sea $\mathcal{B} = \{1, X, X^2, X^3\}$ la base estandar de P_3. Encuentre la matriz que representa a T con respecto a esta base.

[1] $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) Sean $q_1 = X+1,\ q_2 = X-1,\ q_3 = X^2+X,\ q_4 = X^3+1$. Demuestre que $\mathcal{C} = \{q_1, q_2, q_3, q_4\}$ es una base de P_3. .

(e) Encuentre la matriz con respecto a la base $\mathcal{C}$ que representa a T.

Chapter 7

Orthonormal bases and orthogonal projections in $\mathbb{R}^n$

In this chapter we will work in $\mathbb{R}^n$ and not in arbitrary vector spaces since we want to explore in more detail its geometric aspect. In particular we will discuss orthogonality. Note that in an arbitrary vector space, we do not have the concept of angles or orthogonality. Everything that we will discuss here can be extended to inner product spaces where the inner product is used to *define* angles. Recall that we showed in Theorem 2.19 that for non-zero vectors $\vec{x}$, $\vec{y} \in \mathbb{R}^n$ the angle φ between them satisfies the equation

$$\cos\varphi = \frac{\langle \vec{x}\,, \vec{y}\rangle}{\|\vec{x}\|\,\|\vec{y}\|}.$$

In a general inner product space $(V, \langle \cdot\,, \cdot \rangle)$ this equation is used to define the angle between two vectors. In particular, two vectors are said to be orthogonal if their inner product is 0. Inner product spaces are useful for instance in physics, and maybe in some not so distant future there will be chapter in this lecture notes about them.

First we will define what the orthogonal complement of a subspace of $\mathbb{R}^n$ is and we will see that the direct sum of a subspace and its orthogonal complement gives us all of $\mathbb{R}^n$.

We already know what the orthogonal projection of a vector $\vec{x}$ onto another vector $\vec{y} \neq \vec{0}$ is (see Section 2.3). Since it is independent of the norm of $\vec{y}$, we can just as well consider it the orthogonal projection of $\vec{x}$ onto the line generated by $\vec{y}$. In this chapter we will generalise the concept of an orthogonal projection onto a line to the orthogonal projection onto an arbitrary subspace.

As an application, we will discuss the minimal squares method for the approximations of data.

7.1 Orthonormal systems and orthogonal bases

Recall that two vectors $\vec{x}$ and $\vec{y}$ are orthogonal (or perpendicular) to each other if and only if $\langle \vec{x}\,, \vec{y}\rangle = 0$. In this case we write $\vec{x} \perp \vec{y}$.

Definition 7.1. (i) A set of vectors $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$ is called an *orthogonal set* if they are pairwise orthogonal; in formulas we can write this as

$$\langle \vec{x}_j \,, \vec{x}_\ell \rangle = 0 \quad \text{for } j \neq \ell.$$

(ii) A set of vectors $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$ is called an *orthonormal set* if they are pairwise orthonormal; in formulas we can write this as

$$\langle \vec{x}_j \,, \vec{x}_\ell \rangle = \begin{cases} 1 & \text{for } j = \ell, \\ 0 & \text{for } j \neq \ell. \end{cases}$$

The difference between an orthogonal and an orthonormal set is that in the latter we additionally need that each vector of the set satisfies $\langle \vec{x}_j \,, \vec{x}_j \rangle = 1$, that is, that $\|\vec{x}_j\| = 1$. Therefore an orthogonal set may contain vectors of arbitrary lengths, including the vector $\vec{0}$ whereas all vectors in an orthonormal set must have length 1. Note that every orthonormal system is also an orthogonal system. On the other hand, every orthogonal system which does not contain $\vec{0}$ can be converted to an orthonormal one by normalising each vector (that is, by dividing them by their norms).

Examples 7.2. (i) The following systems are orthogonal systems but not orthonormal systems since the norm of at least one of their vectors is different from 1:

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

(ii) The systems following systems are orthonormal systems:

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Lemma 7.3. *Every orthonormal system is linearly independent.*

Proof. Let $\vec{x}_1, \dots, \vec{x}_k$ be an orthonormal system and consider

$$\vec{0} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \cdots + \alpha_{n-1} \vec{x}_{n-1} + \alpha_n \vec{x}_n.$$

We have to show that all α_j must be zero. To do this, we take the inner product on both sides with the vectors $\vec{x}_j$. Let us start with $\vec{x}_1$. We find

$$\begin{aligned} \langle \vec{0} \,, \vec{x}_1 \rangle &= \langle \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \cdots + \alpha_{n-1} \vec{x}_{n-1} + \alpha_n \vec{x}_n \,, \vec{x}_1 \rangle \\ &= \alpha_1 \langle \vec{x}_1 \,, \vec{x}_1 \rangle + \alpha_2 \langle \vec{x}_2 \,, \vec{x}_1 \rangle + \cdots + \alpha_{n-1} \langle \vec{x}_{n-1} \,, \vec{x}_{n-1} \rangle + \alpha_n \langle \vec{x}_n \,, \vec{x}_1 \rangle. \end{aligned}$$

Since $\langle \vec{0} \,, \vec{x}_1 \rangle = 0$, $\langle \vec{x}_1 \,, \vec{x}_1 \rangle = \|\vec{x}_1\|^2 = 1$ and $\langle \vec{x}_2 \,, \vec{x}_1 \rangle = \cdots = \langle \vec{x}_{n-1} \,, \vec{x}_{n-1} \rangle = \langle \vec{x}_n \,, \vec{x}_1 \rangle = 0$, it follows that

$$0 = \alpha_1 + 0 + \cdots + 0 = \alpha_1.$$

Now we can repeat this process with $\vec{x}_2, \vec{x}_3, \dots, \vec{x}_n$ to show that $\alpha_2 = \cdots = \alpha_n = 0$. $\square$

Remark. The lemma shows that every system of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

Definition 7.4. An orthonormal basis of $\mathbb{R}^n$ is a basis whose vectors form an orthogonal set. Occasionally we will write ONB for "orthonormal basis".

Examples 7.5 (Orthonormal bases of $\mathbb{R}^n$).

(i) The canonical basis $\vec{e}_1, \dots, \vec{e}_n$ is an orthonormal basis of $\mathbb{R}^n$.

(ii) The following systems are examples of orthonormal bases of $\mathbb{R}^2$:

$$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\}, \left\{\frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\end{pmatrix}, \frac{1}{\sqrt{14}}\begin{pmatrix}-3\\2\end{pmatrix}, \right\}, \left\{\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}, \frac{1}{5}\begin{pmatrix}-4\\3\end{pmatrix}\right\}.$$

(iii) The following systems are examples of orthonormal bases of $\mathbb{R}^3$:

$$\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}\right\}, \left\{\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}, \frac{1}{10}\begin{pmatrix}-3\\0\\1\end{pmatrix}, \frac{1}{\sqrt{62}}\begin{pmatrix}1\\-5\\6\end{pmatrix}\right\}.$$

Exercise 7.6. Show that every orthonormal basis of $\mathbb{R}^2$ is of the form $\left\{\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}, \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}, \begin{pmatrix}\sin\varphi\\ -\cos\varphi\end{pmatrix}\right\}$ for some $\varphi \in \mathbb{R}$. See also Exercise 7.13.

We will see in Corollary 7.30 that every orthonormal system in $\mathbb{R}^n$ can be completed to an orthonormal basis. In Section 7.5 we will show how to construct an orthonormal basis of a subspace of $\mathbb{R}^n$ from a given basis. In particular it follows that every subspace of $\mathbb{R}^n$ has an orthonormal basis.

Orthonormal bases are very useful. Among other things it is very easy to write a given vector $\vec{w} \in \mathbb{R}^n$ as a linear combinations of such a basis. Recall that if we are given an arbitrary basis $\vec{z}_1, \dots, \vec{z}_n$ of $\mathbb{R}^n$ and we want to write a vector $\vec{x}$ as linear combination of this basis, then we have to find coefficients $\alpha_1, \dots, \alpha_n$ such that $\vec{x} = \alpha_1\vec{z}_1 + \dots + \vec{z}_n\alpha_n$, which means we have to solve a $n \times n$ system in order to determine the coefficients. If however the given basis is an orthonormal basis, then the calculating the coefficients reduces to evaluating n inner products as the following theorem shows.

Theorem 7.7 (Representation of a vector with respect to an ONB). *Let $\vec{x}_1, \dots, \vec{x}_n$ be an orthonormal basis of $\mathbb{R}^n$ and let $\vec{w} \in \mathbb{R}^n$. Then*

$$\vec{w} = \langle \vec{w}, \vec{x}_1\rangle\vec{x}_1 + \langle \vec{w}, \vec{x}_2\rangle\vec{x}_2 + \dots + \langle \vec{w}, \vec{x}_n\rangle\vec{x}_n.$$

Proof. Since $\vec{x}_1, \dots, \vec{x}_n$ is a basis of $\mathbb{R}^n$, there are $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$\vec{w} = \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \dots + \alpha_n\vec{x}_n.$$

Now let us take the inner product on both sides with $\vec{x}_j$ for $j = 1, \dots, n$. Note that $\langle \vec{x}_k \,, \vec{x}_j \rangle = 0$ if $k \neq j$ and that $\langle \vec{x}_j \,, \vec{x}_j \rangle = \|\vec{x}_j\|^2 = 1$.

$$\begin{aligned} \langle \vec{w} \,, \vec{x}_j \rangle &= \langle \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \cdots + \alpha_n \vec{x}_n \,, \vec{x}_j \rangle \\ &= \alpha_1 \langle \vec{x}_1 \,, \vec{x}_j \rangle + \alpha_2 \langle \vec{x}_2 \,, \vec{x}_j \rangle + \cdots + \alpha_n \langle \vec{x}_n \,, \vec{x}_j \rangle \\ &= \alpha_j \langle \vec{x}_j \,, \vec{x}_j \rangle = \alpha_j. \end{aligned}$$ □

Note that the proof of this theorem is basically the same as that of Lemma 7.3. In fact, Lemma 7.3 follows from the theorem above if we choose $\vec{w} = \vec{0}$.

Exercise 7.8. If $\vec{x}_1, \dots, \vec{x}_n$ are an orthogonal, but not necessarily orthonormal basis of $\mathbb{R}^n$, then we have for every $\vec{w} \in \mathbb{R}^n$ that

$$\vec{w} = \frac{\langle \vec{w} \,, \vec{x}_1 \rangle}{\|\vec{x}_1\|^2}\, \vec{x}_1 + \frac{\langle \vec{w} \,, \vec{x}_2 \rangle}{\|\vec{x}_2\|^2}\, \vec{x}_2 + \cdots + \frac{\langle \vec{w} \,, \vec{x}_n \rangle}{\|\vec{x}_n\|^2}\, \vec{x}_n.$$

(You can either use a modified version of the proof of Theorem 7.7 or you define $z_j = \|\vec{x}_j\|^{-1} \vec{x}_j$, show that $\vec{x}_1, \dots, \vec{z}_n$ is an orthogonal basis and apply the formula from Theorem 7.7.)

You should now have understood

- what an orthogonal system is,
- what an orthonormal system is,
- what an orthonormal basis is,
- why orthogonal bases are useful,
- …,

You should now be able to

- check if a given set of vectors is an orthogonal/orthonormal system,
- check if a given set of vectors is an orthogonal/orthonormal basis of the given space,
- check if a given basis is an orthogonal or orthonormal basis,
- give examples of orthonormal basis,
- find the coefficients of a given vector with respect to a given orthonormal or orthogonal basis.
- …

7.2 Orthogonal matrices

We already saw that it is very easy to express a given vector as linear combination of the members of an orthonormal basis. In this section we want to explore the properties of the transition matrices between two orthonormal bases of $\mathbb{R}^n$.

Let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$ and $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_n\}$ be orthonormal bases of $\mathbb{R}^n$. Let $Q = A_{\mathcal{B}\to\mathcal{C}}$ be the transition matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$. We know that its entries q_{ij} are the uniquely determined numbers such that

$$\vec{u}_1 = \begin{pmatrix} q_{11} \\ \vdots \\ q_{n1} \end{pmatrix}_{\mathcal{C}} = q_{11}\vec{w}_1 + \cdots + q_{n1}\vec{w}_n, \qquad \dots \qquad \vec{u}_n = \begin{pmatrix} q_{1n} \\ \vdots \\ q_{nn} \end{pmatrix}_{\mathcal{C}} = q_{1n}\vec{w}_1 + \cdots + q_{nn}\vec{w}_n,$$

Since $\mathcal{C}$ is an orthonormal basis, it follows that $q_{ij} = \langle \vec{u}_j\,, \vec{w}_i \rangle$, see Theorem 7.7. Therefore

$$A_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} \langle \vec{u}_1\,, \vec{w}_1\rangle & \langle \vec{u}_2\,, \vec{w}_1\rangle & \cdots\cdots & \langle \vec{u}_n\,, \vec{w}_1\rangle \\ \langle \vec{u}_1\,, \vec{w}_2\rangle & \langle \vec{u}_2\,, \vec{w}_2\rangle & \cdots\cdots & \langle \vec{u}_n\,, \vec{w}_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{u}_1\,, \vec{w}_n\rangle & \langle \vec{u}_2\,, \vec{w}_n\rangle & \cdots\cdots & \langle \vec{u}_n\,, \vec{w}_n\rangle \end{pmatrix}$$

If we exchange the role of $\mathcal{B}$ and $\mathcal{C}$ and use that $\langle \vec{w}_i\,, \vec{u}_j\rangle = \langle \vec{u}_j\,, \vec{w}_i\rangle$, then we obtain

$$A_{\mathcal{C}\to\mathcal{B}} = \begin{pmatrix} \langle \vec{w}_1\,, \vec{u}_1\rangle & \langle \vec{w}_2\,, \vec{u}_1\rangle & \cdots\cdots & \langle \vec{w}_n\,, \vec{u}_1\rangle \\ \langle \vec{w}_1\,, \vec{u}_2\rangle & \langle \vec{w}_2\,, \vec{u}_2\rangle & \cdots\cdots & \langle \vec{w}_n\,, \vec{u}_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{w}_1\,, \vec{u}_n\rangle & \langle \vec{w}_2\,, \vec{u}_n\rangle & \cdots\cdots & \langle \vec{w}_n\,, \vec{u}_n\rangle \end{pmatrix} = \begin{pmatrix} \langle \vec{u}_1\,, \vec{w}_1\rangle & \langle \vec{u}_1\,, \vec{w}_2\rangle & \cdots\cdots & \langle \vec{u}_1\,, \vec{w}_n\rangle \\ \langle \vec{u}_2\,, \vec{w}_1\rangle & \langle \vec{u}_2\,, \vec{w}_2\rangle & \cdots\cdots & \langle \vec{u}_2\,, \vec{w}_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{u}_n\,, \vec{w}_1\rangle & \langle \vec{u}_n\,, \vec{w}_2\rangle & \cdots\cdots & \langle \vec{u}_n\,, \vec{w}_n\rangle \end{pmatrix}$$

This shows that $A_{\mathcal{C}\to\mathcal{B}} = (A_{\mathcal{B}\to\mathcal{C}})^t$. If we use that $A_{\mathcal{C}\to\mathcal{B}} = (A_{\mathcal{B}\to\mathcal{C}})^{-1}$, then we find that

$$(A_{\mathcal{B}\to\mathcal{C}})^{-1} = (A_{\mathcal{B}\to\mathcal{C}})^t.$$

From these calculations, we obtain the following lemma.

Lemma 7.9. *Let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$ and $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_n\}$ be orthonormal bases of $\mathbb{R}^n$ and let $Q = A_{\mathcal{B}\to\mathcal{C}}$ be the transition matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$. Then*

$$Q^t = Q^{-1}.$$

Definition 7.10. A matrix $A \in M(n \times n)$ is called an *orthogonal matrix* if it is invertible and $A^t = A^{-1}$.

Proposition 7.11. *Let $Q \in M(n \times n)$. Then the following is equivalent:*

(i) *Q is an orthogonal matrix.*

(ii) *Q^t is an orthogonal matrix.*

(iii) *Q^{-1} exists and is an orthogonal matrix.*

Proof. (i) $\implies$ (ii): Assume that Q is orthogonal. Then it is invertible, hence also Q^t is invertible. Moreover, $(Q^t)^{-1} = (Q^{-1})^t = (Q^{-t})^t = Q$ where in the first step we used Theorem 3.50. Hence Q^t is an orthogonal matrix.

(ii) $\Longrightarrow$ (i): Assume that Q^t is an orthogonal matrix. Then $(Q^t)^t = Q$ must be an orthogonal matrix too by what we just proved.

(i) $\Longrightarrow$ (ii): Assume that Q is orthogonal. Then it is invertible and $(Q^{-1})^{-1} = (Q^{-1})^{-1} = (Q^{-1})^t$ where in the second step we used Theorem 3.50. Hence Q^{-1} is an orthogonal matrix.
(ii) $\Longrightarrow$ (i): Assume that Q^{-1} is an orthogonal matrix. Then its inverse $(Q^{-1})^{-1} = Q$ must be an orthogonal matrix too by what we just proved. □

By Lemma 7.9, every transition matrix from one ONB to another ONB is an orthogonal matrix. The reverse is also true as the following theorem shows.

Theorem 7.12. *Let $Q \in M(n \times n)$. Then:*

(i) *Q is an orthogonal matrix if and only if its columns are an orthonormal basis of $\mathbb{R}^n$.*

(ii) *Q is an orthogonal matrix if and only if its rows are an orthonormal basis of $\mathbb{R}^n$.*

(iii) *If Q is an orthgonal matrix, then $|\det Q| = 1$.*

Proof. (i): Assume that Q is an orthogonal matrix and let $\vec{c}_j$ be its columns. We already know that they are a basis of $\mathbb{R}^n$ since Q is invertible. In order to show that they are also an orthonormal system, we calculate

$$\mathrm{id} = Q^tQ = \begin{pmatrix} \vec{c}_1 \\ \vdots \\ \vec{c}_n \end{pmatrix} (\vec{c}_1 \,|\, \cdots \,|\, \vec{c}_n) = \begin{pmatrix} \langle \vec{c}_1\,, \vec{c}_1\rangle & \langle \vec{c}_1\,, \vec{c}_2\rangle & \cdots\cdots & \langle \vec{c}_1\,, \vec{c}_n\rangle \\ \langle \vec{c}_2\,, \vec{c}_1\rangle & \langle \vec{c}_2\,, \vec{c}_2\rangle & \cdots\cdots & \langle \vec{c}_2\,, \vec{c}_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{c}_n\,, \vec{c}_1\rangle & \langle \vec{c}_n\,, \vec{c}_2\rangle & \cdots\cdots & \langle \vec{c}_n\,, \vec{c}_n\rangle \end{pmatrix} \tag{7.1}$$

Since the product is equal to the identity matrix, it follows that all the elements on the diagonal must be equal to 1 and all the other elements must be equal to 0. This means that $\langle \vec{c}_j\,, \vec{c}_j\rangle = 1$ for $j = 1, \dots, n$ and $\langle \vec{c}_j\,, \vec{c}_k\rangle = 0$ for $j \neq k$, hence the columns of Q are an orthonormal basis of $\mathbb{R}^n$.

Now assume that the columns $\vec{c}_1, \dots, \vec{c}_n$ of Q are an orthonormal basis of $\mathbb{R}^n$. Then clearly (7.1) holds which shows that Q is an orthogonal matrix.
(ii): The rows of Q are the columns of Q^t hence the are an orthonormal basis of $\mathbb{R}^n$ by (i) and Proposition 7.11 (ii).
(iii): Recall that $\det Q^t = \det Q$. Therefore we obtain

$$1 = \det \mathrm{id} = \det(QQ^t) = (\det Q)(\det Q^t) = (\det Q)^2,$$

which proves the claim. □

Clearly, not every matrix R with $|\det R| = 1$ is an orthogonal matrix. For instance, if $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $\det R = 1$, but $R^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ is different from $R^t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Question 7.1

Assume that $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^n$ are pairwise orthogonal and let $R \in M(n \times n)$ be the matrix whose columns are the given vectors. Can you calculate R^tR and RR^t? What are the conditions on the vectors such that R is invertible? If it is invertible, what is its inverse? (You should be able to answer the above questions more or less easily if $\|\vec{a}_j\| = 1$ for all $j = 1, \dots, n$ because in this case R is an orthogonal matrix.)

Exercise 7.13. Show that every orthogonal 2×2 matrix is of the form $Q = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ or $Q = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$. Compare this with Exercise 7.6.

It can be shown that every orthogonal matrix represents either a rotation (if its determinant is 1) or the composition of a rotation and a reflection (if its determinant is -1).

Orthogonal matrices in $\mathbb{R}^2$. Let $Q \in M(2 \times 2)$ be an orthogonal matrix with columns $\vec{c}_1$ and $\vec{c}_2$. Recall that $Q\vec{e}_1 = \vec{c}_1$ and $Q\vec{e}_2 = \vec{c}_2$. Since $\vec{c}_1$ is a unit vector, it is of the form $\vec{c}_1 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$ for some $\varphi \in \mathbb{R}$. Since $\vec{c}_2$ is also a unit vector and in addition must be orthogonal to $\vec{c}_1$, there are only the two possible choices $\vec{c}_2^{\,+} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$ or $\vec{c}_2^{\,-} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \end{pmatrix}$, see Figure 7.1.

- In the first case, $\det Q = \det(\vec{c}_1|\vec{c}_2^{\,+}) = \det \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \cos^2\varphi + \sin^2\varphi = 1$ and Q represents the rotation by φ counterclockwise.

- In the second case, $\det Q = \det(\vec{c}_1|\vec{c}_2^{\,-}) = \det \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = -\cos^2\varphi - \sin^2\varphi = -1.$ and Q represents the rotation by φ counterclockwise followed by a reflection on the direction given by $\vec{c}_1$ (or: reflection on the x-axis followed by the rotation by φ counterclockwise).

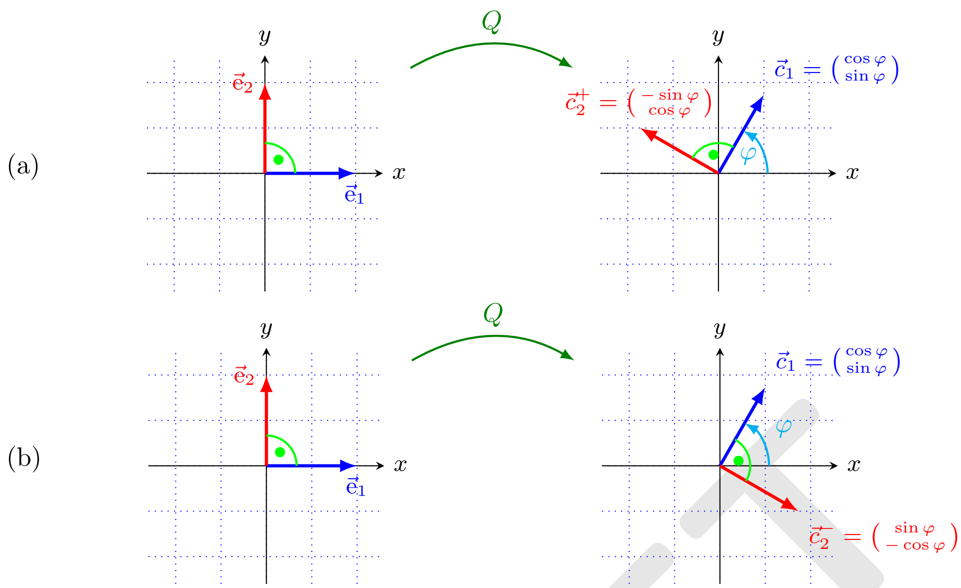


FIGURE 7.1: In case (a), Q represents a rotation and $\det A = 1$. In case (b) it represents rotation followed by a reflection and $\det Q = -1$.

Exercise 7.14. Let Q be an orthogonal $n \times n$ matrix. Show the following.

(i) Q preserves inner products, that is $\langle \vec{x}\,, \vec{y}\rangle = \langle Q\vec{x}\,, Q\vec{y}\rangle$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

(ii) Q preserves lengths, that is $\|\vec{x}\| = \|Q\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$.

(iii) Q preserves angles, that is $\sphericalangle(\vec{x}, \vec{y}) = \sphericalangle(Q\vec{x}, Q\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n \setminus \{\vec{0}\}$.

Exercise 7.15. Let $Q \in M(n \times n)$

(i) Assume that Q preserves inner products, that is $\langle \vec{x}\,, \vec{y}\rangle = \langle Q\vec{x}\,, Q\vec{y}\rangle$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Show that Q is an orthogonal matrix.

(ii) Assume that Q preserves length, that is $\|\vec{x}\| = \|Q\vec{x}\|$ for all. Show that Q is an orthogonal matrix.

Exercise 7.14 together with Exercise 7.15 show the following.

A matrix Q is an orthogonal matrix if and only if it preserves lengths if and only if it preserves angles. That is

$$\begin{aligned} Q \;\text{ is orthogonal} \quad &\iff \quad Q^t = Q^{-1} \\ &\iff \quad \langle Q\vec{x}\,, Q\vec{y}\rangle = \langle \vec{x}\,, \vec{y}\rangle \quad \text{for all} \;\; \vec{x}, \vec{y} \in \mathbb{R}^n \\ &\iff \quad \|Q\vec{x}\| = \|\vec{x}\| \quad \text{for all} \;\; \vec{x} \in \mathbb{R}^n. \end{aligned}$$

Definition 7.16. A linear transformation $T : R^n \to \mathbb{R}^m$ is called an *isometry* if $\|T\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$.

Note that every isometry is injective since $T\vec{x} = \vec{0}$ if and only if $\vec{x} = \vec{0}$, therefore necessarily $n \leq m$.

You should now have understood

- that a matrix is orthogonal if and only if it represents change of bases between two orthonormal bases,
- that an orthogonal matrix represents either a rotation or a rotation composed with a reflection,
- . . . ,

You should now be able to

- check if a given matrix is an orthogonal matrix,
- construct orthogonal matrices,
- . . .

7.3 Orthogonal complements

The first part of this section works for all vector spaces, not necessarily $\mathbb{R}^n$.

Proposition 7.17. *Let U, W be subspaces of a vector space V. Then their intersection $U \cap W$ is a subspace of V.*

Proof. Clearly, $U \cap W \neq \varnothing$ because $\mathbb{O} \in U$ and $\mathbb{O} \in W$, hence $\mathbb{O} \in U \cap W$. Now let $z_1, z_2 \in U \cap W$ and $c \in \mathbb{K}$. Then $z_1, z_2 \in U$ and therefore $z_1 + cz_2 \in U$ because U is a vector space. Analogously it follows that $z_1 + cz_2 \in W$, hence $z_1 + cz_2 \in U \cap W$. □

Observe that $U \cap W$ is the largest subspace which is contained both in U and in V.
For example, the intersection of two planes in $\mathbb{R}^3$ which pass through the origin is either that same plane (if the two original planes are the same plane), or it is a line passing through the origin. In either case, it is a subspace of $\mathbb{R}^3$.

Observe however that in general the union of two vector spaces in general is not a vector space.

Exercise.

- Give an example of two subspaces whose union is not a vector space.
- Give an example of two subspaces whose union is a vector space.

Question 7.2. Union of subspaces.

Can you find a criterion that subspaces must satisfy such that their union is a subspace?

Let us define the *sum* and the *direct sum* of vector spaces.

Definition 7.18. Let U, W be subspaces of a vector space V. Then the *sum* of the vector spaces U and W is defined as

$$U + W = \{u + w : u \in U,\ w \in W\}. \tag{7.2}$$

If in addition $U \cap W = \{\mathbb{O}\}$, then the sum is called the *direct sum of* U and W and one writes $U \oplus W$ instead of $U + W$.

Remark. Let U, W be subspaces of a vector space V. Then the $U + W$ is again a subspace of V.

Proof. Clearly, $U + W \neq \varnothing$ because $\mathbb{O} \in U$ and $\mathbb{O} \in W$, hence $\mathbb{O} + \mathbb{O} = \mathbb{O} \in U + W$. Now let $z_1, z_2 \in U + W$ and $c \in \mathbb{K}$. Then there exist $u_1, u_2 \in U$ and $w_1, w_2 \in W$ with $z_1 = u_1 + w_1$ and $z_2 = u_2 + w_2$. Therefore

$$z_1 + cz_2 = u_1 + w_1 + c(u_2 + w_2) = (u_1 + cu_2) + (w_1 + cw_2) \in U + W$$

and $U + W$ is a subspace by Proposition 5.10. □

Note that $U + W$ consists of all possible linear combinations of vectors from U and from W. We obtain immediately the following observations.

Remark 7.19. (i) Assume that $U = \operatorname{span}\{u_1, \dots, u_k\}$ and that $W = \operatorname{span}\{w_1, \dots, w_j\}$, then $U + W = \operatorname{span}\{u_1, \dots, u_k, w_1, \dots, w_j\}$.

(ii) The space $U + W$ is the smallest vector space which contains both U and W.

Examples 7.20. (i) Let V be a vector space and let $U \subseteq V$ be a subspace. Then we always have:

(a) $U + \{\mathbb{O}\} = U \oplus \{\mathbb{O}\} = U$,
(b) $U + U = U$,
(c) $U + V = V$.

If U and W are subspaces of V, then

(a) $U \subseteq U + W$ and $W \subseteq U + W$.
(b) $U + W = U$ if and only if $W \subseteq U$.

(ii) Let U and W be lines in $\mathbb{R}^2$ passing through the origin. Then they are subspaces of $\mathbb{R}^2$ and we have that $U + W = U$ if the lines are parallel and $U + W = \mathbb{R}^2$ if they are not parallel.

(iii) Let U and W be lines in $\mathbb{R}^3$ passing through the origin. Then they are subspaces of $\mathbb{R}^3$ and we have that $U + W = U$ if the lines are parallel; otherwise $U + W$ is the plane containing both lines.

(iv) Let U be a line and W be a plane in $\mathbb{R}^3$, both passing through the origin. Then they are subspaces of $\mathbb{R}^3$ and we have that $U + W = W$ if the line U is contained in W. If not, then $U + W = \mathbb{R}^3$.

Prove the statements in the examples above.

Recall that the intersection of two subspaces is again a subspace, see Proposition 7.17. The formula for the dimension of the sum of two vector spaces in the next proposition can be understood as follows: If we sum the dimension of the two vector spaces, then we count the part which is common to both spaces twice; therefore we have to subtract its dimension in order to get the correct dimension of the sum of the vector spaces.

Proposition 7.21. *Let U, W be subspaces of a vector space V. Then*

$$\dim(U+W) = \dim U + \dim V - \dim(U \cap W).$$

In particular, $\dim(U+W) = \dim U + \dim V$ *if* $U \cap V = \{\mathbb{O}\}$.

Proof. Let $\dim U = k$ and $\dim W = m$. Recall that $U \cap W$ is a subspace of V. and that $U \cap W \subseteq U$ and $U \cap W \subseteq W$. Let $v_1, \ldots, v_\ell$ be a basis of $U \cap W$. By Theorem 5.42 we can complete it to a basis $v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k$ of U. Similarly, we can complete it to a basis $v_1, \ldots, v_\ell, w_{\ell+1}, \ldots, w_m$ of W. Now we claim that $v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k, w_{\ell+1}, \ldots, w_m$ is a basis of $U+W$.

- First we show that the vectors $v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k, w_{\ell+1}, \ldots, w_m$ generate $U+W$. This follows from Remark 7.19 and

$$\begin{aligned} U+W &= \operatorname{span}\{v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k\} + \operatorname{span}\{v_1, \ldots, v_\ell, w_{\ell+1}, \ldots, w_m\} \\ &= \operatorname{span}\{v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k, v_1, \ldots, v_\ell, w_{\ell+1}, \ldots, w_m\} \\ &= \operatorname{span}\{v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k, w_{\ell+1}, \ldots, w_m\}. \end{aligned}$$

- Now we show that the vectors $v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k, w_{\ell+1}, \ldots, w_m$ are linearly independent. Let $\alpha_1, \ldots, \alpha_n, \beta_{\ell+1}, \ldots, \beta_m \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \cdots + \alpha_\ell v_\ell + \alpha_{\ell+1} u_{\ell+1} + \cdots + \alpha_k u_k + \beta_{\ell+1} w_{\ell+1} + \cdots + \beta_m w_m = \mathbb{O}.$$

It follows that

$$\underbrace{\alpha_1 v_1 + \cdots + \alpha_\ell v_\ell + \alpha_{\ell+1} u_{\ell+1} + \cdots + \alpha_k u_k}_{\in U} = \underbrace{-(\beta_{\ell+1} w_{\ell+1} + \cdots + \beta_m w_m)}_{\in W} \tag{7.3}$$

and therefore $-(\beta_{\ell+1} w_{\ell+1} + \cdots + \beta_m w_m) \in U \cap W$ hence it must be a linear combination of the vectors $v_1, \ldots, v_\ell$ because they are a basis of $U \cap V$. So we can find $\gamma_1, \ldots, \gamma_\ell \in \mathbb{R}$ such that $\gamma_1 v_1 + \cdots + \gamma_\ell v_\ell = -(\beta_{\ell+1} w_{\ell+1} + \cdots + \beta_m w_m)$. This implies that

$$\gamma_1 v_1 + \cdots + \gamma_\ell v_\ell + \beta_{\ell+1} w_{\ell+1} + \cdots + \beta_m w_m = \mathbb{O}.$$

Since $v_1, \ldots, v_\ell, w_{\ell+1}, \ldots, w_m$ is a basis of W, they are linearly independent, and we conclude that $\gamma_1 = \cdots = \gamma_\ell = \beta_{\ell+1} = \cdots = \beta_m = 0$. Inserting in (7.3), we obtain

$$\alpha_1 v_1 + \cdots + \alpha_\ell v_\ell + \alpha_{\ell+1} u_{\ell+1} + \cdots + \alpha_k u_k = \mathbb{O},$$

hence $\alpha_1 = \cdots = \alpha_k = 0$.

It follows that

$$\begin{aligned} \dim(U+W) &= \#\{v_1, \ldots, v_\ell, u_{\ell+1}, \ldots, u_k, w_{\ell+1}, \ldots, w_m\} \\ &= \ell + (k-\ell) + (m-\ell) \\ &= k + m - \ell \\ &= \dim U + \dim W - \dim(U \cap W). \end{aligned}$$

□

For the rest of this section, we will work in $\mathbb{R}^n$. First let us define the orthogonal complement of a given subspace.

Definition 7.22. Let U be a subspace of $\mathbb{R}^n$.

(i) Let U be a subspace of $\mathbb{R}^n$. We say that a vector $\vec{x} \in \mathbb{R}^n$ is perpendicular to U if it is perpendicular to every vector in U. In this case we write $\vec{x} \perp U$.

(ii) The *orthogonal complement* of U is denoted by $U^\perp$ and it is the set of all vectors which are perpendicular to every vector in U, that is

$$U^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{x} \perp U\} = \{\vec{x} \in \mathbb{R}^n : \vec{x} \perp \vec{u} \text{ for every } \vec{u} \in U\}.$$

We start with some easy observations.

Remark 7.23. Let U be a subspace of $\mathbb{R}^n$.

(i) $U^\perp$ is a subspace of $\mathbb{R}^n$.

(ii) $U \cap U^\perp = \{\vec{0}\}$.

(iii) $(\mathbb{R}^n)^\perp = \{\vec{0}\}$, $\{\vec{0}\}^\perp = \mathbb{R}^n$.

Proof. (i) Clearly, $\vec{0} \in U^\perp$. Let $\vec{x}, \vec{y} \in U^\perp$ and let $c \in \mathbb{R}$. Then for every $\vec{u} \in U$ we have that $\langle \vec{x} + c\vec{y}\,, \vec{u}\rangle = \langle \vec{x}\,, \vec{u}\rangle + c\langle \vec{y}\,, \vec{u}\rangle = 0$, hence $\vec{x} + c\vec{y} \in U^\perp$ and $U^\perp$ is a subspace by Theorem 5.10.

(ii) Let $\vec{x} \in U \cap U^\perp$. Then it follows that $\vec{x} \perp \vec{x}$, hence $\|\vec{x}\|^2 = \langle \vec{x}\,, \vec{x}\rangle = 0$ which shows that $\vec{x} = \vec{0}$ and therefore $U \cap U^\perp$ consists only of the vector $\vec{0}$.

(iii) Assume that $\vec{x} \in (\mathbb{R}^n)^\perp$. Then $\vec{x} \perp \vec{y}$ for every $\vec{y} \in \mathbb{R}^n$, in particular also $\vec{x} \perp \vec{x}$. Therefore $\|\vec{x}\|^2 = \langle \vec{x}\,, \vec{x}\rangle = 0$ which shows that $\vec{x} = \vec{0}$. It follows that $\vec{x} \in (\mathbb{R}^n)^\perp$.

It is clear that $\langle \vec{x}\,, \vec{0}\rangle = 0$, hence $\mathbb{R}^n \subseteq \{\vec{0}\}^\perp \subseteq \mathbb{R}^n$ which proves that $\{\vec{0}\}^\perp = \mathbb{R}^n$.

$\square$

Examples 7.24. (i) The orthogonal complement of a line in $\mathbb{R}^2$ is again a line, see Figure 7.2.

(ii) The orthogonal complement of a line in $\mathbb{R}^3$ is the plane perpendicular to the given lines. The orthogonal complement to a plane in $\mathbb{R}^3$ is the line perpendicular to the given plane, see Figure 7.2.

The next goal is to show that $\dim U + \dim U^\perp = n$ and to establish a method for calculating $U^\perp$. To this end, the following lemma is useful. It tells us that in order to verify that some $\vec{x}$ is perpendicular to U we do not have to check that $\vec{x} \perp \vec{u}$ for every $\vec{u} \in U$, but that it is enough to check it for a set of vectors $\vec{u}$ which generate U.

Lemma 7.25. *Let $U = \operatorname{span}\{\vec{u}_1, \dots, \vec{u}_k\} \subseteq \mathbb{R}^n$. Then $\vec{x} \in U^\perp$ if and only if $\vec{x} \perp \vec{u}_j$ for every $j = 1, \dots, k$.*

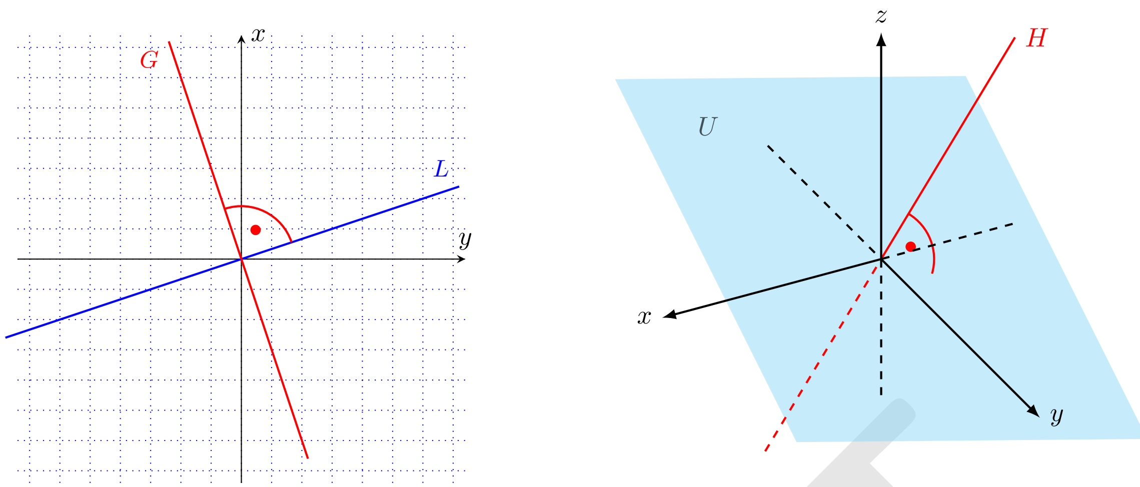


FIGURE 7.2: The figure on the left shows the orthogonal complement of the line L in $\mathbb{R}^2$ which is the line G. The figure on the right shows the orthogonal complement of the plane U in $\mathbb{R}^3$ which is the line H. Note the orhtogonal complement of G is U.

Proof. Suppose that $\vec{x} \perp U$, then $\vec{x} \perp \vec{u}$ for every $\vec{u} \in U$, in particular for the generating vectors $\vec{u}_1, \dots, \vec{u}_k$. Now suppose that $\vec{x} \perp \vec{u}_j$ for all $j = 1, \dots, k$. Let $\vec{u} \in U$ be an arbitrary vector in U. Then there exist $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $\vec{u} = \alpha_1\vec{u}_1 + \dots + \vec{u}_k\alpha_k$. So we obtain

$$\langle \vec{x}\,, \vec{u}\rangle = \langle \vec{x}\,, \alpha_1\vec{u}_1 + \dots + \vec{u}_k\alpha_k\rangle = \langle \vec{x}\,, \alpha_1\vec{u}_1\rangle + \dots + \alpha_k\langle \vec{x}\,, \vec{u}_k\rangle = 0.$$

Since $\vec{u}$ can be chosen arbitrary in U, it follows that $\vec{x} \perp U$. □

The lemma above leads to a method to calculate the orthogonal complement of a given subspace U of $\mathbb{R}^n$ as follows. Note that essentially it was already proved in Theorem 6.40.

Lemma 7.26. *Let $U = \operatorname{span}\{\vec{u}_1, \dots, \vec{u}_k\} \subseteq \mathbb{R}^n$ and let A be the matrix whose rows consist of the vectors $\vec{u}_1, \dots, \vec{u}_k$. Then*

$$U^\perp = \ker A. \tag{7.4}$$

Proof. Let $\vec{x} \in \mathbb{R}^n$. By Lemma 7.25 we know that $\vec{x} \in U^\perp$ if and only if $\vec{x} \perp \vec{u}_j$ for every $j = 1, \dots, k$. This is the case if and only if

$$\begin{aligned} \langle \vec{u}_1\,, \vec{x}\rangle &= 0 \\ \langle \vec{u}_2\,, \vec{x}\rangle &= 0 \\ \vdots \;\; &= \vdots \\ \langle \vec{u}_k\,, \vec{x}\rangle &= 0 \end{aligned} \qquad \text{which can be written in matrix form as} \qquad \begin{pmatrix} \vec{u}_1 \\ \hline \vec{u}_2 \\ \hline \vdots \\ \hline \vec{u}_k \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which is the same as $A\vec{x} = \vec{0}$ by definition of A. In conclusion, $\vec{x} \perp U$ if and only $A\vec{x} = \vec{0}$, that is, if and only if $\vec{x} \in \ker A$. □

The next two theorems are the main results of this section.

Theorem 7.27. *For every subspace $U \subseteq \mathbb{R}^n$ we have that*

$$\dim U + \dim U^\perp = n. \tag{7.5}$$

Proof. Let $\vec{u}_1, \dots, \vec{u}_k$ be a basis of U. Note that $k = \dim U$. Then we have in particular $U = \operatorname{span}\{\vec{u}_1, \dots, \vec{u}_k\}$. As in Lemma 7.25 we consider the matrix $A \in M(k \times n)$ whose rows are the vectors $\vec{u}_1, \dots, \vec{u}_k$. Then $U^\perp = \ker A$, so

$$\dim U^\perp = \dim(\ker A) = n - \dim(\operatorname{Im} A).$$

Note that $\dim(\operatorname{Im} A)$ is the dimension of the column space of A which is equal to the dimension of the row space of A by Proposition 6.32. Since the vectors $\vec{u}_1, \dots, \vec{u}_k$ are linear independent, this dimension is equal to k. Therefore $\dim U^\perp = n - k = n - \dim U$. Rearranging we obtained the desired formula $\dim U^\perp + \dim U = n$.
(We could also have said that the reduced form of A cannot have any zero row because its rows are linearly independent. Therefore the reduced form must have k pivots and we obtain $\dim U^\perp = \dim(\ker A) = n - \#(\text{pivots of the reduced form of } A = n - k = n - \dim U$. We basically re-proved Proposition 6.32.) ☐

Theorem 7.28. *Let $U \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$. Then the following holds.*

(i) $U \oplus U^\perp = \mathbb{R}^n$.

(ii) $(U^\perp)^\perp = U$.

Proof. (i) Recall that $U \cap U^\perp = \{\vec{0}\}$ by Remark 7.23, therefore the sum is a direct sum. Now let us show that $U + U^\perp = \mathbb{R}^n$. Since $U + U^\perp \subseteq \mathbb{R}^n$, we only have to show that $\dim(U + U^\perp) = n$ because the only n-dimensional subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself, see Theorem 5.50. From Proposition 7.21 and Theorem 7.27 we obtain

$$\dim(U + U^\perp) = \dim(U) + \dim(U^\perp) - \dim(U \cap U^\perp) = \dim(U) + \dim(U^\perp) = n$$

where we used that $\dim(U \cap U^\perp) = \dim\{\vec{0}\} = 0$.

(ii) First let us show that $U \subseteq (U^\perp)^\perp$. To this end, fix $\vec{u} \in U$. Then, for every $\vec{y} \in U^\perp$, we hae that $\langle \vec{x}\,, \vec{y} \rangle = 0$, hence $\vec{x} \perp U^\perp$, that is, $\vec{x} \in (U^\perp)^\perp$. Note that $\dim(U^\perp)^\perp = n - \dim U^\perp = n - (n - \dim U) = \dim U$. Since we already know that $U \subseteq (U^\perp)^\perp$, it follows that they must be equal by Theorem 5.50. ☐

The next proposition shows that every subspace of $\mathbb{R}^n$ has an orthonormal basis. Another proof of this fact will be given later when we introduce the Gram-Schmidt process in Section 7.5.

Proposition 7.29. *Every subspace $U \subseteq \mathbb{R}^n$ with $\dim U > 0$ has an orthonormal basis.*

Proof. Let U be a subspace of $\mathbb{R}^n$ with $\dim U = k > 0$. Then $\dim U^\perp = n - k$ and we can choose a basis $\vec{w}_{k+1}, \dots, w_n$ of $U^\perp$. Let $A_0 \in M((n-k) \times n)$ be the matrix whose rows are the vectors $\vec{w}_{k+1}, \dots, w_n$. Since $U = (U^\perp)^\perp$, we know that $U = \ker A_0$. Pick any $\vec{u}_1 \in \ker A_0$ with $\vec{u}_1 \neq \vec{0}$.

Then $\vec{v}_1 \in U$. Now we form the new matrix $A_1 \in M((n-k+1)\times n)$ by adding $\vec{u}_1$ as a new row to the matrix A_0. Note that the rows of A_1 are linearly independent, so $\dim\ker(A_1) = n-(n-k+1) = k-1$. If $k-1 > 0$, then we pick any vector $\vec{v}_2 \in \ker A_1$ with $\vec{v}_2 \neq \vec{0}$. This vector is orthogonal to all the rows of A_1, in particular it belongs to U (since it is orthogonal to $\vec{w}_{k+1}, \dots, \vec{w}_n$) and it is perpendicular to $\vec{u}_1 \in U$. Now we form the matrix $A_2 \in M((n-k+2)\times n)$ by adding the vector $\vec{u}_2$ as a row to A_1. Again, the rows of A_2 are linearly independent and therefore $\dim(\ker A_2) = n-(n-k+2) = k-2$. If $k-2 > 0$, then we pick any vector $\vec{v}_3 \in \ker A_2$ with $\vec{v}_3 \neq \vec{0}$. This vector is orthogonal to all the rows of A_2, in particular it belongs to U (since it is orthogonal to $\vec{w}_{k+1}, \dots, \vec{w}_n$) and it is perpendicular to $\vec{u}_1, \vec{u}_2 \in U$. We continue this process until we have vectors $\vec{u}_1, \dots, \vec{u}_k \in U$ which are pairwise orthogonal and the matrix $A_k \in M(n\times n)$ consists of linearly independent rows, so its kernel is trivial. By construction, $\vec{u}_1, \dots, \vec{u}_k$ is an orthogonal system of k vectors in U with none of them being equal to $\vec{0}$. Hence they are linearly independent and therefore they are an orthonormal basis of U since $\dim U = k$. In order to obtain an orthogonal basis we only have to normalise each of the vectors. $\square$

Corollary 7.30. *Every orthonormal system in $\mathbb{R}^n$ can be completed to an orthonormal basis.*

Proof. Let $\vec{w}_1, \dots, \vec{w}_k$ be an orthonormal system in $\mathbb{R}^n$ and let $W = \operatorname{span}\{\vec{w}_1, \dots, \vec{w}_k\}$. By Proposition 7.29 we can find an orthonormal basis $\vec{u}_1, \dots \vec{u}_{n-k}$ of $W^\perp$ (take $U = W^\perp$ in the proposition). Then $\vec{w}_1, \dots, \vec{w}_k, \vec{u}_1 \dots, \vec{u}_{n-k}$ is then an orthonormal basis of $U \oplus U^\perp = \mathbb{R}^n$. $\square$

We conclude this section with a few examples.

Example 7.31. Find a basis for the orthogonal complement of

$$U = \operatorname{span}\left\{ \begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \right\}.$$

Solution. Recall that $\vec{x} \in U^\perp$ if and only if it is perpendicular to the vectors which generate U. Therefore $\vec{x} \in U^\perp$ if and only if it belongs to the kernel of the matrix whose rows are the generators of U. So we calculate

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 0 & 1 & 0\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 3 & 4\\ 0 & -2 & -2 & -4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 2\end{pmatrix}.$$

Hence a basis of $U^\perp$ is given by

$$\vec{w}_1 = \begin{pmatrix}0\\-2\\0\\1\end{pmatrix}, \qquad \vec{w}_2 = \begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}. \qquad \diamond$$

Example 7.32. Find an orthonormal basis for the orthogonal complement of

$$U = \operatorname{span}\left\{ \begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \right\}.$$

Solution. We will use the method from Proposition 7.29. Another solution of this exercise will be given in Example 7.47. From the solution of Example 7.31 we can take the first basis vector $\vec{w}_1$. We append it to the matrix from the solution of Example 7.31 and reduce the new matrix (note that the first few steps are identical to the reduction of the original matrix). We obtain

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$$

whose kernel is generated by

$$\begin{pmatrix} 5 \\ 1 \\ -5 \\ 2 \end{pmatrix}$$

Hence an orthogonal basis of $U^\perp$ is given by

$$\vec{y}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{y}_2 = \frac{1}{\sqrt{55}} \begin{pmatrix} 5 \\ 1 \\ -5 \\ 2 \end{pmatrix}.$$

$\diamond$

You should now have understood

- the concept of sum and direct sum of two subspaces,
- why the formula $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ makes sense,
- the concept of the orthogonal complement,
- in particular the geometric interpretation of the orthogonal complement of a subspace (at least in $\mathbb{R}^2$ and $\mathbb{R}^3$),
- ...,

You should now be able to

- find the orthogonal complement of a given subspace of $\mathbb{R}^n$,
- find an orthogonal basis of a given subspace of $\mathbb{R}^n$,
- ...

7.4 Orthogonal projections

Recall that in Section 2.3 we discussed the orthogonal projection of one vector onto another in $\mathbb{R}^2$. This can clearly be extended to higher dimensions. Let $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$. Then

$$\operatorname{proj}_{\vec{w}} \vec{v} := \frac{\langle \vec{v} \,, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w} \tag{7.6}$$

is the unique vector in $\mathbb{R}^n$ which is orthogonal to $\vec{w}$ and that that $\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}$ is parallel to $\vec{w}$. We already know that the projection is independent on the length of $\vec{w}$. So $\operatorname{proj}_{\vec{w}} \vec{v}$ should be regarded as the projection of $\vec{v}$ onto the one-dimensional subspace generated by $\vec{w}$.

In this section we want to generalise this to orthogonal projections on higher dimensional subspaces, for instance you could think of the projection in $\mathbb{R}^3$ onto a given plane. Then, given a subspace U of $\mathbb{R}^n$, we want to define the *orthogonal projection* as the function from $\mathbb{R}^n$ to $\mathbb{R}^n$ which assigns to each vector $\vec{v}$ its orthogonal projection onto U. We start with the analogue of Theorem 2.22.

Theorem 7.33 (Orthogonal projection). *Let $U \subseteq \mathbb{R}^n$ be a subspace and let $\vec{v} \in \mathbb{R}^n$. Then there exist uniquely determined vectors $\vec{v}_\|$ and $\vec{v}_\perp$ such that*

$$\vec{v}_\| \in U, \quad \vec{v}_\perp \perp U \qquad \text{and} \qquad \vec{v} = \vec{v}_\| + \vec{v}_\perp. \tag{7.7}$$

The vector $\vec{v}_\|$ is called the orthogonal projection of $\vec{v}$ onto U*; it is denoted by* $\operatorname{proj}_U \vec{v}$.

Proof. First we show the existence of the vectors $\vec{v}_\|$ and $\vec{v}_\perp$. If $U = \mathbb{R}^n$, we take $\vec{v}_\| = \vec{v}$ and $\vec{v}_\perp = \vec{0}$. If $U = \{\vec{0}\}$, we take $\vec{v}_\| = \vec{0}$ and $\vec{v}_\perp = \vec{v}$. Otherwise, let $0 < \dim U = k < n$. Choose orthonormal bases $\vec{u}_1, \dots, \vec{u}_k$ of U and $\vec{w}_{k+1}, \dots, \vec{w}_n$ of $U^\perp$. This is possible by Theorem 7.27 and Proposition 7.29. Then $\vec{u}_1, \dots, \vec{u}_k, \vec{w}_{k+1}, \dots, \vec{w}_n$ is an orthonormal basis of $\mathbb{R}^n$ and for every $\vec{v} \in \mathbb{R}^n$ we find with the help of Theorem 7.7 that

$$\vec{v} = \underbrace{\langle \vec{u}_1 , \vec{v} \rangle \vec{u}_1 + \cdots + \langle \vec{u}_k , \vec{v} \rangle \vec{u}_k}_{\in U} + \underbrace{\langle \vec{w}_{k+1} , \vec{v} \rangle \vec{w}_{k+1} + \cdots + \langle \vec{w}_n , \vec{v} \rangle \vec{w}_n}_{\in U^\perp}.$$

If we set $\vec{v}_\| = \langle \vec{u}_1 , \vec{v} \rangle \vec{u}_1 + \cdots + \langle \vec{u}_k , \vec{v} \rangle \vec{u}_k$ and $\vec{v}_\perp = \langle \vec{w}_{k+1} , \vec{v} \rangle \vec{w}_{k+1} + \cdots + \langle \vec{w}_n , \vec{v} \rangle \vec{w}_n$, then they have the desired properties.

Next we show uniqueness of the decomposition of $\vec{v}$. Assume that there are vectors $\vec{v}_\|$ and $\vec{z}_\| \in U$ and $\vec{v}_\perp$ and $\vec{z}_\perp \in U^\perp$ such that $\vec{v} = \vec{v}_\| + \vec{v}_\perp$ and $\vec{v} = \vec{z}_\| + \vec{z}_\perp$. Then $\vec{v}_\| + \vec{v}_\perp = \vec{z}_\| + \vec{z}_\perp$ and, rearranging, we find that

$$\underbrace{\vec{v}_\| - \vec{z}_\|}_{\in U} = \underbrace{\vec{z}_\perp - \vec{v}_\perp}_{\in U^\perp}$$

Since $U \cap U^\perp = \{\vec{0}\}$, it follows that $\vec{v}_\| - \vec{z}_\| = \vec{0}$ and $\vec{z}_\perp - \vec{v}_\perp = \vec{0}$, and therefore $\vec{z}_\| = \vec{v}_\|$ and $\vec{z}_\perp = \vec{v}_\perp$. ☐

Definition 7.34. Let U be a subspace of $\mathbb{R}^n$. Then we define the orthogonal projection onto U as the map which sends $\vec{v} \in \mathbb{R}^n$ to its orthogonal projection onto U. It is usually denoted by P_U, so

$$P_U : \mathbb{R}^n \to \mathbb{R}^n, \qquad P_U \vec{v} = \operatorname{proj}_U \vec{v}.$$

Remark 7.35 (Formula for the orthogonal projection). The proof of Theorem 7.33 indicates how we can calculate the orthogonal projection onto a given subspace $U \subseteq \mathbb{R}^n$. If $\vec{u}_1, \dots, \vec{u}_k$ is an orthonormal basis of U, then

$$P_U \vec{v} = \langle \vec{u}_1 , \vec{v} \rangle \vec{u}_1 + \cdots + \langle \vec{u}_k , \vec{v} \rangle \vec{u}_k. \tag{7.8}$$

This shows that P_U is a linear transformation since $P_U(\vec{x} + c\vec{y}) = P_U\vec{x} + cP_U\vec{y}$ follows easily from inserting into (7.8).

Exercise. If $\vec{u}_1, \dots, \vec{u}_k$ is an orthogonal basis of U (but not necessarily orthonormal), show that

$$P_U\vec{v} = \frac{\langle \vec{u}_1\,, \vec{v}\rangle}{\|\vec{u}_1\|^2}\vec{u}_1 + \dots + \frac{\langle \vec{u}_k\,, \vec{v}\rangle}{\|\vec{u}_k\|^2}\vec{u}_k. \tag{7.9}$$

Remark 7.36 (Formula for the orthogonal projection for $\dim U = 1$**).** If $\dim U = 1$, we obtain again the formula (7.6) which we already know from Section 2.3. To see this, choose $\vec{w} \in U$ with $\vec{w} \neq \vec{0}$. Then $\vec{w}' = \|\vec{w}\|^{-1}\vec{w}$ is an orthonormal basis of U and according to (7.8) we have that

$$\operatorname{proj}_{\vec{w}} \vec{v} = \operatorname{proj}_U \vec{v} = \langle \vec{w}'\,, \vec{v}\rangle \vec{w}' = \left\langle \|\vec{w}\|^{-1}\vec{w}\,, \vec{v}\right\rangle \left(\|\vec{w}\|^{-1}\vec{w}\right) = \|\vec{w}\|^{-2}\,\langle \vec{w}\,, \vec{v}\rangle\,\vec{w} = \frac{\langle \vec{w}\,, \vec{v}\rangle}{\|\vec{w}\|^2}\vec{w}.$$

Remark 7.37 (Pythagoras's Theorem). Let U be a subspace of $\mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$ and let $\vec{v}_{\|}$ and $\vec{v}_{\perp}$ be as in Theorem 7.33. Then

$$\|\vec{v}\|^2 = \|\vec{v}_{\|}\|^2 + \|\vec{v}_{\perp}\|^2.$$

Proof. Using that $\vec{v}_{\|} \perp \vec{v}_{\perp}$, we find

$$\begin{aligned}\|\vec{v}\|^2 &= \langle \vec{v}\,, \vec{v}\rangle = \langle \vec{v}_{\|} + \vec{v}_{\perp}\,, \vec{v}_{\|} + \vec{v}_{\perp}\rangle = \langle \vec{v}_{\|}\,, \vec{v}_{\|}\rangle + \langle \vec{v}_{\|}\,, \vec{v}_{\perp}\rangle + \langle \vec{v}_{\perp}\,, \vec{v}_{\|}\rangle + \langle \vec{v}_{\perp}\,, \vec{v}_{\perp}\rangle \\ &= \langle \vec{v}_{\|}\,, \vec{v}_{\|}\rangle + \langle \vec{v}_{\perp}\,, \vec{v}_{\perp}\rangle = \|\vec{v}_{\|}\|^2 + \|\vec{v}_{\perp}\|^2.\end{aligned}$$

□

Exercise 7.38. Let U be a subspace of $\mathbb{R}^n$ with basis $\vec{u}_1, \dots, \vec{u}_k$ and let $\vec{w}_{k+1}, \dots, \vec{w}_n$ be a basis of $U^{\perp}$. Find the matrix representation of P_U with respect to the basis $\vec{u}_1, \dots, \vec{u}_k, \vec{w}_{k+1}, \dots, \vec{w}_n$.

Exercise 7.39. Let U be a subspace of $\mathbb{R}^n$. Show that $P_{U^\perp} = \mathrm{id} - P_U$. (You can show this either directly or using the matrix representation of P_U.)

Exercise 7.40. Let U be a subspace of $\mathbb{R}^n$. Show that $(P_U)^2 = P_U$. (You can show this either directly or using the matrix representation of P_U.)

Exercise 7.41. Let U be a subspace of $\mathbb{R}^n$.

(i) Find $\ker P_U$ and $\operatorname{Im} P_U$.

(ii) Find $P_{U^\perp}P_U$ and $P_U P_{U^\perp}$.

In Theorem 7.33 we used the concept of orthogonality to define the orthogonal projection of $\vec{v}$ onto a given subspace. We obtained a decomposition of $\vec{v}$ into a part parallel to the given subspace and a part orthogonal to it. The next theorem shows that the orthogonal projection of $\vec{v}$ onto U gives us the point in U which is closest to $\vec{v}$.

Theorem 7.42. *Let U be a subspace of $\mathbb{R}^n$ and let $\vec{v} \in \mathbb{R}^n$. Then $P_U\vec{v}$ is the point in U which is closest to $\vec{v}$, that is,*

$$\|\vec{v} - P_U\vec{v}\| \leq \|\vec{v} - \vec{u}\| \quad \text{for every} \quad \vec{u} \in U.$$

Proof. Let $\vec{v} \in \mathbb{R}^n$ and $\vec{u} \in U \subseteq \mathbb{R}^n$. Note that $\vec{v} - P_U\vec{v} \in U^{\perp}$ and that $P_U\vec{v} - \vec{u} \in U$ since both vectors belong to U. Therefore, the Pythagoras theorem shows that

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{v} - P_U\vec{v} + P_U\vec{v} - \vec{u}\|^2 = \|\vec{v} - P_U\vec{v}\|^2 + \|P_U\vec{v} - \vec{u}\|^2 \geq \|\vec{v} - P_U\vec{v}\|^2.$$

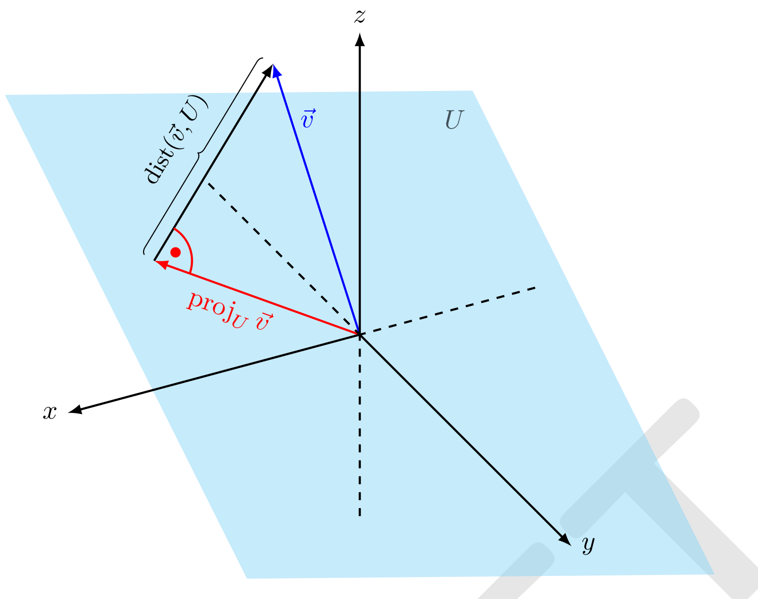


FIGURE 7.3: The figure shows the orthogonal projection of the vector $\vec{v}$ onto the subspace U (which is a vector) and the distance of $\vec{v}$ to U (which is a number. It is the length of the vector $(\vec{v} - \operatorname{proj}_U \vec{v})$).

Taking the square root on both sides shows the desired inequality. $\square$

Definition 7.43. as Let U be a subspace of $\mathbb{R}^n$ and let $\vec{v} \in \mathbb{R}$. The we define the *distance of* $\vec{v}$ *to* U as

$$\operatorname{dist}(\vec{v}, U) := \|\vec{v} - P_U\vec{v}\|.$$

This is the shortest distance of $\vec{v}$ to any point in U.

In Remark 7.35 we already found a formula for the orthogonal projection P_U of a vector $\vec{v}$ to a given subspace U. This formula however requires to have an orthonormal basis of U. We want to give another formula for P_U which does not require the knowledge of an orthonormal basis.

Theorem 7.44. *Let U be a subspace of $\mathbb{R}^n$ with basis $\vec{u}_1, \dots, \vec{u}_k$ and let $B \in M(n \times k)$ be the matrix whose columns are these basis vectors. Then the following holds.*

(i) *B is injective.*

(ii) *$B^tB : \mathbb{R}^k \to \mathbb{R}^k$ is a bijection.*

(iii) *The orthogonal projection onto U is given by the formula*

$$P_U = B(B^tB)B^t.$$

Proof. (i) By construction, the columns of B are linearly independent. Therefore the unique solution of $B\vec{x} = \vec{0}$ is $\vec{x}$ which shows that B is injective.

(ii) Observe that $B^tB \in M(k\times k)$ and assume that $B^tB\vec{x} = \vec{0}$ for some $\vec{x}\in\mathbb{R}^k$. Then it follows for every $\vec{y}\in\mathbb{R}^k$ that $B\vec{y} = \vec{0}$ because

$$0 = \langle \vec{y}\,, B^tB\vec{y}\rangle = \langle (B^t)^t\vec{y}\,, B\vec{y}\rangle = \langle B\vec{y}\,, B\vec{y}\rangle = \|B\vec{y}\|^2.$$

Since B is injective, this implies $\vec{y} = \vec{0}$, so B^tB is injective. Since it is a square matrix, it follows that it is even bijective.

(iii) Observe that by construction $\operatorname{Im} B = U$. Now let $\vec{x}\in\mathbb{R}^n$. Note that $P_U\vec{x}\in\operatorname{Im} B$. Hence there exists exactly one $\vec{z}\in\mathbb{R}^k$ such that $P_U\vec{x} = B\vec{z}$. Moreover, $\vec{x} - P_U\vec{x}\perp U = \operatorname{Im} B$, hence for every $\vec{y}\in\mathbb{R}^k$ we have that

$$0 = \langle \vec{x} - P_U\vec{x}\,, B\vec{y}\rangle = \langle \vec{x} - B\vec{z}\,, B\vec{y}\rangle = \langle B^t\vec{x} - B^tB\vec{z}\,, \vec{y}\rangle.$$

Since this is true for every $\vec{y}\in\mathbb{R}^k$, it follows that $B^t\vec{x} - B^tB\vec{z} = \vec{0}$. Now we recall that B^tB is invertible, so we can solve for $\vec{z}$ and obtain $\vec{z} = (B^tB)^{-1}B\vec{x}$. This finally gives

$$P_U\vec{x} = B\vec{z} = B(B^tB)^{-1}B\vec{x}.$$

Since this holds for every $\vec{x}\in\mathbb{R}^n$, formula (iii) is proved. ☐

You should now have understood

- the concept of an orthogonal projection onto a subspace of $\mathbb{R}^n$,
- the geometric interpretation of orthogonal projections and how it is related to the distance of point to a subspace,
- ...,

You should now be able to

- calculate the orthogonal projection of a point to a subspace,
- calculate the distance of a point to a subspace,
- ...

7.5 The Gram-Schmidt process

In this section we will describe the so-called Gram-Schmidt orthonormalisation process. Roughly speaking, it converts a given basis of a subspace of $\mathbb{R}^n$ into an orthonormal basis, thus providing another proof that every subspace of $\mathbb{R}^n$ has an orthonormal basis (Corollary 7.30).

Theorem 7.45. *Let U be a subspace of $\mathbb{R}^n$ with basis $\vec{u}_1, \dots, \vec{u}_k$. Then there exists an orthonormal basis $\vec{x}_1, \dots, \vec{x}_k$ of U such that*

$$\operatorname{span}\{\vec{u}_1, \dots, \vec{u}_j\} = \operatorname{span}\{\vec{x}_1, \dots, \vec{x}_j\} \quad \textit{for every} \quad j = 1, \dots, k.$$

Proof. The proof is constructive, that is, we do not only prove the existence of such basis, but it tells us how to calculate it. The idea is to construct the new basis $\vec{x}_1, \dots, \vec{x}_k$ step by step. In order to simplify notation a bit, we set $U_j = \operatorname{span}\{\vec{u}_1, \dots, \vec{u}_j\}$ for $j = 1, \dots, k$. Note that $\dim U_j = j$ and that $U_k = U$.

- Set $\vec{x}_1 = \|\vec{v}_1\|^{-1}\vec{v}_1$. Then clearly $\|\vec{x}_1\| = 1$ and $\operatorname{span}\{\vec{u}_1\} = \operatorname{span}\{\vec{x}_1\} = U_1$.
- The vector $\vec{x}_2$ must be a normalised vector in U_2 which is orthogonal to $\vec{x}_1$, that is, it must be orthogonal to U_1. So we simple take $\vec{x}_2$ and subtract its projection onto U_1:

$$\vec{w}_2 = \vec{x}_2 - \operatorname{proj}_{U_1}\vec{w}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{x}_1}\vec{w}_2 = \vec{x}_2 - \langle \vec{x}_1\,, \vec{w}_2\rangle\vec{x}_1.$$

  Clearly $\vec{w}_2 \in U_2$ because it is a linear combination of vectors in U_2. Moreover, $\vec{w}_2 \perp U_1$ because

$$\langle \vec{w}_2\,, \vec{x}_1\rangle = \Big\langle \vec{x}_2 - \langle \vec{x}_1\,, \vec{u}_2\rangle\vec{x}_1\,, \vec{x}_1\Big\rangle = \langle \vec{x}_2\,, \vec{x}_1\rangle - \langle \vec{x}_1\,, \vec{u}_2\rangle\langle \vec{x}_1\,, \vec{x}_1\rangle = \langle \vec{x}_2\,, \vec{x}_1\rangle - \langle \vec{x}_1\,, \vec{u}_2\rangle = 0.$$

  Hence the vector $\vec{x}_2$ that we are looking for is

$$\vec{x}_2 = \|\vec{w}_2\|^{-1}\vec{w}_2.$$

  Since $\vec{x}_2 \in U_2$ it follows that $\operatorname{span}\{\vec{x}_1, \vec{x}_2\} \subseteq U_2$. Since both spaces have dimension 2, they must be equal.
- The vector $\vec{x}_3$ must be a normalised vector in U_3 which is orthogonal to $U_2 = \operatorname{span}\{\vec{x}_1, \vec{x}_2\}$. So we simple take $\vec{x}_3$ and subtract its projection onto U_2:

$$\vec{w}_3 = \vec{x}_3 - \operatorname{proj}_{U_2}\vec{u}_3 = \vec{x}_2 - (\operatorname{proj}_{\vec{x}_1}\vec{u}_3 + \operatorname{proj}_{\vec{x}_2}\vec{u}_3) = \vec{x}_2 - \Big(\langle \vec{x}_1\,, \vec{u}_3\rangle\vec{x}_1 + \langle \vec{x}_1\,, \vec{u}_3\rangle\vec{x}_1\Big)$$

  Clearly $\vec{u}_2 \in U_3$ because it is a linear combination of vectors in U_3. Moreover, $\vec{w}_3 \perp U_2$ because for $j = 1, 2$ we obtain

$$\begin{aligned}\langle \vec{w}_3\,, \vec{x}_j\rangle &= \Big\langle \vec{x}_3 - (\langle \vec{x}_1\,, \vec{w}_3\rangle\vec{x}_1 + \langle \vec{x}_2\,, \vec{w}_3\rangle\vec{x}_2)\,, \vec{x}_j\Big\rangle \\ &= \langle \vec{x}_3\,, \vec{x}_j\rangle - \langle \vec{x}_1\,, \vec{w}_3\rangle\langle \vec{x}_1\,, \vec{x}_j\rangle - \langle \vec{x}_2\,, \vec{w}_3\rangle\langle \vec{x}_2\,, \vec{x}_j\rangle \\ &= \langle \vec{x}_3\,, \vec{x}_j\rangle - \langle \vec{x}_1\,, \vec{w}_3\rangle\langle \vec{x}_j\,, \vec{x}_j\rangle = \langle \vec{x}_3\,, \vec{x}_j\rangle - \langle \vec{x}_1\,, \vec{w}_3\rangle = 0.\end{aligned}$$

  Hence the vector $\vec{x}_2$ that we are looking for is

$$\vec{x}_3 = \|\vec{w}_3\|^{-1}\vec{w}_3.$$

  Since $\vec{x}_3 \in U_3$ it follows that $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \subseteq U_3$. Since both spaces have dimension 3, they must be equal.

We repeat this k times until have constructed the basis $\vec{x}_1, \ldots, \vec{x}_k$.
Note that the general procedure is

- Suppose that we already have constructed $\vec{x}_1, \ldots, \vec{x}_\ell$. Then we first construct

$$\vec{w}_{\ell+1} = \vec{u}_{\ell+1} - P_{U_\ell}\vec{u}_{\ell+1}.$$

  This vector satisfies $\vec{w}_{\ell+1} \in U_{\ell+1}$ and $\vec{w}_{\ell+1} \perp U_\ell$. Note that $\vec{w}_{\ell+1} \neq \vec{0}$ because otherwise we would have that $\vec{u}_{\ell+1} = P_{U_\ell}\vec{u}_{\ell+1} \in U_\ell$ which is impossible because $\vec{u}_{\ell+1}, \vec{u}_\ell, \ldots, \vec{u}_1$ are linearly independent. Then $\vec{x}_{\ell+1} = \|\vec{w}_{\ell+1}\|^{-1}\vec{w}_{\ell+1}$ has all the desired properties. $\square$

Example 7.46. Let $U = \operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ where

$$\vec{u}_1 = \begin{pmatrix} 1\\1\\0\\1\\1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1\\4\\\sqrt{2}\\3\\2 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -2\\5\\0\\0\\1 \end{pmatrix}.$$

We want to find an orthonormal basis $\vec{x}_1, \vec{x}_2, \vec{x}_3$ of U using the Gram-Schmidt process.

Solution. (i) $\vec{x}_1 = \|\vec{v}_1\|^{-1}\vec{v}_1 = \frac{1}{2}\vec{v}_1$.

(ii) $\vec{w}_2 = \vec{u}_2 - \operatorname{proj}_{\vec{x}_1} \vec{u}_2 = \vec{u}_2 - \langle \vec{x}_1 , \vec{u}_2\rangle \vec{x}_1 = \vec{u}_2 - 4\vec{x}_1 = \vec{u}_2 - 2\vec{u}_1 = \begin{pmatrix} -3\\2\\\sqrt{2}\\1\\0 \end{pmatrix}$

$$\implies \qquad \vec{x}_2 = \|\vec{w}_2\|^{-1}\vec{w}_2 = \frac{1}{4}\begin{pmatrix} -3\\2\\\sqrt{2}\\1\\0 \end{pmatrix}$$

(iii) $\vec{w}_3 = \vec{u}_3 - \operatorname{proj}_{\operatorname{span}\{\vec{u}_1,\vec{u}_2\}} \vec{u}_3 = \vec{u}_3 - \big[\langle \vec{x}_1 , \vec{u}_3\rangle \vec{x}_1 + \langle \vec{x}_2 , \vec{u}_3\rangle \vec{x}_1\big] = \vec{u}_3 - \big[2\vec{x}_1 + 4\vec{x}_2\big]$

$$= \begin{pmatrix} -2\\5\\0\\0\\1 \end{pmatrix} - \begin{pmatrix} 1\\1\\0\\1\\1 \end{pmatrix} - \begin{pmatrix} -3\\2\\\sqrt{2}\\1\\0 \end{pmatrix} = \begin{pmatrix} 0\\2\\-\sqrt{2}\\-2\\0 \end{pmatrix}$$

$$\implies \qquad \vec{x}_3 = \|\vec{w}_3\|^{-1}\vec{w}_3 = \tfrac{1}{\sqrt{10}}\begin{pmatrix} 0\\-2\\\sqrt{2}\\2\\0 \end{pmatrix}.$$

Therefore the desired orthonormal basis of U is

$$\vec{x}_1 = \frac{1}{2}\begin{pmatrix} 1\\1\\0\\1\\1 \end{pmatrix}, \quad \vec{x}_2 = \frac{1}{4}\begin{pmatrix} -3\\2\\\sqrt{2}\\1\\0 \end{pmatrix}, \quad \vec{x}_3 = \frac{1}{\sqrt{10}}\begin{pmatrix} 0\\-2\\\sqrt{2}\\2\\0 \end{pmatrix}. \qquad \diamond$$

Note that you obtain a different basis if you change the order of the given basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$.

Example 7.47. We will given another solution of Example 7.32. We were asked to find an orthonormal basis of the orthogonal complement of

$$U = \operatorname{span}\left\{ \begin{pmatrix} 1\\2\\3\\4 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix} \right\}.$$

From Example 7.31 we already know that

$$U^\perp = \operatorname{span}\{\vec{w}_1, \vec{w}_2\} \qquad \text{where} \quad \vec{w}_1 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

We use the Gram-Schmidt process to obtain an orthonormal basis $\vec{x}_1, \vec{x}_2$ of U.

(i) $\vec{x}_1 = \|\vec{v}_1\|^{-1}\vec{v}_1 = \dfrac{1}{\sqrt{5}}\vec{v}_1$.

(ii) $\vec{y}_2 = \vec{w}_2 - \operatorname{proj}_{\vec{x}_1} \vec{w}_2 = \vec{u}_2 - \langle \vec{x}_1 \,, \vec{u}_2 \rangle \vec{x}_1 = \vec{u}_2 - \dfrac{2}{\sqrt{5}}\vec{x}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} - \dfrac{2}{5}\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} -5 \\ -1 \\ 5 \\ -2 \end{pmatrix}$

$$\Longrightarrow \qquad \vec{x}_2 = \|\vec{y}_2\|^{-1}\vec{y}_2 = \frac{1}{\sqrt{55}}\begin{pmatrix} 5 \\ 1 \\ -5 \\ 2 \end{pmatrix}$$

Therefore

$$\vec{x}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \frac{1}{\sqrt{55}}\begin{pmatrix} 5 \\ -1 \\ 5 \\ 2 \end{pmatrix}.$$

You should now have understood

- why the Gram-Schmidt process works
- …,

You should now be able to

- apply the Gram-Schmidt process in order to generate an orthonormal basis of a given subspace,
- …

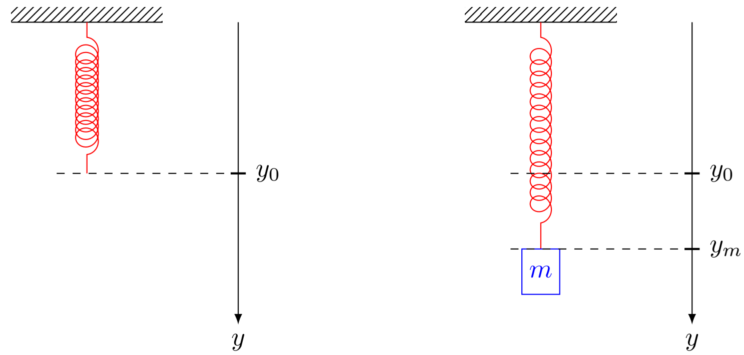
7.6 Application: Least squares

In this section we want to present the least squares method to fit a linear function to certain measurements. Let us see an example.

Example 7.48. Assume that we want to measure the Hook constant k of a spring. By Hook's law we know that

$$y = y_0 + km \tag{7.10}$$

where y_0 is the elongation of the spring without any mass attached and y is the elongation of the spring when we attach the mass m to it.



Assume that we measure the elongation for different masses. If Hook's law is valid and if our measurements were perfect, then our measured points should lie on a line with slope k. However, measurements are never perfect and the points will rather be scattered around a line. Assume that we measured the following.

m	2	3	4	5
y	4.5	5.1	6.1	7.9

Figure 7.4 contains a plot of these measurements in the m-y-plane.

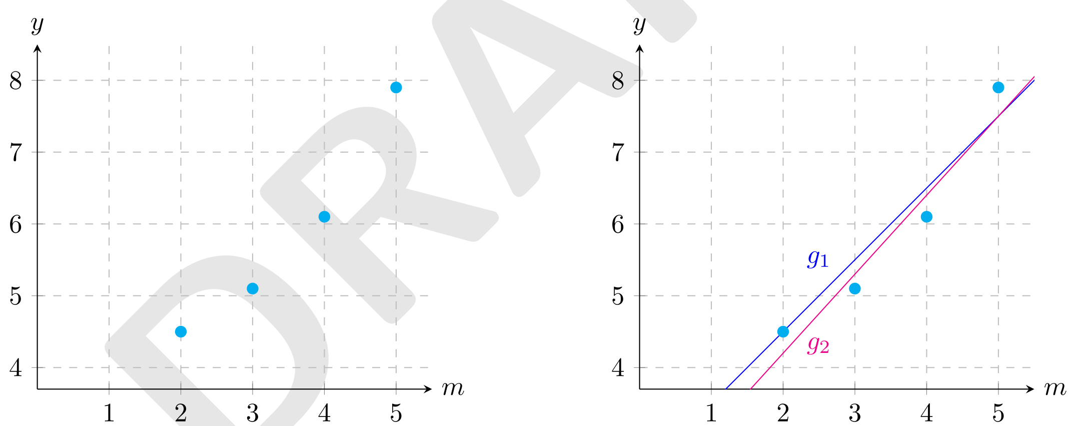


FIGURE 7.4: The left plot shows the measured data. In the plot on the right we added the two functions $g_1(x) = x + 2.5$, $g_2(x) = 1.1x + 2$ which are candidates for linear approximations to the measured data.

The plot gives us some confidence that Hook's law holds since the points seem to lie more or less on a line. How do we best fit a line through the points? The slope seems to be around 1. We could make the following guesses:

$$g_1(x) = x + 2.5 \qquad \text{or} \qquad g_2(x) = 1.1x + 2$$

Which of the two functions is the better approximation? Are there other approximations that are even better?
The answer to this questions depend very much on how we measure how "good" an approximation is. One very common way is the following: For each measured point, we take the difference $\Delta_j := m_j - g(m_j)$ between the measured value and the value of our test function. Then we square all these differences, sum them and then we take the square root $\left(\sum_{j=1}^n (m_j - g(m_j))^2\right)^{\frac{1}{2}}$, see also Figure 7.5. The resulting number will be our measure for how good our guess is.

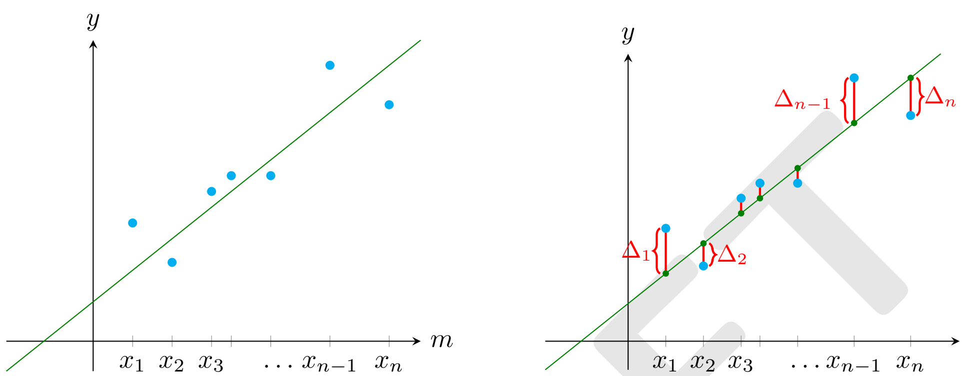


FIGURE 7.5: The graph on the left shows points for which we want to find an approximating linear function. The graph on the right shows such a linear function and how to measure the error or discrepancy between the measured points the proposed line. A measure for the error is $\left(\sum_{j=1}^n \Delta_j^2\right)^{\frac{1}{2}}$.

Before we do this for our data, we make some simple observations.

(i) If all the measured point lie on a line and we take this line as our candidate, then this method gives the total error 0 as it should be.

(ii) We take the squares of the errors in each measured points so that the error is always counted positive. Otherwise it could happen that the errors cancel each other. If we would simply sum the errors, then the total error could be 0 while the approximating line is quite far from all the measure points.

(iii) There are other ways how to measure the error, for example one could use $\sum_{j=1}^n |m_j - g(m_j)|$, but it turns out the methods with the squares has many advantages. (See some course on optimisation for further details.)

Now let us calculate the errors for our measure points and our two proposed functions.

m	2	3	4	5
y (measured)	4.5	5.1	6.1	7.9
$g_1(m)$	4.5	4.5	6.5	7.5
$y - g_1$	0	0.6	-0.4	0.4

m	2	3	4	5
y (measured)	4.5	5.1	6.1	7.9
$g_2(m)$	4.2	5.3	6.4	7.5
$y - g_2$	0.3	-0.2	-0.3	0.4

Therefore we find for the errors

$$\text{Error for function } g_1: \quad \Delta^{(1)} = \left[0^2 + 0.6^2 + (-0.4)^2 + 0.4^2\right]^{\frac{1}{2}} = [0.68]^{\frac{1}{2}} \approx 0.825,$$
$$\text{Error for function } g_2: \quad \Delta^{(2)} = \left[0.3^2 + (-0.2)^2 + (-0.3)^2 + 0.4^2\right]^{\frac{1}{2}} = [0.38]^{\frac{1}{2}} \approx 0.616,$$

so our second guess seems to be closer to the best linear approximation to our measured points than the first guess. This exercise will be continued on p. 261.

Now the question arises how we can find the optimal linear function.

Best linear approximation. Assume we are given measured data $(x_1, y_1), \dots, (x_n, y_n)$ and we want to find a linear function $g(x) = ax + b$ such that the total error

$$\Delta := \Big[\sum_{j=1}^{n} (y_j - g(x_j))^2\Big]^{\frac{1}{2}} \tag{7.11}$$

is minimal. In other words, we have to find the parameters a and b such that Δ becomes as small as possible. The key here is to recognise the right hand side on (7.11) as the norm of a vector (here the particular form of how we chose to measure the error is crucial). Let us rewrite (7.11) as follows:

$$\begin{aligned}\Delta &= \Big[\sum_{j=1}^{n} (y_j - g(x_j))^2\Big]^{\frac{1}{2}} = \Big[\sum_{j=1}^{n} (y_j - (ax_j - b))^2\Big]^{\frac{1}{2}} = \left\| \begin{pmatrix} y_1 - (ax_1 - b) \\ y_2 - (ax_2 - b) \\ \vdots \\ y_n - (ax_n - b) \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} - \left[a \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right] \right\|.\end{aligned}$$

Let us set

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{7.12}$$

Note that these are vectors in $\mathbb{R}^n$. Then

$$\Delta = \|\vec{y} - [a\vec{x} + b\vec{u}]\|$$

and the question is how to choose a and b such that this becomes as small as possible. In other words, we are looking for the point in the vector space spanned by $\vec{x}$ and $\vec{u}$ which is closest to $\vec{y}$. By Theorem 7.42 this point is given by the orthogonal projection of $\vec{y}$ onto that plane.
To calculate this projection, set $U = \operatorname{span}\{\vec{x}, \vec{u}\}$ and let P be the orthogonal projection onto U. Then by our reasoning

$$P\vec{y} = a\vec{x} + b\vec{u}. \tag{7.13}$$

Now let us see how we can calculate a and b easily from (7.13).[1] In the following we will assume that $\vec{x}$ and $\vec{u}$ so that U is a plane. This assumption seems to be reasonable because that they are linearly dependent would mean that $x_1 = \dots = x_n$ (in our example with the spring this would mean that we always used the same mass in the experiment). Observe that if $\vec{x}, \vec{u}$ were linearly independent, then the matrix A below would have only one column; everything else works just the same.
Recall that by Theorem 7.44 the orthogonal projection onto U is given by

$$P = A(A^tA)^{-1}A^t$$

where A is the $n \times 2$ matrix whose columns consist of the vectors $\vec{x}$ and $\vec{u}$. Therefore (7.13) becomes

$$A(A^tA)^{-1}A^t\vec{y} = a\vec{x} + b\vec{u} = A\begin{pmatrix} a \\ b \end{pmatrix}. \tag{7.14}$$

Since by our assumption the columns of A are linearly independent, it is injective. Therefore we can conclude from (7.14) that

$$(A^tA)^{-1}A^t\vec{y} = \begin{pmatrix} a \\ b \end{pmatrix}$$

which is formula for the numbers a and b that we were looking for.

Let us summarise our reasoning above in a theorem.

Theorem 7.49. *Let $(x_1, y_1), \dots, (x_n, y_n)$ be given. The linear function $g(x) = ax + b$ which minimises the total error*

$$\Delta := \left[\sum_{j=1}^{n} (y_j - g(x_j))^2\right]^{\frac{1}{2}} \tag{7.15}$$

is given by

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^tA)^{-1}A^t\vec{y} \tag{7.16}$$

where $\vec{y}, \vec{x}$ and $\vec{u}$ are as in (7.12) and A is the $n \times 2$ matrix whose columns consist of the vectors $\vec{x}$ and $\vec{u}$.

In Remark 7.50 we will show how this formula can be derived with methods from calculus.

Exercise 7.48 continued. . Let us use Theorem 7.49 to calculate the best linear approximation to the data from Exercise 7.48. Note that in this case the m_j correspond to the x_j from the theorem and we will write $\vec{m}$ instead of $\vec{x}$. In this case, we have

$$\vec{m} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad A = (\vec{x}\,|\,\vec{u}) = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 4.5 \\ 5.1 \\ 6.1 \\ 7.9 \end{pmatrix},$$

[1] Of course, you could simply calculate $P\vec{y}$ and then plant the linear $n \times 2$ system to find the coefficients a and b.

hence

$$A^tA = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 54 & 14 \\ 14 & 4 \end{pmatrix}, \qquad (AA^t)^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -7 \\ -7 & 27 \end{pmatrix}$$

and therefore

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^tA)^{-1}A^t\vec{y} = \frac{1}{10}\begin{pmatrix} 2 & -7 \\ -7 & 27 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4.5 \\ 5.1 \\ 6.1 \\ 7.9 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -3 & -1 & 1 & 3 \\ 13 & 6 & -1 & -8 \end{pmatrix}\begin{pmatrix} 4.5 \\ 5.1 \\ 6.1 \\ 7.9 \end{pmatrix}$$

$$= \begin{pmatrix} 1.12 \\ 1.98 \end{pmatrix}.$$

We conclude that the best linear approximation is

$$g(m) = 1.12m + 1.98.$$

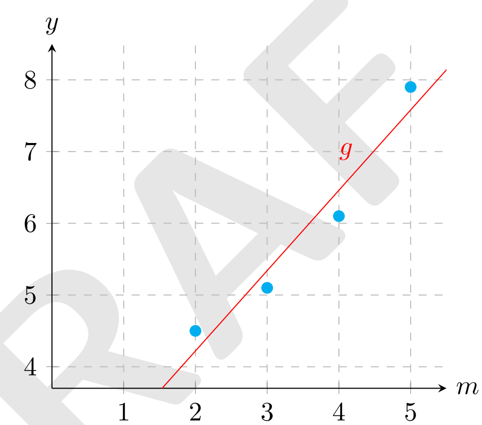


FIGURE 7.6: The plot shows the measured data and the linear approximation $g(m) = 1.12m + 1.98$ calculated with Theorem 7.49.

The method above can be generalised to other types of functions. We will show how it can be adapted to the case of polynomial and to exponential functions.

Polynomial functions.. Assume we are given measured data $(x_1, y_1), \dots, (x_n, y_n)$ and we want to find a polynomial of degree k which best fits the data points. Let $p(x) = a_kx^k + a_{k-1}x^{k-1} + \dots + a_1x + a_0$ be the desired polynomial. We define the vectors

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \vec{\xi}_k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{pmatrix}, \quad \vec{\xi}_{k-1} = \begin{pmatrix} x_1^{k-1} \\ x_2^{k-1} \\ \vdots \\ x_n^{k-1} \end{pmatrix}, \quad \dots, \quad \vec{\xi}_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{\xi}_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

If the vectors $\vec{\xi}_k, \dots, \xi_0$ are linearly independent, then

$$\begin{pmatrix} a_k \\ \vdots \\ a_1 \\ a_0 \end{pmatrix} = (A^t A)^{-1} A^t \vec{y}$$

where $A = (\vec{\xi}_k \mid \dots \mid \vec{\xi}_0)$ is the $n \times (k+1)$ matrix whose columns are the vectors $\vec{\xi}_k, \dots, \vec{\xi}_0$. Note that by our assumption $k < n$ (otherwise the vectors $\vec{\xi}_k, \dots, \vec{\xi}_0$ cannot be linearly independent).

Remark. Generally one should have much more data points than the degree of the polynomial one wants to fit; otherwise the problem of overfitting might occur. For example, assume that the curve we are looking for is $f(x) = 0.1 + 0.2x$. Then a linear fit would give us $g(x) = \frac{2}{7}x + \frac{1}{28} \approx 0.23x + 0.036$. The fit with a quadratic function gives $p(x) = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}$ which matches the data points perfectly but is far away from the curve we look for. The reason is that we have too many free parameters in the polynomial so the fit the data too well. (Note that for any three given $n+1$ points $(x_1, y_1), \dots, (x_{n+1}, y_{n+1})$ with $x_1 z \dots, x_{n+1}$, there exists exactly one polynomial p of degree $\leq n$ such that $p(x_j) = y_j$ for every $j = 1, \dots, n+1$.) If we had a lot more data points and we tried to fit a polynomial to a linear function, then the leading coefficient should become very small but this effect does not appear if we have very few data points.

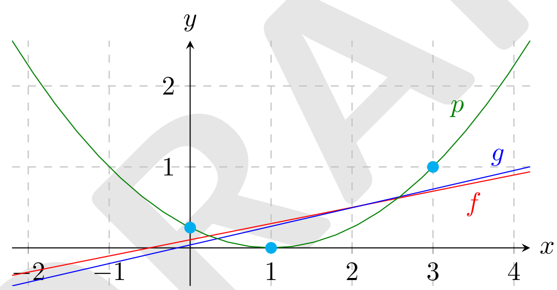


FIGURE 7.7: Example of overfitting when we have too many free variables for a given set of data points. The dots mark the measured points which are supposed to approximate the red curve f. Fitting polynomial p of degree 2 leads to the green curve. The blue curve g is the result of a linear fit.

Exponential functions.. Assume we are given measured data $(x_1, y_1), \dots, (x_n, y_n)$ and we want to find a function of form $g(x) = c\,\mathrm{e}^{kx}$ to fit our data point. Without restriction we may assume that $c > 0$ (otherwise we fit $-g$).

Then we only need to define $h(x) = \ln(g(x)) = \ln c + kx$. Then we use the method to fit a linear function to the data points $(x_1, \ln(y_1)), \dots, (x_n, \ln(y_n))$ in order to obtain c and k.

Remark 7.50. Let us show how the formula in Theorem 7.49 can be derived with analytic methods. Recall that the problem is the following: Let $(x_1, y_1), \dots, (x_n, y_n)$ be given. Find a linear function

$g(x) = ax + b$ which minimises the total error

$$\Delta := \Big[\sum_{j=1}^{n} (y_j - g(x_j))^2 \Big]^{\frac{1}{2}} = \Big[\sum_{j=1}^{n} (y_j - [ax_j + b])^2 \Big]^{\frac{1}{2}}$$

Let us consider Δ as function of a and b. Then we have to find the minimum of

$$\Delta(a, b) = \Big[\sum_{j=1}^{n} (y_j - [ax_j + b])^2 \Big]^{\frac{1}{2}}$$

as a function of the two variables a, b. In order to simplify the calculations a bit, we observe that it is enough to minimise the square of Δ since $\Delta(a, b) \geq 0$ for all a, b, and therefore it is minimal if and only if its square is minimal. So we want to find a, b which minimise

$$F(a, b) := (\Delta(a, b))^2 = \sum_{j=1}^{n} (y_j - ax_j - b)^2. \tag{7.17}$$

To this end, we have to derive F. Since $F : \mathbb{R}^2 \to \mathbb{R}$, the derivative will be vector valued function. We find

$$\begin{aligned} DF(a, b) &= \Big(\frac{\partial F}{\partial a}(a, b),\ \frac{\partial F}{\partial b}(a, b) \Big) = \Big(\sum_{j=1}^{n} -2x_j(y_j - ax_j - b),\ \sum_{j=1}^{n} -2(y_j - ax_j - b) \Big) \\ &= 2\Big(a \sum_{j=1}^{n} x_j^2 + b \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j y_j,\ a \sum_{j=1}^{n} x_j + nb - \sum_{j=1}^{n} y_j \Big). \end{aligned}$$

Now we need to find the critical points, that is, a, b such that $DF(a, b) = 0$. This is the case for

$$\left\{ \begin{aligned} a \sum_{j=1}^{n} x_j^2 + b \sum_{j=1}^{n} x_j &= \sum_{j=1}^{n} x_j y_j \\ a \sum_{j=1}^{n} x_j + bn \qquad &= \sum_{j=1}^{n} y_j \end{aligned} \right\} \quad \text{that is} \quad \begin{pmatrix} \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j \\ \sum_{j=1}^{n} x_j & n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} x_j y_j \\ \sum_{j=1}^{n} y_j \end{pmatrix}. \tag{7.18}$$

Now we can multiply on both sides from the left by the inverse of the matrix and obtain the solution for a, b. This shows that F has only one critical point. Since F tends to infinity for $\|(a, b)\| \to \infty$, the function F must indeed have a minimum in this critical point. For details, see a course on vector calculus or optimisation.

We observe the following: If, as before, we set

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad A = (\vec{x} \,|\, \vec{u}) = \begin{pmatrix} x_1 & 1 \\ \vdots & \\ x_n & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

then

$$\sum_{j=1}^n x_j^2 = \langle \vec{x}\,, \vec{x}\rangle, \qquad \sum_{j=1}^n x_j = \langle \vec{x}\,, \vec{u}\rangle, \qquad n = \langle \vec{u}\,, \vec{u}\rangle, \qquad \sum_{j=1}^n x_j y_j = \langle \vec{x}\,, \vec{y}\rangle, \qquad \sum_{j=1}^n y_j = \langle \vec{u}\,, \vec{y}\rangle.$$

Therefore the expressions in equation (7.18) can be rewritten as

$$\begin{pmatrix} \sum_{j=1}^n x_j^2 & \sum_{j=1}^n x_j \\ \sum_{j=1}^n x_j & n \end{pmatrix} = \begin{pmatrix} \langle \vec{x}\,, \vec{x}\rangle & \langle \vec{x}\,, \vec{u}\rangle \\ \langle \vec{u}\,, \vec{x}\rangle & \langle \vec{u}\,, \vec{u}\rangle \end{pmatrix} = \begin{pmatrix} \vec{x} \\ \vec{u} \end{pmatrix} (\vec{x} \,|\, \vec{u}) = A^t A$$

$$\begin{pmatrix} \sum_{j=1}^n x_j y_j \\ \sum_{j=1}^n y_j \end{pmatrix} = \begin{pmatrix} \langle \vec{x}\,, \vec{y}\rangle \\ \langle \vec{u}\,, \vec{y}\rangle \end{pmatrix} = \begin{pmatrix} \vec{x} \\ \vec{u} \end{pmatrix} \vec{y} = A^t \vec{y}$$

and we recognise that equation (7.18) is the same as

$$A^t A \begin{pmatrix} a \\ b \end{pmatrix} = A^t \vec{y}$$

which becomes our equation (7.16) if we multiply both sides of the equation from the left by $(A^t A)^{-1}$.

You should now have understood

- what the least square method is,
- how it is related to orthogonal projections,
- what overfitting is,
- . . . ,

You should now be able to

- fit a linear function to given data points,
- fit a polynomial to given data points,
- fit an exponential function to given data points,
- . . .

7.7 Summary

Let U be a subspace of $\mathbb{R}^n$. Then its orthogonal complement is defined by

$$U^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{x} \perp \vec{u} \ \text{ for all } \ \vec{u} \in U\}.$$

For any subspace $U \subseteq \mathbb{R}^n$ the following is true:

- $U^\perp$ is a vector space.
- $U^\perp = \ker A$ where A is any matrix whose rows are the formed by a basis of U.

- $(U^\perp)^\perp = U$.
- $\dim U + \dim U^\perp = n$.
- $U \oplus U^\perp = \mathbb{R}^n$.
- U has an orthonormal basis. One way to construct such a basis is to first construct an arbitrary basis of U and then apply the Gram-Schmidt orthogonalisation process to obtain an orthonormal basis.

Orthogonal projection onto a subspace $U \subseteq \mathbb{R}^n$

Let $P_U : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto U. Then

- P_U is a linear transformation.
- $P_U\vec{x} \parallel U$ for every $\vec{x} \in \mathbb{R}^n$.
- $\vec{x} - P_U\vec{x} \perp U$ for every $\vec{x} \in \mathbb{R}^n$.
- For every $\vec{x} \in \mathbb{R}^n$ the point in U nearest to $\vec{x}$ is given by $\vec{x} - P_U\vec{x}$ and $\operatorname{dist}(\vec{x}, U) = \|\vec{x} - P_U\vec{x}\|$.
- Formulas for P_U:
  - If $\vec{u}_1, \dots, \vec{u}_k$ is a basis of U, then
  $$P_U = \langle \vec{u}_1 \,, \cdot\rangle + \cdots + \langle \vec{u}_k \,, \cdot\rangle,$$
  that is $P_U\vec{x} = \langle \vec{u}_1 \,, \vec{x}\rangle + \cdots + \langle \vec{u}_k \,, \vec{x}\rangle$ for every $\vec{x} \in \mathbb{R}^n$.
  - $P_U = B(B^tB)B^t$ where B is any matrix whose columns form a basis of U.

Orthogonal matrices

A matrix $Q \in M(n \times n)$ is called an orthogonal matrix if it is invertible and if $Q^{-1} = Q^t$. Note that the following assertions for a matrix $Q \in M(n \times n)$ are equivalent:

(i) Q is an orthogonal matrix.

(ii) Q^t is an orthogonal matrix.

(iii) Q^{-1} is an orthogonal matrix.

(iv) The columns of Q are an orthonormal basis of $\mathbb{R}^n$.

(v) The rows of Q are an orthonormal basis of $\mathbb{R}^n$.

(vi) Q preserves inner products, that is $\langle \vec{x} \,, \vec{y}\rangle = \langle Q\vec{x} \,, Q\vec{y}\rangle$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

(vii) Q preserves lengths, that is $\|\vec{x}\| = \|Q\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$.

Every orthogonal matrix represents either a rotation (in this case its determinant is 1) or a composition of a rotation with a reflection (in this case its determinant is -1).

7.8 Exercises

1. (a) Complete $\begin{pmatrix} 1/4 \\ \sqrt{15/16} \end{pmatrix}$ a una base ortonormal para $\mathbb{R}^2$. ¿Cuántas posibilidades hay para hacerlo?

   (b) Complete $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ a una base ortonormal para $\mathbb{R}^3$. ¿Cuántas posibilidades hay para hacerlo?

   (c) Complete $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ a una base ortonormal para $\mathbb{R}^3$. ¿Cuántas posibilidades hay para hacerlo?

2. Encuentre una base para el complemento ortogonal de los siguientes espacios vectoriales. Encuentre la dimensión del espacio y la dimensión de su complemento ortogonal.

   (a) $U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\} \subseteq \mathbb{R}^4$, (b) $U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\} \subseteq \mathbb{R}^4$.

3. (a) Sea $U = \{(x, y, z)^t \in \mathbb{R}^3 : x + 2y + 3z = 0\} \subseteq \mathbb{R}^3$.

   (i) Sea $\vec{v} = (0, 2, 5)^t$. Ecuentre el punto $\vec{x} \in U$ que esté más cercano a $\vec{v}$ y calcule la distancia entre $\vec{v}$ y $\vec{x}$.

   (ii) ¿Hay un punto $\vec{y} \in U$ que esté a una distancia máximal de $\vec{v}$?

   (iii) Encuentre la matriz que representa la proyección ortogonal sobre U (en la base estandar).

   (b) Sea $W = \operatorname{gen}\{(1, 1, 1, 1)^t,\ (2, 1, 1, 0)^t\} \subseteq \mathbb{R}^4$.

   (i) Encuentre una base ortogonal de W.

   (ii) Sean $\vec{a}_1 = (1, 2, 0, 1)^t$, $\vec{a}_2 = (11, 4, 4, -3)^t$, $\vec{a}_3 = (0, -1, -1, 0)^t$. Para cada $j = 1, 2, 3$ encuentre el punto $\vec{w}_j \in W$ que esté más cercano a $\vec{a}_j$ y calcule la distancia entre $\vec{a}_j$ y $\vec{w}_j$.

   (iii) Encuentre la matriz que representa la proyección ortogonal sobre W (en la base estandar).

4. Sean $\vec{v} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$, $\vec{a} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

   (a) Demuestre que $\vec{v}$ y $\vec{w}$ son linealmente independientes y encuentre una base ortonormal de $U = \operatorname{span}\{\vec{v}, \vec{w}\} \subseteq \mathbb{R}^4$.

(b) Demuestre que $\vec{a}$, $\vec{b}$, $\vec{c}$ y $\vec{d}$ son linealmente independientes. Use el proceso de Gram-Schmidt para encontrar una base ortonormal de $U = \operatorname{span}\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\} \subseteq \mathbb{R}^5$. Encuentre una base de $U^\perp$.

5. Encuentre una base ortonormal de $U^\perp$ donde $U = \operatorname{gen}\{(1, 0, 2, 4)^t\} \subseteq \mathbb{R}^4$.

6. Una bola rueda a lo largo del eje x con velocidad constante. A lo largo de la trayectoria de la bola se miden las coordenadas x de la bola en ciertos tiempos t. Las siguientes mediciones son (t en segundos, x en metros):

x	1.5	2.0	3.0	4.0	4.5	6
t	1.4	2.3	4.7	6.6	7.4	10.8

(a) Dibuje los puntos en el plano tx.

(b) Use el método de mínimos cuadrados para econtrar la posición inicial x_0 y la velocidad v de la bola.

(c) Dibuje la recta en el bosquejo anterior. ¿Dónde/Cómo se ven x_0 y v?

Hint. Recuerde que $x(t) = x_0 + vt$ para un movimiento con velocidad constante.

7. Se supone que una sustancia química inestable decaye según la ley $P(t) = P_0\,\mathrm{e}^{kt}$. Suponga que se hicieron las siguientes mediciones:

t	1	2	3	4	5
P	7.4	6.5	5.7	5.2	4.9

Con el método de mínimos cuadrados aplicado a $\ln(P(t))$, encuentre P_0 y k que mejor corresponden con las mediciones. Dé una estimada para $P(8)$.

8. Con el método de mínimos cuadrados encuentre el polínomio $y = p(x)$ de grado 2 que mejor aproxima los siguientes datos:

x	-2	-1	0	1	2	3	4
y	15	8	2.8	-1.2	-4.9	-7.9	-8.7

9. Sea $n \in \mathbb{N}$ y sean $Q, T \in M(n \times n)$.

(a) Demuestre que T es una isometría si y solo si $\langle T\vec{x}, T\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$ para todo $\vec{x}, \vec{y} \in \mathbb{R}^n$ (es decir: una isometría mantiene ángulos).

(b) Demuestre que Q es una matriz ortogonal si y solo si Q es una isometría.

10. (a) Sea $\varphi \in \mathbb{R}$ y sean $\vec{v}_1 = \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix}$. Demuestre que $\vec{v}_1, \vec{v}_2$ es una base ortonormal de $\mathbb{R}^2$.

(b) Sea $\alpha \in \mathbb{R}$. Encuentre la matriz $Q(\alpha) \in M(2 \times 2)$ que describe rotación por α contra las manecillas del reloj.

(c) Sean $\alpha, \beta \in \mathbb{R}$. Explique por qué es claro que $Q(\alpha)Q(\beta) = Q(\alpha + \beta)$. Use esta relación para concluir las identidades trigonométricas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

11. Sean $O(n) = \{Q \in M(n \times n) : Q \text{ es matriz ortogonal}\}$ y $SO(n) = \{Q \in O(n) : \det Q = 1\}$.

(a) Demuestre que $O(n)$ con la composición es un grupo. Es decir, hay que probar que:

(i) Para todo $Q, R \in O(n)$, la composición QR es un elemento en $O(n)$.

(ii) Existe un $E \in O(n)$ tal que $QE = Q$ y $EQ = Q$ para todo $Q \in O(n)$.

(iii) Para todo $Q \in O(n)$ existe un elemento inverso $\widetilde{Q}$ tal que $\widetilde{Q}Q = Q\widetilde{Q} = E$.

(b) ¿Es $O(n)$ conmutativo (es decir, se tiene $QR = RQ$ para todo $Q, R \in O(n)$)?

(c) Demuestre que $SO(n)$ con la composición es un grupo.

12. Ses $T : \mathbb{R}^n \to \mathbb{R}^m$ una isometría. Demuestre que T es inyectivo y que $m \geq n$.

13. Sea V un espacio vectorial y sean $U, W \subseteq V$ subespacios.

(a) Demuestre que $U \cap W$ es un subespacio.

(b) Demuestre que $\dim U + W = \dim U + \dim V - \dim(U \cap W)$.

(c) Suponga que $U \cap W = \{0\}$. Demuestre que $\dim U \oplus W = \dim U + \dim V$.

(d) Demuestre que $U^\perp$ es un subespacio de V y que $(U^\perp)^\perp = U$.

Chapter 8

Symmetric matrices and diagonalisation

In this chapter we work mostly in $\mathbb{R}^n$ and in $\mathbb{C}^n$. We write $M_\mathbb{R}(n \times n)$ or $M_\mathbb{C}(n \times n)$ only if it is important if the matrix under consideration is a real or a complex matrix.

The first section is dedicated to $\mathbb{C}^n$. We already know that it is a vector space. But know we introduce an inner product on it. Moreover we define hermitian and unitary matrices on $\mathbb{C}^n$ which are analogous to symmetric and orthogonal matrices in $\mathbb{R}^n$. We define eigenvalues and eigenvectors in Section 8.3. It turns out that it is more convenient to work over $\mathbb{C}$ because the eigenvalues are zeros of the so-called characteristic polynomial and in $\mathbb{C}$ every polynomial has a zero. The main theorem is Theorem 8.48 which says that a $n \times n$ matrix is diagonalisable if it has enough eigenvectors to generate $\mathbb{C}^n$ (or $\mathbb{R}^n$). It turns out that every symmetric and every hermitian matrix is diagonalisable.

We end the chapter with an application of orthogonal diagonalisation to the solution of quadratic equations in two variables.

8.1 Complex vector spaces

In this section we introduce $\mathbb{C}^n$ as an inner product space because some calculations about eigenvalues later in this chapter are more natural in $\mathbb{C}^n$ than in $\mathbb{R}^n$. Most of this section may be skipped. The important part is the definition of the inner product on $\mathbb{C}^n$ and the notion of orthogonality derived from it, and the concept of hermitian and unitary matrices.

Similar as for $\mathbb{R}^n$, we define the *vector space* $\mathbb{C}^n$ to the set

$$\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \,:\, z_1, \dots, z_n \in \mathbb{C} \right\}$$

together with the sum and multiplication by a scalar $c \in \mathbb{C}$:

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} + \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} := \begin{pmatrix} w_1 + z_1 \\ \vdots \\ w_n + z_n \end{pmatrix}, \qquad c\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} := \begin{pmatrix} cz_1 \\ \vdots \\ cz_n \end{pmatrix}.$$

It is not hard to check that $\mathbb{C}^n$ together with these operations satisfies the vector space axioms from Definition 5.1 with $\mathbb{K} = \mathbb{C}$, hence it is a complex vector space. In particular, we have the concepts like linear independence of vectors, basis and dimension of $\mathbb{C}^n$, etc.

Next we introduce an inner product on $\mathbb{C}^n$. As in the case of real vectors, we would like to interprete $\langle \vec{z}\,, \vec{z}\rangle$ as the square of the norm of $\vec{z}$. In particular it should be a nonnegative real number. In particular, for $\mathbb{C}^1 = \mathbb{C}$, the vectors are just complex numbers $\vec{z} = z_1$ and we would like to have $\langle \vec{z}\,, \vec{z}\rangle = |z_1|^2 = z_1\overline{z_1}$ where $\overline{z}$ is the complex conjugate of the complex number z. This motivates us to define the inner product in $\mathbb{C}^n$ as follows.

Definition 8.1 (Inner product and norm of a vector in $\mathbb{C}^n$). For vectors $\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$ the *inner product* (or *scalar product* or *dot product*) is defined as

$$\langle \vec{z}\,, \vec{w}\rangle = \left\langle \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} z_j\overline{w_j} = z_1\overline{w_1} + \cdots + z_n\overline{w_n}.$$

The *length* of $\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$ is denoted by $\|\vec{z}\|$ and it is given by

$$\|\vec{z}\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}\ .$$

Other names for the length of $\vec{z}$ are *magnitude of* $\vec{z}$ or *norm of* $\vec{z}$.

Exercise 8.2. Show that the scalar product from Definition 8.1 can be viewed as an extension of the scalar product in $\mathbb{R}^n$ in the following sense: If the components of $\vec{z}$ and $\vec{v}$ happen to be real, then they can also be seen as vectors in $\mathbb{R}^n$. The claim is that their scalar product as vectors in $\mathbb{R}^n$ is equal to their scalar product in $\mathbb{C}^n$. The same is true for their norms.

Properties 8.3. (i) Norm of a vector: *For all vectors $\vec{z} \in \mathbb{C}^n$, we have that*

$$\langle \vec{z}\,, \vec{z}\rangle = \|\vec{z}\|^2.$$

(ii) Symmetry of the inner product: *For all vectors $\vec{v}, \vec{w} \in \mathbb{C}^n$, we have (note the complex conjugation on the right hand side!)*

$$\langle \vec{v}\,, \vec{w}\rangle = \overline{\langle \vec{w}\,, \vec{v}\rangle}.$$

(iii) Sesqulinearity of the inner product: *For all vectors $\vec{u}, \vec{v}, \vec{z} \in \mathbb{C}^n$ and all $c \in \mathbb{C}$, we have that*

$$\langle \vec{v} + c\vec{w}\,, \vec{z}\rangle = \langle \vec{v}\,, \vec{z}\rangle + c\langle \vec{w}\,, \vec{z}\rangle \quad \textit{and} \quad \langle \vec{v}\,, \vec{w} + c\vec{z}\rangle = \langle \vec{v}\,, \vec{w}\rangle + \overline{c}\langle \vec{v}\,, \vec{z}\rangle.$$

(iv) *For all vectors $\vec{v}, \vec{w} \in \mathbb{C}^n$ and $c \in \mathbb{C}$, we have that $\|c\vec{v}\| = |c|\|\vec{v}\|$.*

Proof. Let $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$, $\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n$ and let $c \in \mathbb{C}$.

(i) $\langle \vec{z}\,, \vec{z}\rangle = z_1\overline{z_1} + \cdots + z_n\overline{z_n} = |z_1|^2 + \cdots + |z_n|^2 = \|\vec{z}\|^2$.

(ii) $\langle \vec{v}\,, \vec{w}\rangle = v_1\overline{w_1} + \cdots + v_n\overline{w_n} = \overline{\overline{v_1}w_1 + \cdots + \overline{v_n}w_n} = \overline{w_1\overline{v_1} + \cdots + w_n\overline{v_n}} = \overline{\langle \vec{w}\,, \vec{v}\rangle}$.

(iii) A straightforward calculation shows

$$\begin{aligned} \langle \vec{v} + c\vec{w}\,, \vec{z}\rangle &= (v_1 + cw_1)\overline{w_1} + \cdots + (v_n + cw_n)\overline{w_n} \\ &= v_1\overline{w_1} + \cdots + v_n\overline{w_n} + cw_1\overline{w_1} + \cdots + cw_n\overline{w_n} \\ &= \langle \vec{v}\,, \vec{z}\rangle + c\langle \vec{w}\,, \vec{z}\rangle. \end{aligned}$$

The second equation can be shown by an analogous calculation. Instead of repeating them, we can also use the symmetry property of the inner product:

$$\langle \vec{v}\,, \vec{w} + c\vec{z}\rangle = \overline{\langle \vec{w} + c\vec{z}\,, \vec{v}\rangle} = \overline{\langle \vec{w}\,, \vec{v}\rangle + c\langle \vec{z}\,, \vec{v}\rangle} = \overline{\langle \vec{w}\,, \vec{v}\rangle} + \overline{c}\overline{\langle \vec{z}\,, \vec{v}\rangle} = \langle \vec{v}\,, \vec{z}\rangle + \overline{c}\langle \vec{v}\,, \vec{z}\rangle.$$

(iv) $\|c\vec{z}\|^2 = \langle c\vec{z}\,, c\vec{z}\rangle = c\overline{c}\langle \vec{z}\,, \vec{z}\rangle = |c|^2\|\vec{z}\|^2$. Taking the square root on both sides, we obtain the desired equality $\|c\vec{z}\| = |c|\|\vec{z}\|$. □

For $\mathbb{C}^n$ there is no cosine theorem and in general it does not make too much sense to speak about the angle between two complex vectors (orthogonality still makes sense!).

Definition 8.4. Let $\vec{z}, \vec{v} \in \mathbb{C}^n$.

(i) The vectors $\vec{z}, \vec{v}$ are called *orthogonal* or *perpendicular* if $\langle \vec{z}\,, \vec{v}\rangle = 0$. In this case we write $\vec{z} \perp \vec{v}$.

(ii) If $\vec{v} \neq \vec{0}$, then the *orthogonal projection of $\vec{z}$ onto $\vec{v}$* is $\operatorname{proj}_{\vec{v}} \vec{z} = \frac{\langle \vec{z}\,, \vec{v}\rangle}{\|\vec{v}\|^2}\vec{v}$.

The next proposition shows that orthogonality works $\mathbb{C}^n$ as expected.

Proposition 8.5. *Let $\vec{z}, \vec{v} \in \mathbb{C}^n$.*

(i) Pythagoras theorem: *If $\vec{z} \perp \vec{v}$, then $\|\vec{z} + \vec{v}\|^2 = \|\vec{z}\|^2 + \|\vec{v}\|^2$.*

(ii) *If $\vec{v} \neq \vec{0}$, then $\vec{z} = \vec{z}_\| + \vec{z}_\perp$ with $\vec{z}_\| := \operatorname{proj}_{\vec{v}} \vec{z}$ and $\vec{z}_\perp := \vec{z} - \operatorname{proj}_{\vec{v}} \vec{z}$ and*

$$\operatorname{proj}_{\vec{v}} \vec{z} \parallel \vec{v}, \quad \textit{and} \quad \vec{z} - \operatorname{proj}_{\vec{v}} \vec{z} \perp \vec{v}.$$

Moreover, $\|\operatorname{proj}_{\vec{v}} \vec{z}\| \leq \|\vec{z}\|$.

(iii) *If $\vec{v} \neq \vec{0}$, then $\mathbb{C}^n \to \mathbb{C}^n,\ \vec{z} \mapsto \operatorname{proj}_{\vec{v}} \vec{z}$ is a linear map.*

Proof. (i) If $\vec{z} \perp \vec{v}$, then $\|\vec{z}+\vec{v}\|^2 = \langle \vec{z}\,, \vec{z}\rangle + \langle \vec{z}\,, \vec{v}\rangle + \langle \vec{v}\,, \vec{z}\rangle + \langle \vec{v}\,, \vec{v}\rangle = \langle \vec{z}\,, \vec{z}\rangle + \langle \vec{v}\,, \vec{v}\rangle = \|\vec{z}\|^2 + \|\vec{v}\|^2$.

(ii) It is clear that $\vec{z} = \vec{z}_{\|} + \vec{z}_{\perp}$ and that $\vec{z}_{\|} \parallel \vec{v}$ by definition of $\vec{z}_{\|}$ and $\vec{z}_{\perp}$. That $\vec{z}_{\perp} \perp \vec{v}$ follows from

$$\langle \vec{z}_{\perp}\,, \vec{v}\rangle = \langle \vec{z} - \operatorname{proj}_{\vec{v}} \vec{z}\,, \vec{v}\rangle = \langle \vec{z}\,, \vec{v}\rangle - \langle \operatorname{proj}_{\vec{v}} \vec{z}\,, \vec{v}\rangle = \langle \vec{z}\,, \vec{v}\rangle - \frac{\langle \vec{z}\,, \vec{v}\rangle}{\|\vec{v}\|^2}\langle \vec{v}\,, \vec{v}\rangle = \langle \vec{z}\,, \vec{v}\rangle - \langle \vec{z}\,, \vec{v}\rangle = 0.$$

Finally, by the Pythagoras theorem,

$$\|\vec{z}\|^2 = \|(\vec{z} - \operatorname{proj}_{\vec{v}} \vec{z}) + \operatorname{proj}_{\vec{v}} \vec{z}\|^2 = \|\vec{z} - \operatorname{proj}_{\vec{v}} \vec{z}\|^2 + \|\operatorname{proj}_{\vec{v}} \vec{z}\|^2 \geq \|\operatorname{proj}_{\vec{v}} \vec{z}\|^2.$$

(iii) Assume that $\vec{v} \neq \vec{0}$ and let $\vec{z}_1, \vec{z}_2 \in \mathbb{C}^n$ and $c \in \mathbb{C}$. Then

$$\begin{aligned}\operatorname{proj}_{\vec{v}}(\vec{z}_1 + c\vec{z}_2) &= \frac{\langle z_1 + c\vec{z}_2\,, \vec{v}\rangle}{\|\vec{v}\|^2} = \frac{\langle z_1\,, \vec{v}\rangle + c\langle \vec{z}_2\,, \vec{v}\rangle}{\|\vec{v}\|^2} = \frac{\langle z_1\,, \vec{v}\rangle \vec{v}}{\|\vec{v}\|^2} = \frac{c\langle \vec{z}_2\,, \vec{v}\rangle}{\|\vec{v}\|^2} \\ &= \operatorname{proj}_{\vec{v}} \vec{z}_1 + c \operatorname{proj}_{\vec{v}} \vec{z}_2.\end{aligned}$$

□

Question 8.1

What changes if in the definition of the orthogonal projection we put $\langle \vec{v}\,, \vec{z}\rangle$ instead of $\langle \vec{z}\,, \vec{v}\rangle$?

Now let us show the triangle inequality. Note the the following inequalities (8.1) and (8.2) were proved for real vector spaces in Corollary 2.20 using the cosine theorem.

Proposition 8.6. *For all vectors $\vec{v}, \vec{w} \in \mathbb{C}^n$ and $c \in \mathbb{C}$, we have the* Cauchy-Schwarz inequality *(which is a special case of the so-called* Hölder inequality*)*

$$|\langle \vec{v}\,, \vec{w}\rangle| \leq \|\vec{v}\|\,\|\vec{w}\| \tag{8.1}$$

and the triangle inequality

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|. \tag{8.2}$$

Proof. We will first show (8.1). It is obviously true if $\vec{w} = \vec{0}$ because in this case both sides of the inequality are equal to 0. So let us assume now that $\vec{w} \neq \vec{0}$. Note that for any $\lambda \in \mathbb{C}$ we have that

$$0 \leq \|\vec{v} - \lambda\vec{w}\|^2 = \langle \vec{v} - \lambda\vec{w}\,, \vec{v} - \lambda\vec{w}\rangle = \|v\|^2 - \lambda\langle \vec{w}\,, \vec{v}\rangle - \overline{\lambda}\langle \vec{v}\,, \vec{w}\rangle + |\lambda|^2\|w\|^2.$$

If we chose $\lambda = -\frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}$, we obtain

$$\begin{aligned} 0 &\leq \|v\|^2 - \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\langle \vec{w}\,, \vec{v}\rangle - \frac{\overline{\langle \vec{v}\,, \vec{w}\rangle}}{\|\vec{w}\|^2}\langle \vec{v}\,, \vec{w}\rangle + \frac{|\langle \vec{v}\,, \vec{w}\rangle|^2}{\|\vec{w}\|^4}\,\|w\|^2 \\ &= \|v\|^2 - 2\frac{|\langle \vec{v}\,, \vec{w}\rangle|^2}{\|\vec{w}\|^2} + \frac{|\langle \vec{v}\,, \vec{w}\rangle|^2}{\|\vec{w}\|^4}\,\|w\|^2 \\ &= \|v\|^2 - \frac{|\langle \vec{v}\,, \vec{w}\rangle|^2}{\|\vec{w}\|^2} = \frac{1}{\|\vec{w}\|^2}\Big[\|v\|^2\|w\|^2 - |\langle \vec{v}\,, \vec{w}\rangle|^2\Big] \end{aligned}$$

It follows that $\|v\|^2\|w\|^2 - |\langle \vec{v}\,, \vec{w}\rangle|^2 \geq 0$, hence $\|v\|^2\|w\|^2 \geq |\langle \vec{v}\,, \vec{w}\rangle|^2$. We obtain the desired inequality by taking the square root.

Now let us show the triangle inequality. It is essentially the same the for vectors in $\mathbb{R}^n$, cf. Corollary 2.20.

$$\begin{aligned}
\|\vec{v}+\vec{w}\|^2 &= \langle \vec{v}+\vec{w}\,, \vec{v}+\vec{w}\rangle = \langle \vec{v}\,, \vec{v}\rangle + \langle \vec{v}\,, \vec{w}\rangle + \langle \vec{w}\,, \vec{v}\rangle + \langle \vec{w}\,, \vec{w}\rangle \\
&= \langle \vec{v}\,, \vec{v}\rangle + \langle \vec{v}\,, \vec{w}\rangle + \overline{\langle \vec{v}\,, \vec{w}\rangle} + \langle \vec{w}\,, \vec{w}\rangle \\
&= \|\vec{v}\|^2 + 2\operatorname{Re}\langle \vec{v}\,, \vec{w}\rangle + \|\vec{w}\|^2 \\
&\leq \|\vec{v}\|^2 + 2|\langle \vec{v}\,, \vec{w}\rangle| + \|\vec{w}\|^2 \leq \|\vec{v}\|^2 + 2\|\vec{v}\|\,\|\vec{w}\| + \|\vec{w}\|^2 = (\|\vec{v}\| + \|\vec{w}|)^2.
\end{aligned}$$

In the first inequality we used that $\operatorname{Re} a \leq |a|$ for any complex number a and in the second inequality we used (8.1). If we take the square root on both sides we get the triangle inequality. □

Remark 8.7. Observe that the choice of λ in the proof of (8.1) is not as arbitrary as it may seem. Note that for this particular λ

$$\vec{v} - \lambda\vec{w} = \vec{v} - \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\,\vec{w} = \vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}.$$

Hence this choice of λ minimises the norm of $\vec{v}-\lambda\vec{w}$ and $\vec{v}-\operatorname{proj}_{\vec{w}}\vec{v} \perp \vec{w}$. Therefore, by Pythagoras,

$$\begin{aligned}
\|\vec{v}\|^2 &= \|(\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}) + \operatorname{proj}_{\vec{w}} \vec{v}\|^2 = \|\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}\|^2 + \|\operatorname{proj}_{\vec{w}} \vec{v}\|^2 \\
&\geq \|\operatorname{proj}_{\vec{w}} \vec{v}\|^2 = \left\| \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\,\vec{w} \right\|^2 = \frac{|\langle \vec{v}\,, \vec{w}\rangle|^2}{\|\vec{w}\|^2}
\end{aligned}$$

which shows that $\|\vec{v}\|^2\|\vec{w}\|^2 \geq |\langle \vec{v}\,, \vec{w}\rangle|^2$.
Another way to see this inequality is

$$\begin{aligned}
0 &\leq \|\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}\|^2 = \langle \vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}\,,\ \vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}\rangle = \langle \vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}\,, \vec{v}\rangle = \|\vec{v}\|^2 - \langle \operatorname{proj}_{\vec{w}} \vec{v}\,, \vec{v}\rangle \\
&= \|\vec{v}\|^2 - \frac{\langle \vec{v}\,, \vec{w}\rangle}{\|\vec{w}\|^2}\langle \vec{w}\,, \vec{v}\rangle = \|\vec{v}\|^2 - \frac{|\langle \vec{v}\,, \vec{w}\rangle|^2}{\|\vec{w}\|^2}
\end{aligned}$$

which again gives $\|\vec{v}\|^2\|\vec{w}\|^2 \geq |\langle \vec{v}\,, \vec{w}\rangle|^2$.

Important classes of matrices

Recall that for a matrix $A \in M_{\mathbb{R}}(m \times n)$ we defined its transpose A^t. The important property of A^t is that it is the unique matrix such that

$$\langle A\vec{x}\,, \vec{y}\rangle = \langle \vec{x}\,, A^t\vec{y}\rangle \qquad \text{for all } \ \vec{x} \in \mathbb{R}^n,\ \vec{y} \in \mathbb{R}^m.$$

In the complex case, we want for a given matrix $A \in M_{\mathbb{C}}(m \times n)$ a matrix A^* such that

$$\langle A\vec{x}\,, \vec{y}\rangle = \langle \vec{x}\,, A^*\vec{y}\rangle \qquad \text{for all } \ \vec{x} \in \mathbb{C}^n,\ \vec{y} \in \mathbb{C}^m.$$

It is easy to check that we have to take $A^* = \overline{A}^t$, where $\overline{A}$ is the matrix we obtain from A by taking the complex conjugate of every entry. Clearly, if all entries in A are real numbers, then $A^t = A^*$.

Definition 8.8. The matrix A^* is called the *adjoint matrix* of A.

Lemma 8.9. *Let $A \in M(n \times n)$. Then $\det A^* = \overline{\det A}$ = complex conjugate of $\det A$.*

Proof. $\det A^* = \det(\overline{A})^t = \det \overline{A} = \overline{\det A}$. The last equality follows directly from the definition of the determinant. □

A matrix with real entries is symmetric if and only if $A = A^t$. The analogue for complex matrices are hermitian matrices.

Definition 8.10. A matrix $A \in M(n \times n)$ is called the *hermitian* if $A = A^*$.

Examples 8.11.

- $A = \begin{pmatrix} 1 & 2+3\mathrm{i} \\ 5 & 1-7I \end{pmatrix} \implies A^* = \begin{pmatrix} 1 & 5 \\ 2-3\mathrm{i} & 1+7I \end{pmatrix}$. The matrix A is not hermitian.
- $A = \begin{pmatrix} 1 & 2+3\mathrm{i} \\ 2-3\mathrm{i} & 5 \end{pmatrix} \implies A^* = \begin{pmatrix} 1 & 5 \\ 2-3\mathrm{i} & 1+7I \end{pmatrix}$. The matrix A is hermitian.

Exercise 8.12.

- Show that the entries on the diagonal of a hermitian matrix must be real.
- Show that the determinant of a hermitian matrix is a real number.

Another important class of real matrices are the orthogonal matrices. Recall that a matrix $Q \in M_{\mathbb{R}}(n \times n)$ is an orthogonal matrix if and only if $Q^t = Q^{-1}$. We saw that if Q is orthogonal, then its columns (or rows) form an orthonormal basis for $\mathbb{R}^n$ and that $|\det Q| = 1$, hence $\det Q = \pm 1$. The analogue in complex vector spaces are so-called unitary matrices.

Definition 8.13. A matrix $Q \in M(n \times n)$ is called *unitary* if $Q^* = Q^{-1}$.

It is clear from the definition that a matrix ! is unitary if and only if its columns (or rows) form an orthonormal basis for $\mathbb{C}^n$, cf. Theorem 7.12.

Proposition 8.14. *Let $Q \in M(n \times n)$.*

(i) *The following is equivalent:*

  (a) *Q is unitary.*
  (b) *$\langle Q\vec{x}\,, Q\vec{y}\rangle = \langle \vec{x}\,, \vec{y}\rangle$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.*
  (c) *$\|Q\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$.*

(ii) *If Q is unitary, then $|\det Q| = 1$.*

Proof. (i) (a) $\implies$ (b): Assume that Q is a unitary matrix and let $\vec{x}$, $\vec{y} \in \mathbb{C}^n$. Then

$$\langle Q\vec{x}\,, Q\vec{y}\rangle = \langle Q^*Q\vec{x}\,, \vec{y}\rangle = \langle \vec{x}\,, \vec{y}\rangle.$$

(b) $\implies$ (a): Fix $\vec{x} \in \mathbb{C}^n$. Then we have $\langle Q\vec{x}\,, Q\vec{y}\rangle = \langle \vec{x}\,, \vec{y}\rangle$ for all $\vec{y} \in \mathbb{C}^n$, hence

$$0 = \langle Q\vec{x}\,, Q\vec{y}\rangle - \langle \vec{x}\,, \vec{y}\rangle = \langle Q^*Q\vec{x}\,, \vec{y}\rangle - \langle \vec{x}\,, \vec{y}\rangle = \langle Q^*Q\vec{x} - \vec{x}\,, \vec{y}\rangle. = \langle (Q^*Q - \mathrm{id})\vec{x}\,, \vec{y}\rangle.$$

Since this is true for any $\vec{y} \in \mathbb{C}^n$, it follows that $(Q^*Q - \mathrm{id})\vec{x} = 0$. Since $\vec{x} \in \mathbb{C}^n$ was arbitrary, we conclude that $Q^*Q - \mathrm{id} = 0$, in other words, that $Q^*Q = \mathrm{id}$.

(b) $\Longrightarrow$ (c): It follows from (b) that $\|Q\vec{x}\|^2 = \langle Q\vec{x}\,, Q\vec{x}\rangle = \langle \vec{x}\,, \vec{x}\rangle = \|\vec{x}\|^2$, hence $\|Q\vec{x}\| = \|\vec{x}\|$.

(c) $\Longrightarrow$ (b): Observe that the inner product of two vectors in $\mathbb{C}^n$ can be expressed completely in terms of norms as follows

$$\langle \vec{a}\,, \vec{b}\rangle = \frac{1}{4}\Big[\|\vec{a}+\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2 + \mathrm{i}\|\vec{a}+\mathrm{i}\vec{b}\|^2 - \mathrm{i}\|\vec{a}-\mathrm{i}\vec{b}\|^2\Big]$$

as can be easily verified. Hence we find

$$\begin{aligned}
\langle Q\vec{x}\,, Q\vec{y}\rangle &= \frac{1}{4}\Big[\|Q\vec{x}+Q\vec{y}\|^2 - \|Q\vec{x}-Q\vec{y}\|^2 + \mathrm{i}\|Q\vec{x}+\mathrm{i}Q\vec{y}\|^2 - \mathrm{i}\|Q\vec{x}-\mathrm{i}Q\vec{y}\|^2\Big] \\
&= \frac{1}{4}\Big[\|Q(\vec{x}+\vec{y})\|^2 - \|Q(\vec{x}-\vec{y})\|^2 + \mathrm{i}\|Q(\vec{x}+\mathrm{i}\vec{y})\|^2 - \mathrm{i}\|Q(\vec{x}-\mathrm{i}\vec{y})\|^2\Big] \\
&= \frac{1}{4}\Big[\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 + \mathrm{i}\|\vec{x}+\mathrm{i}\vec{y}\|^2 - \mathrm{i}\|\vec{x}-\mathrm{i}\vec{y}\|^2\Big] \\
&= \langle \vec{x}\,, \vec{y}\rangle.
\end{aligned}$$

(ii) Assume that Q is unitary. Then

$$1 = \det \mathrm{id} = \det QQ^* = (\det Q)(\det Q^*) = (\det Q)\bar{(}\det Q) = |\det Q|^2. \qquad \square$$

Examples 8.15.

- The matrix $Q = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ is unitary because $QQ^* = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, hence $Q^* = Q^{-1}$. Note that $\det Q = -\mathrm{i}^2 = 1$.
- The matrix $Q = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{\beta} \end{pmatrix}$ is unitary because $QQ^* = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\beta} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, hence $Q^* = Q^{-1}$. Note that $\det Q = \mathrm{e}^{\mathrm{i}(\alpha+\beta)}$, hence $|\det Q| = 1$.

You should now have understood

- the vector space structure of $\mathbb{C}^n$,
- the inner product on $\mathbb{C}^n$,
- that the concept of orthogonality makes sense in $\mathbb{C}^n$ and works as in $\mathbb{R}^n$,
- why hermitian matrices in $\mathbb{C}^n$ play the role of symmetric matrices in $\mathbb{R}^n$,
- why unitary matrices in $\mathbb{C}^n$ play the role of orthogonal matrices in $\mathbb{R}^n$,
- ...,

You should now be able to

- calculate with vectors in $\mathbb{C}^n$,
- check if vectors in $\mathbb{C}^n$ are orthogonal,

- calculate the orthogonal projection of one vector onto another,
- check if a given matrix is hermitian,
- check if a given matrix is unitary,
- ...

8.2 Similar matrices

:

Definition 8.16. Let $A, B \in M(n \times n)$ be (real or complex) matrices. They are called *similar* if there exists an invertible matrix C such that

$$A = C^{-1}BC. \tag{8.3}$$

In this case, we write $A \sim B$.

Exercise 8.17. Show that $A \sim B$ if and only if there exists an invertible matrix $\widetilde{C}$ such that

$$A = \widetilde{C}B\widetilde{C}^{-1}. \tag{8.4}$$

Question 8.2

Assume that A and B are similar. Is the matrix C in (8.3) unique or is it possible that there are different invertible matrices $C_1 \neq C_2$ such that $A = C_1^{-1}BC_1 = C_2^{-1}BC_2$?

Remark 8.18. Similarity is an *equivalence relation* on the set of all square matrices. This means that it satisfies the following three properties. Let $A_1, A_2, A_3 \in M(n \times n)$. Then:

(i) *Reflexivity:* $A \sim A$ for every $A \in M(n \times n)$.

(ii) *Symmetry:* If $A_1 \sim A_2$, then also $A_2 \sim A_1$.

(iii) *Transitivity:* If $A_1 \sim A_2$ and $A_2 \sim A_3$, then also $A_1 \sim A_3$.

Proof. (i) Reflexivity is clear. We only need to choose $C = \mathrm{id}$.

(ii) Assume that $A_1 \sim A_2$. Then there exists an invertible matrix C such that $A_1 = C^{-1}A_2C$. Multiplication from the left by C and from the right by C^{-1} gives $CA_1C^{-1} = A_2$. Let $\widetilde{C} = C^{-1}$. Then $\widetilde{C}$ is invertible and $\widetilde{C}^{-1} = C$. Hence we obtain $\widetilde{C}^{-1}A_1\widetilde{C} = A_2$ which shows that $A_2 \sim A_1$.

(iii) *Transitivity:* If $A_1 \sim A_2$ and $A_2 \sim A_3$, then there exist invertible matrices C_1 and C_2 such that $A_1 = C_1^{-1}A_2C_1$ and $A_2 = C_2^{-1}A_3C_2$. It follows that

$$A_1 = C_1^{-1}A_2C_1 = C_1^{-1}C_2^{-1}A_3C_2C_1 = (C_1C_2)^{-1}A_3C_1C_2.$$

Setting $C = C_1C_2$ shows that $A_1 = C^{-1}A_3C$, hence $A_1 \sim A_3$. □

We can interpret $A \sim B$ as follows: Let C be an invertible matrix with $A = C^{-1}BC$. Since C is an invertible matrix, its columns $\vec{c}_1, \dots, \vec{c}_n$ form a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and we can view C as the transition matrix from the canonical basis to the basis $\vec{c}_1, \dots, \vec{c}_n$. Since B is the matrix representation of the map $\vec{x} \mapsto B\vec{x}$ with respect to the canonical basis of $\mathbb{R}^n$, the equation $A = C^{-1}BC$ says that A represents the same linear map but with respect to the basis $\vec{c}_1, \dots, \vec{c}_n$.
On the other hand, if A and B are matrix representations of the same linear transformation but with respect to possibly different bases, then $A = C^{-1}BC$ where C is the transition matrix between the two bases. Hence A and B are similar.
So we showed:

Two matrices A and $B \in M(n \times n)$ are similar if and only if they represent the same linear transformation. The matrix C in $A = C^{-1}BC$ is the transition matrix between the two bases used in the representations A and B.

Hence the following fact is not very surprising.

Proposition 8.19. *If $A, B \in M(n \times n)$ are similar, then* $\det A = \det B$.

Proof. Let $C \in M(n \times n)$ invertible such that $A = C^{-1}BC$. Then

$$\det A = \det C^{-1}BC = \det(C^{-1}) \det B \det C = (\det C)^{-1} \det B \det C = \det B. \qquad \square$$

Exercise 8.20. Show that $\det A = \det B$ does not imply that A and B are similar.

Exercise 8.21. Assume that A and B are similar. Show that $\dim(\ker A) = \dim(\ker B)$ and that $\dim(\operatorname{Im} A) = \dim(\operatorname{Im} B)$. Why is this no surprise?

Question 8.3

Assume that A and B are similar. What is the relation between $\ker A$ and $\ker B$? What is the relation between $\operatorname{Im} A$ and $\operatorname{Im} B$?
Hint. Theorem 6.18.

A very nice class of matrices are the diagonal matrices because it is rather easy to calculate with them. Closely are the so-called diagonalisable matrices.

Definition 8.22. A matrix $A \in M(n \times n)$ is called *diagonalisable* if it is similar to a diagonal matrix.

In other words, A is diagonalisable if there exists a diagonal matrix D and an invertible matrix C with

$$C^{-1}AC = D. \tag{8.5}$$

How can we decide if a matrix A is diagonalisable? We know that it is diagonalisable if and only if it similar to a diagonal matrix, that is, if and only if there exists a basis $\vec{c}_1, \dots, \vec{c}_n$ such that the representation of A with respect to these vectors is a diagonal matrix. In this case, (8.5) is satisfied if the columns of C are the basis vectors $\vec{c}_1, \dots, \vec{c}_n$.

Denote the diagonal entries of D by $d_1, \dots, d_n$. Then it easy to see that $D\vec{e}_j = d_j\vec{e}_j$. This means that if we apply D to some $\vec{e}_j$, then the image $D\vec{e}_j$ is parallel to $\vec{e}_j$. Since D is nothing else than the representation of A with respect to the basis $\vec{c}_1, \dots, \vec{c}_n$, we have $A\vec{c}_j = d_j\vec{c}_j$.
We can make this more formal: Take equation (8.5) and multiply both sides from the left by C so that we obtain $AC = CD$. Recall that for any matrix B, we have that $B\vec{e}_j = j$th column of B. If we obtain

$$\begin{aligned} AC\vec{e}_j &= A\vec{c}_j, \\ CD\vec{e}_j &= C(d_j\vec{e}_j) = d_jC(\vec{e}_j) = d_j\vec{c}_j. \end{aligned} \qquad \xRightarrow{AC=CD} \qquad A\vec{c}_j = d\vec{c}_j.$$

In summary, we found:

A matrix $A \in M(n \times n)$ is diagonalisable if and only we can find a basis $\vec{c}_1, \dots, \vec{c}_n$ of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and numbers $d_1, \dots, d_n$ such that

$$A\vec{c}_j = d_j\vec{c}_j, \qquad j = 1, \dots, n.$$

In this case $C^{-1}AC = D$ (or equivalently $A = CDC^{-1}$) where $D = \operatorname{diag}(d_1, \dots, d_n)$ and $C = (\vec{c}_1|\cdots|\vec{c}_n)$.

The vectors $\vec{c}_j$ are called eigenvectors of A and the numbers d_j are called eigenvalues of A.o They will be discussed in greater detail in the next section where we will also see how we can calculate them.
Diagonalization of a matrix is very useful when we want to calculate powers of the matrix.

Proposition 8.23. *Let $A \in M(n \times n)$ be a diagnalizable matrix and let C be an invertible matrix and $D = \operatorname{diag}(d_1, \dots, d_n$ such that $A = CDC^{-1}$. Then $A^k = C\operatorname{diag}(d_1^k, \dots, d_n^k)C^{-1}$ for all $k \in \mathbb{N}_0$. If A is invertible, tne all d_j are different from 0 and the formula is true for all $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{N}_0$. Then

$$\begin{aligned} A^k &= \big(C\operatorname{diag}(d_1, \dots, d_n)C^{-1}\big)^k \\ &= C\operatorname{diag}(d_1, \dots, d_n)C^{-1}C\operatorname{diag}(d_1, \dots, d_n)C^{-1}\cdots C\operatorname{diag}(d_1, \dots, d_n)C^{-1} \\ &= C\operatorname{diag}(d_1, \dots, d_n)\operatorname{diag}(d_1, \dots, d_n)\cdots\operatorname{diag}(d_1, \dots, d_n)C^{-1} \\ &= C\big(\operatorname{diag}(d_1, \dots, d_n)\big)^kC^{-1} \\ &= C\operatorname{diag}(d_1^k, \dots, d_n^k)C^{-1} \end{aligned}$$

. If all $d_j \neq 0$, then D is invertible with inverse $D^{-1} = \big(\operatorname{diag}(d_1, \dots, d_n)\big)^{-1} = \operatorname{diag}(d_1^{-1}, \dots, d_n^{-1})$. Hence A is invertible and $A^{-1} = \big(CDC^{-1}\big)^{-1} = CD^{-1}C^{-1}$ and we obtain for $k \in \mathbb{Z}$ with $k < 0$

$$\begin{aligned} A^k &= A^{-|k|} = (A^{-1})^{|k|} = \big(CD^{-1}C^{-1}\big)^{|k|} = C(D^{-1})^{|k|}C^{-1} = CD^kC^{-1} = CD^{-|k|}C^{-1} \\ &= C\operatorname{diag}(d_1^k, \dots, d_n^k)C^{-1}. \end{aligned}$$ □

Proposition 8.23 is useful for example when we describe dynamical systems by matrices or when we solve linear differential equations with constant coefficients in higher dimensions.

You should now have understood

- that similar matrices represent the same linear transformation,
- why similar matrices have the same determinant,
- why a matrix is diagonalisable if and only if $\mathbb{R}^n$ (or $\mathbb{C}^n$) admits a basis consisting of eigenvectors of A,
- $\dots$,

You should now be able to

- $\dots$

8.3 Eigenvalues and eigenvectors

Definition 8.24. Let V be a vector space and let $T : V \to V$ be linear transformation. A number λ is called an *eigenvalue* of T if there exists a vector $\vec{v} \neq \vec{0}$ such that

$$Tv = \lambda v. \tag{8.6}$$

The vector v is then called a *eigenvector*.

The reason why we exclude $v = \mathbb{O}$ in the definition above is because for every λ it is true that $T\mathbb{O} = \mathbb{O} = \lambda\mathbb{O}$, so (8.8) would be satisfied for any λ if we were allowed to choose $v = \mathbb{O}$, in which case the definition would not make too much sense.

Exercise 8.25. Show that 0 is an eigenvalue of T if and only if $\dim(\ker T) \geq 1$, that is, if and only if T is not invertible. Show that v is an eigenvector with eigenvalue 0 if and only if $v \in \ker T \setminus \{\mathbb{O}\}$.

Exercise 8.26. Show that all eigenvalues of a unitary matrix have norm 1.

Question 8.4

Let V, W be vector spaces and let $T : V \to W$ be a linear transformation. Why does in **not** make sense to speak of eigenvalues of T if $V \neq W$?

Let us list some properties of eigenvectors that are easy to see.

(i) A vector v is an eigenvector of T if and only if $Tv \parallel v$.

(ii) If v is an eigenvector of T with eigenvalue $\lambda \neq 0$, then $v \in \operatorname{Im} T$ because $v = \frac{1}{\lambda}Tv$.

(iii) If v is an eigenvector of T with eigenvalue λ, then every non-zero multiple of v is an eigenvector with the same eigenvalue because

$$T(cv) = cTv = c\lambda v = \lambda(cv).$$

(iv) We can generalise (iii) as follows: If $v_1, \dots, v_k$ are eigenvectors of T with the same eigenvalue λ, then every non-zero linear combination is an eigenvector with the same eigenvalue because

$$T(\alpha_1 v_1 + \dots \alpha_k v_k) = \alpha_1 T v_1 + \dots \alpha_k T v_k = \alpha_1 \lambda v_1 + \dots + \alpha_k \lambda v_k = \lambda(\alpha_1 v_1 + \dots + \alpha_k v_k).$$

(iv) says that the set of all eigenvectors with the same eigenvalue is almost a subspace. The only thing missing is the zero vector $\mathbb{O}$. This motivates the following definition.

Definition 8.27. Let V be a vector space and let $T : V \to V$ be a linear map with eigenvalue λ. Then the *eigenspace* of T corresponding to λ is

$$\begin{aligned} \operatorname{Eig}_\lambda(T) := \operatorname{Eig}(T, \lambda) &:= \{v \in V : v \text{ is eigenvector of } T \text{ with eigenvalue } \lambda\} \cup \{\mathbb{O}\} \\ &= \{v \in V : Tv = \lambda v\}. \end{aligned}$$

The dimension of $\operatorname{Eig}_\lambda(T)$ is called the *geometric multiplicity* of λ.

Proposition 8.28. *Let $T : V \to V$ be a linear map and let λ be an eigenvalue of T. Then*

$$\operatorname{Eig}_\lambda(T) = \ker(T - \lambda \operatorname{id}).$$

Proof. Let $v \in V$. Then

$$\begin{aligned} v \in \operatorname{Eig}_\lambda(T) \iff Tv = \lambda v \iff Tv - \lambda v = \mathbb{O} \iff Tv - \lambda \operatorname{id} v = \mathbb{O} \\ \iff (T - \lambda \operatorname{id})v = \mathbb{O} \iff v \in \ker(T - \lambda \operatorname{id}). \end{aligned}$$ □

Note that Proposition 8.28 shows again that $\operatorname{Eig}_\lambda(T)$ is a subspace of V. Moreover it shows that that λ is an eigenvalue of T if and only if $T - \lambda \operatorname{id}$ is not invertible. For the special case $\lambda = 0$ we have that $\operatorname{Eig}_0(T) = \ker T$.

Examples 8.29. (a) Let V be a vector space and let $T = \operatorname{id}$. Then for every $v \in V$ we have that $Tv = v = 1v$. Hence T has only one eigenvalue, namely $\lambda = 1$ and $\operatorname{Eig}_1(T) = \ker(T - \operatorname{id}) = \ker 0 = V$. Its geometric multiplicity is $\dim(\operatorname{Eig}_1(T) = \dim V$.

(b) Let $V = \mathbb{R}^2$ and let R be reflection on the x-axis. If $\vec v$ is an eigenvector of R, then $R\vec v$ must be parallel to $\vec v$. This happens if and only if $\vec v$ is parallel to the x-axis in which case $R\vec v = \vec v$, or if $\vec v$ is perpendicular to the x-axis in which case $R\vec v = -\vec v$. All other vectors change directions under a reflection. Hence we have the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ and $\operatorname{Eig}_1(R) = \operatorname{span}\{\vec e_1\}$, $\operatorname{Eig}_{-1}(R) = \operatorname{span}\{\vec e_2\}$. Each eigenvalue has geometric multiplicity 1.

Note that the matrix representation of R with respect to the canonical basis of $\mathbb{R}^2$ is $A_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A_R\vec x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. Hence $A_R\vec x$ is parallel to $\vec x$ if and only if $x_1 = 0$ (in which case $\vec x \in \operatorname{span}\{\vec e_2\}$) or $x_2 = 0$ (in which case $\vec x \in \operatorname{span}\{\vec e_1\}$).

(c) Let $V = \mathbb{R}^2$ and let R be rotation about $90°$. Then clearly $R\vec v \not\parallel \vec v$ for any $\vec v \in \mathbb{R}^2 \setminus \{\vec 0\}$. Hence R has no eigenvalues.

Note that the matrix representation of R with respect to the canonical basis of $\mathbb{R}^2$ is $A_R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If we consider A_R as a real matrix, then it has no eigenvalues. However, if consider A_R as a complex matrix, then it has the eigenvalues $\pm \mathrm{i}$ as we shall see later.

(d) Let $A = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&5&0&0&0&0 \\ 0&0&5&0&0&0 \\ 0&0&0&5&0&0 \\ 0&0&0&0&8&0 \\ 0&0&0&0&0&8 \end{pmatrix}$. As always, we identify A with the linear map $\mathbb{R}^6 \to \mathbb{R}^7$, $\vec{x} \mapsto A\vec{x}$. It is not hard to see that the eigenvalues and eigenspaces of A are

$$\begin{aligned} &\lambda_1 = 1, && \mathrm{Eig}_1(A) = \mathrm{span}\{\vec{e}_1\}, && \text{geom. multiplicity: } 1, \\ &\lambda_2 = 5, && \mathrm{Eig}_5(A) = \mathrm{span}\{\vec{e}_2, \vec{e}_3, \vec{e}_4\}, && \text{geom. multiplicity: } 3, \\ &\lambda_3 = 8, && \mathrm{Eig}_8(A) = \mathrm{span}\{\vec{e}_6, \vec{e}_7\}. && \text{geom. multiplicity: } 2. \end{aligned}$$

Show the claims above.

(e) Let $V = C^\infty(\mathbb{R})$ be the space of all infinitely many times differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ and let $T : V \to V$, $Tf = f'$. Analogously to Example 6.5 we can show that T is a linear transformation. The eigenvalues of T are those $\lambda \in \mathbb{R}$ such that there exists function $f \in C^\infty(R)$ with $f' = \lambda f$. We know that for every $\lambda \in \mathbb{R}$ this differential equation has a solution and that every solution is of the form $f_\lambda(x) = c\,\mathrm{e}^{\lambda x}$ for some real number c. Therefore every $\lambda \in \mathbb{R}$ is an eigenvalue of T with eigenspace $\mathrm{Eig}_\lambda(T) = \mathrm{span}\{g_\lambda\}$ where g_λ is the function given by $g_\lambda(x) = \mathrm{e}^{\lambda x}$. In particular, the geometric multiplicity of any $\lambda \in \mathbb{R}$ is 1.

(f) Let $V = C^\infty(\mathbb{R})$ be the space of all infinitely many times differentiable functions from $\mathbb{R}$ to $\mathbb{C}$ and let $T : V \to V$, $Tf = f'$. The eigenvalues of T are those numbers $\lambda \in \mathbb{C}$ such that there exists function $f \in C^\infty(R)$ with $f' = \lambda f$. As before, it follows that every $\lambda \in \mathbb{C}$ is an eigenvalue of T with eigenspace $\mathrm{Eig}_\lambda(T) = \mathrm{span}\{g_\lambda\}$ where g_λ is the function given by $g_\lambda(x) = \mathrm{e}^{\lambda x}$. In particular, the geometric multiplicity of any $\lambda \in \mathbb{R}$ is 1.

(g) Let $V = C^\infty(\mathbb{R})$ be the space of all infinitely many times differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ and let $T : V \to V$, $Tf = f''$. It is easy to see that T is a linear transformation The eigenvalues of T are those $\lambda \in \mathbb{R}$ such that there exists function $f \in C^\infty(R)$ with $f'' = \lambda f$. If $\lambda > 0$, then the general solution of this differential equation is $f_\lambda(x) = a\,\mathrm{e}^{\sqrt{\lambda}x} + b\,\mathrm{e}^{\sqrt{\lambda}x}$. If $\lambda < 0$, the general solution is $f_\lambda(x) = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$. If $\lambda = 0$, the general solution is $f_0(x) = ax + b$. Hence every $\lambda \in \mathbb{R}$ is an eigenvalue of T with geometric multiplicity 2.

Write down the eigenspaces for a given λ.

Find the eigenvalues and eigenspaces if we consider the vector space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{C}$.

In the examples above if was relatively easy to guess the eigenvalues. But how do we calculate the eigenvalues of, e.g., $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ or of the linear transformation $T : M(n \times n) \to M(n \times n)$, $T(A) = A + A^t$?

Since any linear transformation on a finite dimensional vector space V can be "translated" to a matrix by choosing a basis on V, it is sufficient to find eigenvalues of matrices as the next theorem shows.

Theorem 8.30. *Let V be a finite dimensional vector space with basis $\mathcal{B} = \{v_1, \dots, v_n\}$ and let $T : V \to V$ be a linear transformation. If A_T is the matrix representation of T with respect to* *the basis $\mathcal{B}$, then the eigenvalues of T and A_T coincide and a vector $v = c_1v_1 + \cdots + c_nv_n$ is an eigenvector of T with eigenvalue λ if and only if $\vec{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an eigenvector of A_T with the same eigenvalue λ. In particular, the dimensions of the eigenspaces of T and of A_T coincide.*

Proof. Let $\mathbb{K} = \mathbb{R}$ if V is a real vector space and $\mathbb{K} = \mathbb{C}$ if V is a complex vector space and let $\Phi : V \to \mathbb{R}^n$ be the linear map defined by $\Phi(v_j) = \vec{e}_j$, $(j = 1, \dots, n)$. That means that Φ "translates" a vector $v = c_1v_1 + \cdots + c_nv_n$ into the column vector $\vec{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, cf. Section 6.4.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi"'] & V \arrow[d, "\Phi"] \\ \mathbb{K}^n \arrow[r, "A_T"] & \mathbb{K}^m \end{tikzcd}$$

Recall that $T = \Phi^{-1}A_T\Phi$. Let λ be an eigenvalue of T with eigenvector v, that is, $Tv = \lambda v$. We express v as linear combination of the basis vectors from $\mathcal{B}$ as $v = c_1v_1 + \cdots + c_nv_n$. Hence

$$Tv = \lambda v \iff \Phi^{-1}A_T\Phi v = \lambda v \iff A_T\Phi v = \Phi\lambda v \iff A_T(\Phi v) = \lambda(\Phi v)$$

which is the case if and only if λ is an eigenvalue of A_T and $\Phi v \in \mathrm{Eig}_\lambda(A_T)$. ☐

The proof shows that $\mathrm{Eig}_\lambda(A_T) = \Phi(\mathrm{Eig}_\lambda(T))$ as was to be expected.

Corollary 8.31. *Assume that A and B are similar matrices and let C be an invertible matrix with $A = C^{-1}BC$. Then A and B have the same eigenvalues and for every eigenvalue λ we have that $\mathrm{Eig}_\lambda(B) = C\,\mathrm{Eig}_\lambda(A)$.*

Now back to the question about how to calculate the eigenvalues and eigenvectors of a given matrix A. Recall that λ is an eigenvalue of A if and only if $\ker(A - \lambda\,\mathrm{id}) \neq \{\vec{0}\}$, see Proposition 8.28. Since $A - \lambda\,\mathrm{id}$ is a square matrix, this is the case if and only if $\det(A - \lambda\,\mathrm{id}) = 0$.

Definition 8.32. The function $\lambda \mapsto \det(A - \lambda\,\mathrm{id})$ is called the *characteristic polynomial* of A. It is usually denoted by p_A.

Before we discuss the characteristic polynomial and show that it is indeed a polynomial, we will describe how to find the eigenvalues and eigenvectors of a given square matrix A.

Procedure to find the eigenvalues and eigenvectors of a given square matrix A.

- Calculate the characteristic polynomial $p_A(\lambda) := \det(A - \lambda\,\mathrm{id})$.
- Find the zeros $\lambda_1, \dots, \lambda_k$ of the characteristic polynomial. They are the **eigenvalues** of A.
- For each eigenvalue λ_j calculate $\ker(A - \lambda_j)$, for instance using Gauß-Jordan elimination. This gives the **eigenspaces**.

Example 8.33. Find the eigenvalues and eigenspaces of $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

Solution.

- The characteristic polynomial of A is
$$p_A(\lambda) = \det(A - \lambda \operatorname{id}) = \det \begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5.$$

- Now we can either complete the square or use the solution formula for quadratic equations to find the zeros of p_A. Here we choose to complete the square.
$$p_A(\lambda) = \lambda^2 - 6\lambda + 5 = (\lambda - 3)^2 - 4 = (\lambda - 5)(\lambda - 1).$$
Hence the **eigenvalues** of A are $\lambda_1 = 5$ and $\lambda_2 = 1$.

- Now we calculate the eigenspaces using Gauß elimination.
  * $A - 5\operatorname{id} = \begin{pmatrix} 2-5 & 1 \\ 3 & 4-5 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to -R_1} \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$.
  Therefore, $\ker(A - 5\operatorname{id}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$.
  * $A - 1\operatorname{id} = \begin{pmatrix} 2-1 & 1 \\ 3 & 4-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.
  Therefore, $\ker(A - 1\operatorname{id}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

In summary, we have two eigenvalues,

$$\lambda_1 = 5, \quad \operatorname{Eig}_5(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}, \qquad \text{geom. multiplicity: } 1,$$
$$\lambda_2 = 1, \quad \operatorname{Eig}_1(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \qquad \text{geom. multiplicity: } 1.$$

◇

If we set $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ we can check our result by calculating

$$A\vec{v}_1 = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 5\vec{v}_1,$$
$$A\vec{v}_2 = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{v}_2.$$

Before we give more examples, we show that the characteristic polynomial is indeed a polynomial. First we need a definition.

Definition 8.34. Let $A = (a_{ij})_{i,j=1}^n \in M(n \times n)$. The *trace of A* is the sum of its entries on the diagonal:
$$\operatorname{tr} A := a_{11} + a_{22} + \dots a_{nn}.$$

Theorem 8.35. *Let $A = (a_{ij})_{i,j=1}^n \in M(n \times n)$ and let $p_A(\lambda) = \det(A - \lambda \operatorname{id})$ be the characteristic polynomial of A. Then the following is true.*

(i) *p_A is a polynomial of degree n.*

(i) *Let $p_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. Then we have formulas for the coefficients c_n, c_{n-1} and c_0:*

$$c_n = (-1)^n, \qquad c_{n-1} = (-1)^{n-1} \operatorname{tr} A, \qquad c_0 = \det A.$$

Proof. By definition,

$$p_A(\lambda) = \det(A - \lambda \operatorname{id}) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}.$$

According to Remark 4.4, the determinant is sum of products where each product consists of a sign and n factors chosen such that it contains one entry of $A - \lambda \operatorname{id}$ from each row and from each column. Therefore it is clear that p_A is a polynomial in λ. The term with the most λ in it is the one of the form

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda). \tag{8.7}$$

All the other terms contain at most $n - 2$ factors with λ. To see this, assume for example that in one of the terms the factor from the first row is not $(a_{11} - \lambda)$ but some a_{1j}. Then there cannot be another factor from the jth column, in particular the factor $(a_{jj} - \lambda)$ cannot appear. So this term has already two factors without λ, hence the degree of the term as polynomial in λ can be at most $n - 2$. This shows that

$$p_A(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) + \text{ terms of order at most } n - 2. \tag{8.8}$$

If we expand the first term and sort by powers of λ, we obtain

$$\begin{aligned} (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) &= (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}(a_{11} + \cdots + a_{nn}) \\ &\quad + \text{ terms of order at most } n - 2. \end{aligned}$$

Inserting this in (8.7), we find that

$$p_A(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}(a_{11} + \cdots + a_{nn}) + \text{terms of order at most } n - 2, \tag{8.9}$$

hence $\deg(p_A) = n$.

Formula (8.9) also shows the claim about c_n and c_{n-1}. The formula for c_0 follows from

$$c_0 = p_A(0) = \det(A - 0 \operatorname{id}) = \det A. \qquad \square$$

We immediately obtain the following very important corollary.

Corollary 8.36. *An $n \times n$ matrix can have at most n different eigenvalues.*

Proof. Let $A \in M(n \times n)$. Then the eigenvalues of A are exactly the zeros of its characteristic polynomial. Since it has degree n, it can have at most n zeros. □

Now we understand why working with complex vector spaces is more suitable when we are interested in eigenvalues. They are precisely the zeros of the characteristic polynomial. While a polynomial may not have real zeros, it always has zeros when we allow them to be complex numbers. Indeed, any polynomial can always be factorised over $\mathbb{C}$.

Let $A \in M(n \times n)$ and let p_A be its characteristic polynomial. Then there exist complex numbers $\lambda_1, \dots, \lambda_k$ and integers $m_1, \dots, m_k \geq 1$ such that

$$p_a(\lambda) = (\lambda_1 - \lambda)^{m_1} \cdot (\lambda_2 - \lambda)^{m_2} \cdots (\lambda_k - \lambda)^{m_k}.$$

The numbers $\lambda_1, \dots, \lambda_k$ are precisely the complex eigenvalues of A and $m_1 + \cdots + m_k = \deg p_A = n$.

Definition 8.37. The integer m_j is called the *algebraic multiplicity* of the eigenvalue λ_j.

The following theorem is very important but we omit its proof.

Theorem 8.38. *Let $A \in M(n \times n)$ and let λ be an eigenvalue of A. Then*

$$\textit{geometric multiplicity of } \lambda \leq \textit{algebraic multiplicity of } \lambda.$$

Example 8.39. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}$. Since $A - \lambda\,\mathrm{id}$ is an upper triangular matrix, its determinant is the product of the entries on the diagonal. We we obtain

$$p_A(\lambda) = \det(A - \lambda\,\mathrm{id}) = (1-\lambda)(5-\lambda)^3(8-\lambda)^2.$$

Therefore the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 5$, $\lambda_3 = 8$. Let us calculate the eigenspaces.

- $A - 1\,\mathrm{id} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix} \xrightarrow{\text{permute rows}} \begin{pmatrix} 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. This matrix is in row echelon form and we can see easily that $\mathrm{Eig}_1(A) = \ker(A - 1\,\mathrm{id}) = \operatorname{span}\{\vec{e}_1\}$ which has dimension 1.
- $A - 5\,\mathrm{id} = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{\text{permute rows}} \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. This matrix is in row echelon form and we can see easily that $\mathrm{Eig}_5(A) = \ker(A - 5\,\mathrm{id}) = \operatorname{span}\{\vec{e}_2\}$ which has dimension 1.
- $A - 8\,\mathrm{id} = \begin{pmatrix} -7 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. This matrix is in row echelon form and we can see easily that $\mathrm{Eig}_8(A) = \ker(A - 8\,\mathrm{id}) = \operatorname{span}\{\vec{e}_5,\ \vec{e}_6\}$ which has dimension 2.

In summary, we have

$$\begin{array}{llll}
\lambda_1 = 1, & \operatorname{Eig}_1(A) = \operatorname{span}\{\vec{e}_1\}, & \text{geom. multiplicity: } 1, & \text{alg. multiplicity: } 1,\\
\lambda_2 = 5, & \operatorname{Eig}_5(A) = \operatorname{span}\{\vec{e}_2\}, & \text{geom. multiplicity: } 1, & \text{alg. multiplicity: } 3,\\
\lambda_3 = 8, & \operatorname{Eig}_8(A) = \operatorname{span}\{\vec{e}_6,\, \vec{e}_7\}, & \text{geom. multiplicity: } 2, & \text{alg. multiplicity: } 2.
\end{array}$$

Example 8.40. Find the complex eigenvalues and eigenspaces of $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution. From Example 8.29 we already know that R has no real eigenvalues. The characteristic polynomial of R is

$$p_R(\lambda) = \det(R - \lambda) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = (\lambda^2 - \mathrm{i})(\lambda^2 + \mathrm{i}).$$

Hence the eigenvalues are $\lambda_1 = -\mathrm{i}$ and $\lambda_2 = \mathrm{i}$. Let us calculate the eigenspaces.

- $R-(-\mathrm{i})\,\mathrm{id} = \begin{pmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{pmatrix} \xrightarrow{R_2 \to R_2 + \mathrm{i} R_1} \begin{pmatrix} \mathrm{i} & -1 \\ 0 & 0 \end{pmatrix}$. Hence $\operatorname{Eig}_{-\mathrm{i}}(R) = \ker(R + \mathrm{i}\,\mathrm{id}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \right\}$.
- $R-\mathrm{i}\,\mathrm{id} = \begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \xrightarrow{R_2 \to R_2 + \mathrm{i} R_1} \begin{pmatrix} -\mathrm{i} & -1 \\ 0 & 0 \end{pmatrix}$. Hence $\operatorname{Eig}_{\mathrm{i}}(R) = \ker(R - \mathrm{i}\,\mathrm{id}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \right\}$.

$\diamond$

Example 8.41. Find the diagonalisation of $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

Solution. We need to find an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$. By Example 8.33, A has the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$, hence A is indeed diagonalisable. We know that the diagonal entries of D are the eigenvalues of A, hence $D = \operatorname{diag}(5, 1)$ and the columns of C are the corresponding eigenvalues $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, hence

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \quad \text{and} \quad D = C^{-1}AC.$$

Alternatively, we could have chosen $\widetilde{D} = \operatorname{diag}(1, 5)$. Then the corresponding $\widetilde{C}$ is $\widetilde{C} = (\vec{v}_2|\vec{v}_1$ because the jth column of the invertible matrix must be an eigenvector corresponding the the jth entry of the diagonal matrix, hence

$$\widetilde{D} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \quad \widetilde{C} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad \widetilde{D} = \widetilde{C}^{-1}A\widetilde{C}. \qquad \diamond$$

Observe that up to ordering the diagonal elements, the matrix D is uniquely determined by A. For the matrix C however we have more choices. For instance, if we multiply each column of C by an arbitrary constant different from 0, it still works.

Example 8.42. Let $V = M(2 \times 2$ and let $T : V \to V,\ T(M) = M + M^t$. Find the eigenvalues and eigenspaces of T.

Solution. Let $M_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $M_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $M_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $M_4 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\mathcal{B} = \{M_1, M_2, M_3, M_4\}$ is a basis of $M(2 \times 2)$. The matrix representation of T with respect to it is

$$A_T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The characteristic polynomial is

$$\det(A_T - \lambda \operatorname{id}) = \det \begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda) \det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)^2 \big[(1-\lambda)^2 - 1\big] = \lambda(\lambda-2)^3.$$

Hence there are two eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = 2$.
Let us find the eigenspaces.

- $A_T - 0 \operatorname{id} = A_T \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow[\substack{R_1 \to \frac{1}{2} R_1 \\ R_4 \to \frac{1}{2} R_4}]{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$

- $A_T - 2 \operatorname{id} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

Hence $\operatorname{Eig}_0(A_T) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $\operatorname{Eig}_2(A_T) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

This means that the eigenvalues of T are 0 and 2 and that the eigenspaces are $\operatorname{Eig}_0(T) = \operatorname{span}\{M_2 - M_3\} = \operatorname{span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ and

$$\operatorname{Eig}_0(T) = \operatorname{span}\{M_2 - M_3\} = \operatorname{span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} = M_{\text{asym}}(2 \times 2),$$

$$\operatorname{Eig}_2(T) = \operatorname{span}\{M_1,\ M_2 + M_3,\ M_4\} = \operatorname{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = M_{\text{sym}}(2 \times 2),$$

Remark. We could have calculated the eigenspaces or T directly without calculating those of A_T first as follows.

- A matrix M belongs to $\operatorname{Eig}_0(T)$ if and only if $T(M) = 0$. This is the case if and only if $M + M^t = 0$ which means that $M = -M^t$. So $\operatorname{Eig}_0(T)$ is the space of all antisymmetric 2×2 matrices.

- A matrix M belongs to $\mathrm{Eig}_2(T)$ if and only if $T(M) = 2M$. This means that $M + M^t = 2M$. This is the case if and only if $M = M^t$. So $\mathrm{Eig}_0(T)$ is the space of all symmetric 2×2 matrices.

◇

You should now have understood

- the concept of eigenvalues and eigenvectors,
- why an $n \times n$ matrix can have at most n eigenvalues,
- why the restriction of A to any of its eigenspaces acts as a multiple of the identity,
- what the characteristic polynomial of a matrix says about its eigenvalues,
- why a $n \times n$ matrix is diagonalisable if and only if $\mathbb{K}^n$ has a basis consisting of eigenvectors of A,
- $\dots$,

You should now be able to

- calculate the characteristic polynomial of a square matrix A,
- calculate the eigenvalues and eigenvectors of a square matrix A,
- diagonalise a diagonalisable matrix,
- $\dots$

8.4 Properties of the eigenvalues and eigenvectors

In this section we collect important properties of eigenvectors.

Proposition 8.43. *Let $A \in M(n \times n)$ and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be pairwise different eigenvalues of A with eigenvectors $\vec v_1, \vec v_2, \dots, \vec v_k$. Then the vectors $\vec v_1, \vec v_2, \dots, \vec v_k$ are linearly independent.*

Proof. We proof the claim by induction.

Basis of the induction: $k = 2$. Assume that $\lambda_1 \neq \lambda_2$ are eigenvalues of A with eigenvectors $\vec v_1$ and $\vec v_2$. Hence $A\vec v_1 = \lambda_1 \vec 1$ and $A\vec v_2 = \lambda_2 \vec 2$ and $\vec v_1 \neq \vec 0 \neq \vec v_2$. Let α_1, α_2 numbers such that

$$\alpha_1 \vec v_1 + \alpha_2 \vec v_2 = \vec 0. \tag{8.10}$$

Assume that $\alpha_1 \neq 0$. Then $\vec v_1 = \frac{\alpha_2}{\alpha_1}\vec v_2$ and

$$\lambda_1 \vec v_1 = A\vec v_1 = A\left(\frac{\alpha_2}{\alpha_1}\vec v_2\right) = \frac{\alpha_2}{\alpha_1} A\vec v_2 = \frac{\alpha_2}{\alpha_1}\lambda_2 \vec v_2 = \lambda_2 \frac{\alpha_2}{\alpha_1}\vec v_2 = \lambda_2 \vec v_1 \quad\Longrightarrow\quad \vec 0 = (\lambda_1 - \lambda_2)\vec v_1.$$

Since $\lambda_1 \neq \lambda_2$ and $\vec v_1 \neq \vec 0$, the last equality is false and therefore we must have $\alpha_1 = 0$. Then, by (8.10), $\vec 0 = \alpha_1 \vec v_1 + \alpha_2 \vec v_2 = \alpha_2 \vec v_2$, hence also $\alpha_2 = 0$ which proves that $\vec v_1$ and $\vec v_2$ are linearly independent.

Induction step: Assume that we already know for some $j < k$ that the vectors $\vec{v}_1, \dots, \vec{v}_j$ are linearly independent. We have to show that then also the vectors $\vec{v}_1, \dots, \vec{v}_{j+1}$ are linearly independent. To this end, let Let $\alpha_1, \alpha_2, \dots, \alpha_{j+1}$ such that

$$\vec{0} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_j\vec{v}_j + \alpha_{j+1}\vec{v}_{j+1}. \tag{8.11}$$

On the one hand we apply A on both sides of the equation and use the fact that vectors are eigenvectors. On the other hand we multiply both sides by λ_{j+1} and then we compare the two results.

$$\begin{aligned}
\text{apply } A: \qquad \vec{0} &= A(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_j\vec{v}_j + \alpha_{j+1}\vec{v}_{j+1}) \\
&= \alpha_1 A\vec{v}_1 + \alpha_2 A\vec{v}_2 + \dots + \alpha_j A\vec{v}_j + \alpha_{j+1} A\vec{v}_{j+1} \\
&= \alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\vec{v}_2 + \dots + \alpha_j\lambda_j\vec{v}_j + \alpha_{j+1}\lambda_{j+1}\vec{v}_{j+1} \qquad ①
\end{aligned}$$

$$\text{multiply by } \lambda_{j+1}: \qquad \vec{0} = \alpha_1\lambda_{j+1}\vec{v}_1 + \alpha_2\lambda_{j+1}\vec{v}_2 + \dots + \alpha_j\lambda_{j+1}\vec{v}_j + \alpha_{j+1}\lambda_{j+1}\vec{v}_{j+1} \qquad ②$$

The subtraction ①-① gives

$$\alpha_1(\lambda_1 - \lambda_{j+1})\vec{v}_1 + \alpha_2(\lambda_1 - \lambda_{j+1})\vec{v}_2 + \dots + \alpha_j(\lambda_1 - \lambda_{j+1})\vec{v}_j.$$

Note that the term with $\vec{v}_{j+1}$ cancelled. By the induction hypothesis, the vectors $\vec{v}_1, \dots, \vec{v}_j$ are linearly independent, hence

$$\alpha_1(\lambda_1 - \lambda_{j+1}) = 0, \quad \alpha_2(\lambda_1 - \lambda_{j+1}) = 0, \quad \dots, \quad \alpha_j(\lambda_1 - \lambda_{j+1}) = 0.$$

We also know that λ_{j+1} is not equal to any of the other λ_ℓ, hence it follows that

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \dots, \quad \alpha_j = 0.$$

Inserting this in (8.11) gives that also $\alpha_{j+1} = 0$ and the proof is complete. □

Note that the proposition shows again that an $n \times n$ matrix can have at most n different eigenvalues.

Corollary 8.44. *Let $A \in M(n \times n)$ and let $\mu_1 \dots, \mu_k$ be the different eigenvalues of A. If in each $\mathrm{Eig}_{\mu_j}(A)$ we choose linearly independent vectors $\vec{v}_1^{j}, \dots, \vec{v}_{\ell_1}^{j}$, then the system of all those vectors in linearly independent. In particular, if we choose bases in $\mathrm{Eig}_{\mu_j}(A)$, we see that the sum of eigenspaces is a direct sum*

$$\mathrm{Eig}_{\mu_1}(A) \oplus \dots \oplus \mathrm{Eig}_{\mu_k}(A)$$

and $\dim(\mathrm{Eig}_{\mu_1}(A) \oplus \dots \oplus \mathrm{Eig}_{\mu_k}(A)) = \dim(\mathrm{Eig}_{\mu_1}(A) + \dots + \dim \mathrm{Eig}_{\mu_k}(A))$.

Proof. Let $\alpha_j^{(m)}$ be numbers such that

$$\begin{aligned}
\vec{0} &= \alpha_1^{(1)}\vec{v}_1^{1} + \dots + \alpha_{\ell_1}^{(1)}\vec{v}_{\ell_1}^{1} + \alpha_1^{(2)}\vec{v}_1^{2} + \dots + \alpha_{\ell_2}^{(2)}\vec{v}_{\ell_2}^{2} + \dots \alpha_1^{(k)}\vec{v}_1^{k} + \dots + \alpha_{\ell_k}^{(k)}\vec{v}_{\ell_k}^{k} \\
&= \vec{w}_1 + \vec{w}_2 + \dots \vec{w}_k
\end{aligned}$$

with $\vec{w}_j = \alpha_1^{(j)}\vec{v}_1^{j} + \dots + \alpha_{\ell_1}^{(j)}\vec{v}_{\ell_1}^{j} \in \mathrm{Eig}_{\mu_j}$. Proposition 8.43 implies that $\vec{w}_1 = \dots = \vec{w}_k = \vec{0}$. But then also all coefficients $\alpha_j^{(m)} = 0$ because for fixed m, the vectors $\vec{v}_1^{(m)}, \dots, \vec{v}_{\ell_m}^{(m)}$ are linearly independent. Now all the assertions are clear. □

A very special class of matrices are the diagonal matrices.

Theorem 8.45. (i) *Let* $D = \operatorname{diag}(d_1, \dots, d_n) = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ *be a diagonal matrix. Then the eigenvalues of* D *are precisely the numbers* $d_1, \dots, d_n$ *and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.*

(ii) *Let* $B = \begin{pmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ *and* $C = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ * & & d_n \end{pmatrix}$ *be upper and lower triangular matrices respectively. Then the eigenvalues of* D *are precisely the numbers* $d_1, \dots, d_n$ *and the algebraic multiplicity of an eigenvalue is equal to the number of times it appears on the diagonal. In general, nothing can be said about the geometric multiplicities.*

Proof. (i) Since the determinant of a diagonal matrix is the product of its diagonal elements, we obtain for the characteristic polynomial of D

$$p_D(\lambda) = \det(D - \lambda) = \det \begin{pmatrix} d_1 - \lambda & & 0 \\ & \ddots & \\ 0 & & d_n - \lambda \end{pmatrix} = (d_1 - \lambda) \cdot \dots \cdot (d_n - \lambda).$$

Since the zeros of the characteristic polynomial are the eigenvalues of D, we showed that the numbers on the diagonal of D are precisely its eigenvalues. The algebraic multiplicity of an eigenvalue μ is equal to the number of times it is repeated on the diagonal of D. The algebraic multiplicity of μ is equal to $\dim(\ker(D - \mu \operatorname{id})$. Note that $D - \mu \operatorname{id}$ is a diagonal matrix and the jth entry on its diagonal is 0 if and only if $\mu = d_j$. it is not hard to see that the dimension of the kernel of a diagonal matrix is equal to the number of zeros on its diagonal. So, in summary we have for an eigenvalue μ of A:

$$\begin{aligned} \text{algebraic multiplicity of } \mu &= \text{number of times } \mu \text{ appears in the diagonal of } D \\ &= \text{geometric multiplicity of } \mu. \end{aligned}$$

(ii) Since the determinant of a triangular matrix is the product of its diagonal elements, we obtain for the characteristic polynomial of B

$$p_B(\lambda) = \det(B - \lambda) = \det \begin{pmatrix} d_1 - \lambda & & * \\ & \ddots & \\ 0 & & d_n - \lambda \end{pmatrix} = (d_1 - \lambda) \cdot \dots \cdot (d_n - \lambda).$$

and analogously for C. The reasoning for the algebraic multiplicities of the eigenvalues is as in the case of a diagonal matrix. However, in general the algebraic and geometric multiplicity of an eigenvalue of a triangular matrix may be different as Example 8.39 show.

$\square$

Example 8.46. Let $D = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$. Then $p_D(\lambda) = (1-\lambda)(5-\lambda)^3(5-\lambda)^2$.

The eigenvalues are 1 (with geom. mult = alg. mult = 1), 5 (with geom. mult = alg. mult = 3) and 8 (with geom. mult = alg. mult = 2),

Theorem 8.47. *If A and B are similar matrices, then they have the same characteristic polynomial. In particular, they have the same eigenvalues with the same algebraic multiplicities. Moreover, also the geometric multiplicities are equal.*

Proof. Let C be an invertible matrix such that $A = C^{-1}BC$. Hence

$$A - \lambda\,\mathrm{id} = C^{-1}BC - \lambda\,\mathrm{id} = C^{-1}BC - \lambda C^{-1}C = C^{-1}(B - \lambda\,\mathrm{id})C$$

and we obtain for the characteristic polynomial of A

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda\,\mathrm{id}) = \det(C^{-1}(B-\lambda\,\mathrm{id})C) = \det(C^{-1})\det(B-\lambda\,\mathrm{id})\det C = \det(B - \lambda\,\mathrm{id}) \\ &= p_B(\lambda). \end{aligned}$$

This shows that A and B have the same eigenvalues and that their algebraic multiplicities coincide.

Now let μ be an eigenvalue. Then

$$\begin{aligned} \mathrm{Eig}_\mu(A) &= \ker(A - \mu\,\mathrm{id}) = \ker(C^{-1}(B - \mu\,\mathrm{id})C) = \ker((B-\mu\,\mathrm{id})C) = C^{-1}\ker(B - \mu\,\mathrm{id}) \\ &= C^{-1}\,\mathrm{Eig}_\mu(B) \end{aligned}$$

where in the second to last step we used that C^{-1} is invertible. The invertibility of C^{-1} also shows that $\dim(C^{-1}\,\mathrm{Eig}_\mu(B)) = \dim(\mathrm{Eig}_\mu(B)$, hence $\dim \mathrm{Eig}_\mu(A) = \dim(\mathrm{Eig}_\mu(B)$, which shows that the geometric multiplicity of μ as eigenvalue of A is equal to that of B. $\square$

Next we prove a very important theorem about the diagonalisation of matrices.

Theorem 8.48. *Let $A \in M_{\mathbb{K}}(n \times n)$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then the following is equivalent.*

(i) *A is diagonalisable, that means that there exists a diagonal matrix D and an invertible matrix C such that $C^{-1}AC = D$.*

(ii) *For every eigenvalue of A, its geometric and algebraic multiplicities are equal.*

(iii) *A has a set of n linearly independent eigenvectors.*

(iv) *$\mathbb{K}^n$ has a basis consisting of eigenvectors of A.*

Proof. Let $\mu_1, \dots, \mu_k$ be the different eigenvalues of A and let us denote the algebraic multiplicities of μ_j by $m_j(A)$ and $m_j(D)$ and and the geometric multiplicities by $n_j(A)$ and $n_j(D)$.

(i) $\Longrightarrow$ (ii): By assumption A and D are similar so they have the same eigenvalues by Theorem 8.47 and

$$m_j(A) = m_j(D) \quad \text{and} \quad n_j(A) = n_j(D) \qquad \text{for all } j = 1, \dots, k,$$

and Theorem 8.45 shows that

$$m_j(D) = n_j(D) \qquad \text{for all } j = 1, \dots, k,$$

because D is a diagonal matrix. Hence we conclude that also

$$m_j(A) = n_j(A) \qquad \text{for all } j = 1, \dots, k.$$

(ii) $\Longrightarrow$ (iii): Recall that the geometric multiplicities $n_j(A)$ are the dimensions of the kernel of $A - \mu_j \operatorname{id}$. So in each $\ker(A - \mu_j)$ we may choose a basis consisting of $n_j(A)$ vectors. In total we have $n_1(A)+\dots+n_k(A) = m_1(A)+\dots+m_k(A) = n$ such vectors and they are linearly independent by Corollary 8.44.

(iii) $\Longrightarrow$ (iv): This is clear because $\dim \mathbb{K}^n = n$.

(iv) $\Longrightarrow$ (i): Let $\mathcal{B} = \{\vec{c}_1, \dots, \vec{c}_n\}$ be a basis of $\mathbb{K}^n$ consisting of eigenvectors of A and let $d_1, \dots, d_n$ be the corresponding eigenvalues, that is, $A\vec{c}_j = d_j\vec{c}_j$. Note that the d_j are not necessarily pairwise different. Then the matrix $C = (\vec{c}_1|\cdots|\vec{c}_n)$ is invertible and $C^{-1}AC$ is the representation of A in the basis $\mathcal{B}$, hence $C^{-1}AC = \operatorname{diag}(d_1, \dots, d_n)$. In more detail, using that $\vec{c}_j = C\vec{e}_j$ and $C^{-1}\vec{c}_j = \vec{e}_j$,

$$j\text{th column of } C^{-1}AC = C^{-1}A\vec{c}_j = C^{-1}(d_j\vec{c}_j) = d_jC^{-1}\vec{c}_j = d_j\vec{e}_j,$$

hence $D = (d_1\vec{e}_1|\cdots|d_n\vec{e}_n) = \operatorname{diag}(d_1, \dots, d_n)$. □

An immediate consequence of Theorem 8.48 is the following.

Corollary 8.49. *If a matrix $A \in M(n \times n)$ has n different eigenvalues, then it is diagonalisable.*

Proof. If A has n different eigenvalues $\lambda_1, \dots, \lambda_n$, then for each of them the algebraic multiplicity is equal to 1. Moreover,

$$1 \le \text{geometric multiplicity} \le \text{algebraic multiplicity} = 1$$

for each eigenvalue. Hence the algebraic and the geometric multiplicity for each eigenvalue are equal (both are equal to 1) and the claim follows from Theorem 8.48. □

Corollary 8.50. *If the matrix $A \in M(n \times n)$ has n is diagonalisable, then its determinant is equal to the product of its eigenvalues.*

Proof. Let $\lambda_1, \dots, \lambda_n$ be the (not necessarily different) eigenvalues of A and let C be an invertible matrix such that $C^{-1}AC = D := \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$\det A = \det(CDC^{-1}) = (\det C)(\det D)(\det C^{-1}) = \det D = \prod_{j=1}^{n} \lambda_j.$$

□

Theorem 8.51. *Let $A \in M(n \times n)$ and let $\mu_1, \dots, \mu_k$ be its different eigenvalues. Then A is diagonalisable if and only if*

$$\mathbb{K}^n = \operatorname{Eig}_{\mu_1}(A) \oplus \cdots \oplus \operatorname{Eig}_{\mu_k}(A). \tag{8.12}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether A is acting on $\mathbb{R}$ or on $\mathbb{C}$.

Proof. Let us denote the algebraic multiplicity of each μ_j by $m_j(A)$ and its geometric multiplicity by $n_j(A)$.
If A is diagonalisable, then the geometric and algebraic multiplicities are equal for each eigenvalue. Hence

$$\begin{aligned}\dim(\operatorname{Eig}_{\mu_1}(A) \oplus \cdots \oplus \operatorname{Eig}_{\mu_k}(A)) &= \dim(\operatorname{Eig}_{\mu_1}(A)) + \cdots + \dim(\operatorname{Eig}_{\mu_k}(A)) \\ &= n_1(A) + \cdots + n_k(A) = m_1(A) + \cdots + m_k(A) = n.\end{aligned}$$

Since every n-dimensional subspace of $\mathbb{K}^n$ is equal to $\mathbb{K}^n$, (8.12) is proved.

Now assume that (8.12) is true. We have to show that A is diagonalisable. In each $\operatorname{Eig}_{\mu_j}$ we choose a basis $\mathcal{B}_j$. By (8.12) the collection of all those basis vectors form a basis of $\mathbb{K}^n$. Therefore we found a basis of $\mathbb{K}^n$ consisting of eigenvectors of A. Hence A is diagonalisable by Theorem 8.48. □

The above theorem says that A is diagonalisable if and only if there are enough eigenvalues of A to span $\mathbb{K}^n$. This is the case if and only if $\mathbb{K}^n$ splits in the direct sum of subspaces on each of which A acts simply by multiplying each vector with the number (namely with the corresponding eigenvalue).

To practice a bit the notions of algebraic and geometric multiplicities, finish this section with an alternative proof of Theorem 8.48.

Alternative proof of Theorem 8.48. Let us prove (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).
(i) $\Longrightarrow$ (iv): This was already discussed after Definition 8.22. Let $D = \operatorname{diag}(d_1, \dots, d_n)$ and let $\vec{c}_1, \dots, \vec{c}_n$ be the columns of C. Clearly they are form a basis of $\mathbb{K}^n$ because C in invertible. By assumption we know that $AC = CD$. Hence we have that

$$A\vec{c}_j = j\text{th column of } AC = j\text{th column of } CD = d_j \cdot (j\text{th column of } C) = d_j\vec{c}_j.$$

Therefore the vectors $\vec{c}_1, \dots, \vec{c}_n$ are linearly independent and are all eigenvalues of A and hence they are even a basis of $\mathbb{K}^n$.
(iv) $\Longrightarrow$ (iii): Clear.
(iii) $\Longrightarrow$ (ii): Suppose that $\vec{v}_1 \ \dots, \vec{v}_n$ is a basis of K^n consisting of eigenvectors of A. Clearly, each of them must belong to some eigenspace of A. Let ℓ_j be the number of those vectors which belong to $\operatorname{Eig}_{\mu_j}(A)$. Hence it follows that $\ell_j \le n_j(A)$ because the vectors are linearly independent and $n_j(A) = \dim \operatorname{Eig}_{\mu_j}(A)$. So by Theorem 8.38 we have $\ell_j \le n_j(A) \le m_j(A)$ where $m_j(A)$ is the algebraic multiplicity of μ_j. Summing over all eigenvectors, we obtain

$$n = \ell_1 + \cdots + \ell_k \le n_1(A) + \cdots + n_k(A) \le m_1(A) + \cdots + m_k(A) = n$$

The first equality holds because the vectors are a basis of $\mathbb{K}^n$ and the last equality holds by definition of the algebraic multiplicity. Hence all the $\le$ signs are in reality equalities and $n_1(A) + \cdots + n_k(A) = m_1(A) + \cdots + m_k(A)$. Therefore

$$\begin{aligned}0 &= n_1(A) + \cdots + n_k(A) - \big[m_1(A) + \cdots + m_k(A)\big] \\ &= \big[n_1(A) - m_1(A)\big] + \cdots + \big[n_k(A) - m_k(A)\big].\end{aligned}$$

Since $n_j(A)-m_j(A) \leq 0$ for all $j = 1, \dots, k$, each of the terms must be zero which shows that $n_j(A)-m_j(A)$ as desired.

(ii) $\Longrightarrow$ (i): For each $j = 1, \dots, k$ let us choose a basis $\mathcal{B}_j$ of $\operatorname{Eig}_{\mu_j}(A)$. Observe that each basis has $n_j(A)$ vectors. By Corollary 8.44, the system consisting of all these basis vectors is linearly independent. Moreover, the total number of these vectors is $n_1(A) + \dots + n_k(A) = m_1(A) + \dots + m_k(A) = n$ where we used the assumption that the algebraic and geometric multiplicities are equal for each eigenvalue. Hence the collection of all those vectors form a basis of $\mathbb{K}^n$. That A is diagonalisable follows now as in the proof of (iv) $\Longrightarrow$ (i): □

You should now have understood

- why the eigenvectors of different eigenvalues of a matrix A are linearly independent,
- more generally, why the sum of the eigenspaces is even a direct sum,
- why a matrix is diagonalisable if and only if the vector space has a basis consisting of eigenvectors of A,
- algebraic and geometric multiplicities,
- . . . ,

You should now be able to

- verify if a given matrix is diagonalisable,
- if it is diagonalisable, find its diagonalisation,
- . . .

8.5 Symmetric and Hermitian matrices

In this section we will deal with symmetric and hermitian matrices. The main results are that all eigenvalues of a hermitian matrix are real, that eigenvectors corresponding to different eigenvalues are orthogonal and that every hermitian matrix is diagonalisable. Note the symmetric matrices are a special case of hermitian ones, so whenever we show something about hermitian matrices, the same is true for symmetric matrices.

Theorem 8.52. *Let A be a hermitian matrix. Then every eigenvalue λ of A is real.*

Proof. Let A be hermitian, that is, $A^* = A$ and let λ be an eigenvalue of A with eigenvector $\vec{v}$. Then $\vec{v} \neq \vec{0}$ and $A\vec{v} = \lambda\vec{v}$. We have to show that $\lambda = \overline{\lambda}$. Therefore

$$\lambda\|\vec{v}\|^2 = \lambda\langle\vec{v}\,,\vec{v}\rangle = \langle\lambda\vec{v}\,,\vec{v}\rangle = \langle A\vec{v}\,,\vec{v}\rangle = \langle\vec{v}\,,A^*\vec{v}\rangle = \langle\vec{v}\,,A\vec{v}\rangle = \langle\vec{v}\,,\lambda\vec{v}\rangle = \overline{\lambda}\langle\vec{v}\,,\vec{v}\rangle = \overline{\lambda}\|\vec{v}\|^2.$$

Since $\vec{v} \neq \vec{0}$, it follows that $\lambda = \overline{\lambda}$ which means that the imaginary part of λ is 0, hence $\lambda \in \mathbb{R}$. □

Theorem 8.53. *Let A be a hermitian matrix and let λ_1, λ_2 be two different eigenvalues of A with eigenvectors $\vec{v}_1$ and $\vec{v}_2$, that is $A\vec{v}_1 = \lambda\vec{v}_1$ and $A\vec{v}_2 = \lambda\vec{v}_2$. Then $\vec{v}_1 \perp \vec{v}_2$.*

Proof. The prove is similar to the proof of Theorem 8.52. We have to show that $\langle \vec{v}_1 \,, \vec{v}_2 \rangle = 0$. Note that by Theorem 8.52, the eigenvalues λ_1, λ_2 are real.

$$\lambda_1 \langle \vec{v}_1 \,, \vec{v}_2 \rangle = \langle \lambda_1 \vec{v}_1 \,, \vec{v}_2 \rangle = \langle A\vec{v}_1 \,, \vec{v}_2 \rangle = \langle \vec{v}_1 \,, A^* \vec{v}_2 \rangle = \langle \vec{v}_1 \,, A\vec{v}_2 \rangle = \langle \vec{v}_1 \,, \lambda_2 \vec{v}_2 \rangle = \overline{\lambda_2} \langle \vec{v} \,, \vec{v}_2 \rangle = \lambda_2 \langle \vec{v}_1 \,, \vec{v}_2 \rangle.$$

Since $\lambda_1 \neq \lambda_2$ by assumption it follows that $\langle \vec{v}_1 \,, \vec{v}_2 \rangle = 0$. □

Corollary 8.54. *Let A be a hermitian matrix and let λ_1, λ_2 be two different eigenvalues of A. Then $\mathrm{Eig}_{\lambda_1}(A) \perp \mathrm{Eig}_{\lambda_2}(A)$.*

The next theorem is one of the most important theorems in Linear Algebra.

Theorem 8.55. *Every hermitian matrix is diagonalisable.*

Theorem 8.55*. *Every symmetric matrix is diagonalisable.*

As a corollary we obtain the following very important theorem.

Theorem 8.57. *A matrix is hermitian if and only if it is orthogonally diagonalisable, that is, there exists a unitary matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ = Q^*AQ$.*

The formulation of the above theorem for real matrices is:

Theorem 8.57*. *A matrix is symmetric if and only if it is orthogonally diagonalisable, that is, there exists an orthogonal matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ = Q^tAQ$.*

In both cases, $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where the $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and the columns of Q are the corresponding eigenvectors.

Proof. Let A be a hermitian matrix. From Theorem 8.55 we know that A is diagonalisable. Hence

$$\mathbb{C}^n = \mathrm{Eig}_{\mu_1}(A) \oplus \ldots \mathrm{Eig}_{\mu_k}(A)$$

where $\mu_1, \ldots, \mu_k$ are the different eigenvalues of A. In each eigenspace $\mathrm{Eig}_{\mu_j}(A)$ we can choose an orthogonal basis $\mathcal{B}_j$ consisting of n_j vectors $\vec{v}_1^j, \ldots, \vec{v}_{n_j}^j$ where n_j is the geometric multiplicity of μ_j. We know that the eigenspaces are pairwise orthogonal by Corollary 8.54. Hence the system of all these vectors form an orthogonal basis $\mathcal{B}$ of $\mathbb{C}^n$. Let Q be the matrix whose columns are the vectors of this basis. Since they are pairwise orthogonal, Q is unitary.

Now assume that A is orthogonally diagonalisable. We have to show that A is hermitian. Let Q be an orthogonal matrix and let D be a diagonal matrix such that $D = Q^*AQ$. Then $A = QDQ^*$ and

$$A^* = (QDQ^*)^* = (Q^*)^*D^*Q^* = QDQ^* = A$$

where we used that $D^* = D$ because D is a diagonal matrix whose entries on the diagonal are real numbers because they are the eigenvalues of A. □

The proof of Theorem 8.57* is the same.

Corollary 8.59. *If a matrix A is hermitian (or symmetric), then its determinant is the product of its eigenvalues.*

Proof. This follows from Theorem 8.55 (or Theorem 8.55*) and Corollary 8.50. $\square$

Proof of Theorem 8.55. Let $A \in M_{\mathbb{C}}(n \times n)$ be a hermitian matrix and let $\mu_1, \dots, \mu_k$ be the different eigenvalues of A with geometric multiplicities $n_1, \dots, n_k$. By Theorem 8.51 it suffices to show that

$$\mathbb{C}^n = \mathrm{Eig}_{\mu_1}(A) \oplus \cdots \oplus \mathrm{Eig}_{\mu_k}(A).$$

Let us denote the right hand side by U, that is, $U := \mathrm{Eig}_{\mu_1}(A) \oplus \cdots \oplus \mathrm{Eig}_{\mu_k}(A)$. Then we have to show that $U^\perp = \{\vec{0}\}$. For the sake of a contradiction, assume that this is not true and let $\ell = \dim(U^\perp)$. In each $\mathrm{Eig}_{\mu_j}(A)$ we choose an orthogonal basis $\vec{v}_1^{(j)}, \dots, \vec{v}_{n_j}^{(j)}$ and we choose and orthogonal basis $\vec{w}_1, \dots, \vec{w}_\ell$ in $U^\perp$. The set of all these vectors is an orthonormal basis $\mathcal{B}$ of $\mathbb{C}^n$ because the all the eigenspaces are orthogonal to each other and to $U^\perp$. Let Q be the matrix whose columns are these vectors: $Q = (\vec{v}_1^{(1)} | \cdots | \vec{v}_{n_k}^{(k)} | \vec{w}_1 | \cdots | \vec{w}_\ell)$. Then Q is a unitary matrix because its columns are an orthogonal basis of $\mathbb{C}^n$. Next let us define $B = Q^{-1}AQ$. Then B is symmetric because $B^* = (Q^{-1}AQ)^* = Q^*A^*(Q^{-1})^* = Q^{-1}AQ = B$ where we used that $A = A^*$ by assumption and that $Q^{-1} = Q^*$ because it is a unitary matrix. On the other hand, B being the matrix representation of A with respect to the basis $\mathcal{B}$, it is of the form

$$B = \left(\begin{array}{ccccc:c}
\mu_1 & & & & & \\
 & \ddots & & & & \\
 & & \mu_1 & & & \\
 & & & \mu_2 & & \\
 & & & & \ddots & \\
 & & & & \mu_k & \\
\hdashline
 & & & & & C
\end{array}\right).$$

All the empty spaces are 0 and C is an $\ell \times \ell$ matrix (it is the matrix representation of the restriction of A to $U^\perp$ with respect to the basis $\vec{w}_1, \dots, \vec{w}_\ell$). The characteristic polynomial of C has at least one zero, hence C has at least one eigenvalue λ. Clearly, λ is then also an eigenvalue of B and if $\vec{y} \in \mathbb{C}^\ell$ is an eigenvector of C, we obtain an eigenvector of B with the same eigenvalue by putting 0s as its first $n - \ell$ components and $\vec{y}$ as its last ℓ components. Since A and B have the same eigenvalues, λ must be equal to one of the eigenvectors $\mu_1, \dots, \mu_k$, say $\lambda = \mu_{j_0}$. But then the dimension of the eigenspace $\mathrm{Eig}_{\mu_{j_0}}(B)$ is strictly larger than the dimension of $\mathrm{Eig}_{\mu_{j_0}}(A)$ which contradicts Theorem 8.47. Therefore $U^\perp = \{\vec{0}\}$ and the theorem is proved. $\square$

Proof of Theorem 8.55.* The proof is essentially the same as that for Theorem 8.55. We only have to note that the matrix C is symmetric (because B is symmetric). If we view C as a complex matrix, it has at least one eigenvalue λ because in $\mathbb{C}$ its characteristic polynomial has at least one complex zero. However, since C is hermitian, all its eigenvalues are real, hence λ is real, so it is an eigenvalue of C if we view it as a real matrix. $\square$

You should now have understood

- why hermitian and symmetric matrices have real eigenvalues,
- why eigenvectors for different eigenvalues of a hermitian matrix are perpendicular to each other,
- why a hermitian/symmetric matrix is orthogonally diagonalisable,
- that up to a rotation and maybe reflection, the eigenspaces of a hermitian matrix are generated by the coordinate axes,
- . . . ,

You should now be able to

- find eigenvalues and eigenvectors of hermitian/symmetric matrices,
- diagonalise symmetric matrices,
- write $\mathbb{C}^n$ (or $\mathbb{R}^n$) as direct sum of the eigenspaces of a given hermitian (or symmetric) matrix,
- . . .

8.6 Application: Conic Sections

In this section we will study quadratic equations in x and y. Recall that we know how to deal with linear equations in two variables. The most general form is

$$ax + by = d \tag{8.13}$$

with constants a, b, d. A solution is a tuple (x, y) which satisfies (8.13). We can view the set of all solutions at as a subset in the plane $\mathbb{R}^2$. Since (8.13) is a linear equation (a 1×2 system of linear equations), we know that we have the following possibilities:

(a) a *line* if $a \neq 0$ or $b \neq 0$,

(b) the *plane* $\mathbb{R}^2$ if $a = 0$, $b = 0$ and $d = 0$,

(c) the *empty set* (no solution) if $a = 0$, $b = 0$ and $d \neq 0$,

Now we will consider the quadratic equation

$$\boxed{ax^2 + bxy + cy^2 = d} \tag{8.14}$$

with constants a, b, c, d.

In the following we will always assume that $d \geq 0$. This is no loss of generality because if $d < 0$, we can multiply both sides of (8.14) by -1 and replace a, b, c by $-a, -b, -c$. The set of solutions does not change.

Again, we can identify the solutions in $\mathbb{R}^2$ and the question is what type of figures they are. The equation (8.14) is not linear, so we have to see what relation (8.14) has with what we studied so far. It turns out that the left hand side of (8.14) can be written as an inner product

$$\left\langle G\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \quad \text{with} \quad G = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}. \tag{8.15}$$

Question 8.5

The matrix G from (8.15) is not the only possible choice. Find all possible matrices G such that $\langle G\binom{x}{y}, \binom{x}{y} \rangle = ax^2 + bxy + cy^2$.

The matrix G is very convenient because it is symmetric. This means that up to an *orthogonal* transformation, it is a diagonal matrix. So once we know how to solve the problem when G is diagonal, then we know it for the general case since the solutions differ only by a rotation and maybe a reflection. This motivates us to first study the case when G is diagonal, that is, when $b = 0$.

Quadratic equation without mixed term ($b = 0$).

If $b = 0$, then (8.14) becomes

$$\boxed{ax^2 + cy^2 = d} \tag{8.16}$$

with constants $d \geq 0$ and $a, c \in \mathbb{R}$.

Remark 8.60. The solution set is symmetric with respect to the x-axis and the y-axis because if some (x, y) is a solution of (8.16), then so are $(-x, y)$ and $(x, -y)$.

Let us define

$$\alpha := \sqrt{|a|}, \qquad \gamma := \sqrt{|c|}, \qquad \text{hence} \qquad \alpha^2 = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0 \end{cases} \quad \text{and} \quad \gamma^2 = \begin{cases} c & \text{if } c \geq 0, \\ -c & \text{if } c < 0. \end{cases}$$

We have to distinguish several cases according to whether the coefficients a, c are positive, negative or 0.

Case 1.1: $a > 0$ and $c > 0$. In this case, the equation (8.16) becomes

$$\alpha^2 x^2 + \gamma^2 y^2 = d. \tag{8.16.1.1}$$

(i) If $d > 0$, then (8.16.1.1) is the equation of an ellipse whose axes are parallel to the x and the y-axis. The intersection with the x-axis is at $\pm\frac{\sqrt{d}}{\alpha} = \pm\sqrt{d/a}$ and the intersection with the y-axis is at $\pm\frac{\sqrt{d}}{\gamma} = \pm\sqrt{d/c}$.

(ii) If $d = 0$, then the only solution of (8.16.1.1) is the point $(0, 0)$.

Remark 8.61. Note that the length of the semiaxes of the ellipse is proportional to $\sqrt{d}$. Hence as d decreases, the ellipse from (i) becomes smaller and for $d = 0$ it degenerates to the point $(0, 0)$ from (ii).

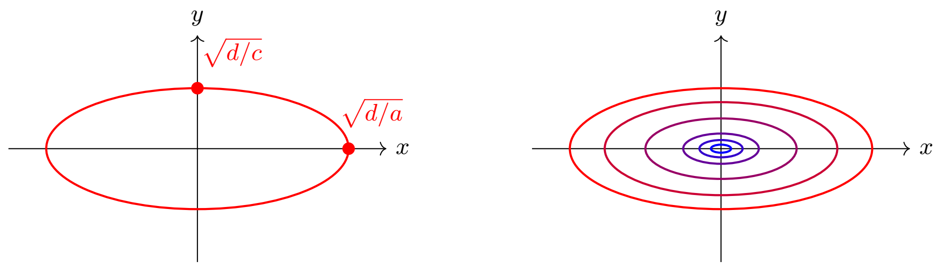


FIGURE 8.1: Solution of (8.16) for $\det G > 0$. If $a > 0, b > 0$, then the solution is an ellipse (if $d > 0$) or the point $(0, 0)$ (if $d = 0$). The right picture shows ellipses with a and c fixed but decreasing d (from red to blue). If $a < 0, b < 0, d > 0$, then there is no solution.

Case 1.2: $a < 0$ and $c < 0$. In this case, the equation (8.16) becomes

$$-\alpha^2 x^2 - \gamma^2 y^2 = d. \tag{8.16.1.2}$$

(i) If $d > 0$, then (8.16.1.2) has no solution because the left hand side is always less or equal to 0 while the right hand side is strictly positive.

(ii) If $d = 0$, then the only solution of (8.16.1.2) is the point $(0, 0)$.

Case 2.1: $a > 0$ and $c < 0$. In this case, the equation (8.16) becomes

$$\alpha^2 x^2 - \gamma^2 y^2 = d. \tag{8.16.2.1}$$

(i) If $d > 0$, then (8.16.2.1) is the equation of a hyperbola. If $x = 0$, the equation has no solution. Indeed, we need $|x| \geq \frac{\sqrt{r}}{\alpha}$ such that the equation has a solution. Therefore the hyperpola does not intersect the y-axes (the hyperbola cannot pass through a strip $-\frac{\sqrt{d}}{\alpha} < y < \frac{\sqrt{d}}{\alpha}$).

- *Intersection with the coordinate axes*: No intersection with the y-axis. Intersection with the x-axis at $x = \pm\frac{\sqrt{d}}{\alpha} = \pm\sqrt{d/a}$.
- *Asymptotics*: For $|x| \to \infty$ and $|y| \to \infty$, the hyperbola has the asymptotes

$$y = \pm\frac{\alpha}{\gamma}x.$$

Note that the asymptote does not depend on d.

Proof. It follows from (8.16.2.1) that $|x| \to \infty$ if and only if $|y| \to \infty$ because otherwise the difference $\alpha^2 x^2 - \gamma^2 y^2$ cannot be constant. Dividing (8.16.2.1) by x^2 and by γ^2 and rearranging leads to

$$y^2 = \frac{\alpha^2}{\gamma^2}x^2 - \frac{d}{\gamma^2 x^2} \overset{x \text{ large}}{\approx} \frac{\alpha^2}{\gamma^2}x^2, \qquad \text{hence} \qquad y \approx \pm\frac{\alpha}{\gamma}x. \qquad \square$$

(ii) If $d = 0$, then (8.16.2.1), becomes $\alpha^2 x^2 + \gamma^2 y^2 = 0$, and its solution is the pair of lines $y = \pm\frac{\alpha}{\gamma}x$.

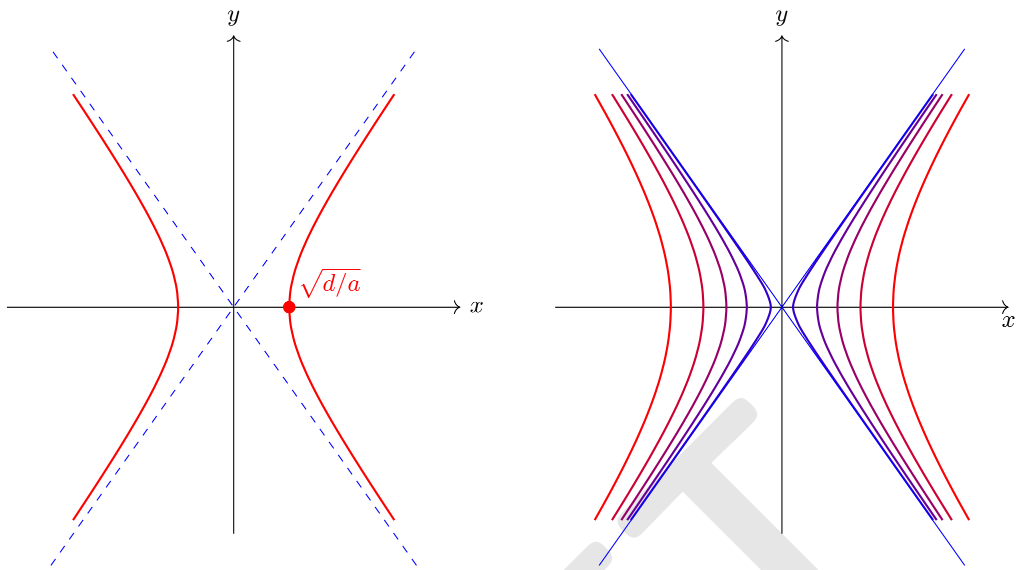


FIGURE 8.2: Solution of (8.16) for $\det G < 0$. The solutions are hyperbola (if $d > 0$) or a set of two intersecting lines. The left picture shows a solution for $a > 0$, $c < 0$ and $d > 0$. The right picture shows a hyperbola for fixed a and c but decreasing d. The blue pair of lines passing through the origin correspond to the case $d = 0$.

Remark 8.62. Note that the intersection point of the hyperbola with the x-axis is proportional to $\sqrt{d}$. Hence as d decreases, the intersection points moves closer to the 0 and the turn becomes sharper. If $d = 0$, the intersection point reaches 0 and the hyperbola become to angles which look like two crossing lines.

Case 2.2: $a < 0$ and $c > 0$. In this case, the equation (8.16) becomes

$$-\alpha^2 x^2 + \gamma^2 y^2 = d. \tag{8.16.2.2}$$

This case is the same as Case 2.1, only with the roles of x and y interchanged. So we find:

(i) If $d > 0$, then (8.16.2.1) is the equation of a hyperbola.

- *Intersection with the coordinate axes*: No intersection with the x-axis. Intersection with the y-axis at $y = \pm\frac{\sqrt{d}}{\gamma} = \pm\sqrt{d/c}$.
- *Asymptotics*: For $|x| \to \infty$ and $|y| \to \infty$, the hyperbola has the asymptotes $y = \pm\frac{\alpha}{\gamma}x$.

(ii) If $d = 0$, then (8.16.2.1), becomes $\alpha^2x^2+\gamma^2y^2 = 0$, and its solution is the pair of lines $y = \pm\frac{\alpha}{\gamma}x$.

Case 3.1: $a > 0$ and $c = 0$. Then (8.16) becomes $\alpha^2 x^2 = d$.

- If $d > 0$, the solutions are the two parallel lines $x = \pm\frac{\sqrt{d}}{\alpha}$.
- If $d = 0$, the solution is the line $x = 0$.

Case 3.2: $a = 0$ and $c > 0$. Then (8.16) becomes $\gamma^2 y^2 = d$.

- If $d > 0$, the solutions are the two parallel lines $y = \pm\frac{\sqrt{d}}{\gamma}$.
- If $d = 0$, the solution is the line $y = 0$.

Case 3.3: $a < 0$ and $c = 0$. Then (8.16) becomes $-\alpha^2 x^2 = d$.

- If $d > 0$, there is no solution.
- If $d = 0$, the solution is the point $(0, 0)$.

Case 3.4: $a = 0$ and $c < 0$. Then (8.16) becomes $-\gamma^2 x^2 = d$.

- If $d > 0$, there is no solution.
- If $d = 0$, the solution is the point $(0, 0)$.

Case 3.5: $a = 0$ and $c = 0$. Then (8.16) becomes $0 = d$.

- If $d > 0$, there is no solution.
- If $d = 0$, the solution is $\mathbb{R}^2$.

Note that in the Cases 1.1 and 1.2, $\det G = ac > 0$, in the Cases 2.1 and 2.2, $\det G = ac < 0$ and in all remaining cases $\det G = 0$.

Quadratic equation with mixed term.

Now we want to solve (8.14) without the assumption that $b = 0$. Let $G = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then (8.14) is equivalent to

$$\langle G\vec{x}\,, \vec{x}\rangle = d. \tag{8.17}$$

If G was diagonal, then we immediately could give the solution. We know that G is symmetric, hence we know that G can be orthogonally diagonalized. In other words, there exists an orthogonal basis of $\mathbb{R}^2$ with respect to which G has a representation as a diagonal matrix. We can even choose this basis such that they are a rotation of the canonical basis $\vec{e}_1$ and $\vec{e}_2$ (without an additional reflection).

Let λ_1, λ_2 be eigenvalues of G and let $D = \operatorname{diag}(\lambda_1, \lambda_2)$. We choose and orthogonal matrix Q such that

$$D = Q^{-1}GQ. \tag{8.18}$$

Denote the columns of Q by $\vec{v}_1$ and $\vec{v}_2$. They are normalised eigenvectors of G with eigenvalues λ_1 and λ_2 respectively. Recall that for an orthogonal matrix Q we always have that $\det Q = \pm 1$. We may assume that $\det Q = 1$, because if not we can simply multiply one of its columns by -1. This column then is still a normalised eigenvector of G with the same eigenvalue, hence (8.18) is still valid. With this choice we guarantee that Q is a rotation.

From (8.18) it follows that $G = QDQ^{-1} = QDQ^*$. So we obtain from (8.17) that

$$d = \langle G\vec{x}\,, \vec{x}\rangle = \langle QDQ^*\vec{x}\,, \vec{x}\rangle = \langle DQ^*\vec{x}\,, Q^*\vec{x}\rangle = \langle D\vec{x}\,'\,, \vec{x}\,'\rangle = \langle D\vec{x}\,'\,, \vec{x}\,'\rangle = \lambda_1 x'^2 + \lambda_2 y'^2$$

where $\vec{x}\,' = \begin{pmatrix} x' \\ y' \end{pmatrix} = Q^*\vec{x} = Q^{-1}\vec{x}$.

Observe that the column vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the representation of $\vec{x}$ with respect to the basis $\vec{v}_1, \vec{v}_2$ (recall that they are eigenvectors of G). Therefore the solution of (8.14) is one of the solutions we found for the case $b = 0$ only now the symmetry axis of the figures are no longer the x- and y-axis, but they are the directions of the eigenvectors of G. In other words: Since Q is a rotation, we obtain the solutions of $ax^2 + bxy + cy^2 = d$ by rotating the solutions of $ax^2 + cy^2 = d$ with the matrix Q.

Procedure to find the solutions of $ax^2 + bxy + cy^2 = d$**.**

- Write down the symmetric matrix $G = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.
- Find the eigenvalues λ_1 and λ_2 and eigenvectors of G and define the diagonal matrix $D = \operatorname{diag}(\lambda_1, \lambda_2)$. and the orthogonal matrix Q such that $\det Q = 1$ and $D = Q^{-1}GQ$.
- Quadratic form without mixed terms: $d = \lambda_1 x'^2 + \lambda_2 y'^2$ where x', y' are the components of $\vec{x}\,' = Q^{-1}\vec{x}$.
- Graphic of the solution: In the xy-coordinate system, indicate the x'-axis (parallel to $\vec{v}_1$) and the y'-axis (parallel to $\vec{v}_2$). Note that these axes are a rotation of the x- and the y-axis. The solutions are then, depending on the eigenvalues, an ellipse, hyperbola, etc. whose symmetry axis are the x'- and y'-axis.

If we want to know only the shape of the solution, it is enough to calculate the eigenvalues λ_1, λ_2 of G, or even only $\det G$. Recall that we always assume $d \geq 0$.

- If $\det G > 0$, then we obtain an **ellipse** (which may be degenerate).
  - If $\lambda_1 > 0$ and $\lambda_2 > 0$, then the solution is an ellipse with length of its axes $\sqrt{d/\lambda_1}$ and $\sqrt{d/\lambda_2}$. If $d = 0$ the ellipse is only the point $(0, 0)$.
  - If $\lambda_1 < 0$ and $\lambda_2 < 0$, then there is either no solution (if $d > 0$) or the solution is only the point $(0, 0)$ (if $d = 0$).
- If $\det G < 0$, then we obtain a **hyperbola** (which may be degenerate).
  - If $\lambda_1 > 0$ and $\lambda_2 < 0$, then the solution is a hyperbola which with intersections with the x'-axis at $\sqrt{d/\lambda_1}$ and no intersection with the y'-axis.
  - If $\lambda_1 < 0$ and $\lambda_2 > 0$, then the solution is a hyperbola which with intersections with the x'-axis at $\sqrt{d/\lambda_2}$ and no intersection with the x'-axis.

  In both cases, the asymptotes of the hyperbola have slope $\pm\sqrt{\lambda_1/\lambda_2}$. If $d = 0$, the hyperbola degenerate to the pair of lines $y = \pm\sqrt{\lambda_1/\lambda_2}\ x$.
- If $\det G = 0$, then we obtain either the empty set, one of the axes, two lines parallel to one of the axes, or $\mathbb{R}^2$.

Definition 8.63. The axis of symmetry are called the *principal axes*.

Example 8.64. Consider the equation

$$10x^2 + 6xy + 2y^2 = 4. \tag{8.19}$$

(i) Write the equation in matrix form.

(ii) Make a change of coordinates so that the quadratic equation (8.19) has no mixed term.

(iii) Describe the solution of (8.19) in geometrical terms and sketch it. Indicate the principal axes and important intersections.

Solution. (i) First we write (8.19) in the form $\langle G\vec{x}, \vec{x}\rangle$ with symmetric matrix G. Let us define $G = \begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}$. Then (8.19) is equivalent to

$$\left\langle G \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = 4. \tag{8.20}$$

(ii) Now we calculate the **eigenvalues** of G. They are the roots of the characteristic polynomial $\det(G - \lambda)$.

$$0 = \det(G - \lambda) = (10 - \lambda)(2 - \lambda) - 9 = \lambda^2 - 12\lambda + 11 = (\lambda - 6)^2 - 25 = (\lambda - 1)(\lambda - 11).$$

Hence the eigenvalues of G are

$$\lambda_1 = 1, \quad \lambda_2 = 11.$$

Next we need the normalised **eigenvectors**. To this end, we calculate $\ker(G - \lambda_j)$ Gauß elimination:

- $G - \lambda_1 = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \vec{v}_1 = \dfrac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix},$
- $G - \lambda_2 = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \vec{v}_2 = \dfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

(Recall that for symmetric matrices the eigenvectors for different eigenvalues are orthogonal. If you solve such an exercise it might be a good idea to check if the vectors are indeed orthogonal to each other.)

Observation. With the information obtained so far, we already can sketch the solution.

- The solution is an ellipse because both eigenvalues are positive.
- The principal axes (symmetry axes) are parallel to the vectors $\vec{v}_1$ u $\vec{v}_2$. The ellipse intersects them in $\pm\sqrt{4/1} = \pm 2$ along the axis parallel to $\vec{v}_1$ and in $\pm\sqrt{4/11} = \pm 2/\sqrt{1/11}$ along the axis parallel to $\vec{v}_2$.

Set

$$Q = (\vec{v}_1|\vec{v}_2) = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$Q^{-1} = Q^t \quad \text{y} \quad D = Q^{-1}GQ = Q^tGQ.$$

Observe that $\det Q = 1$, so it is a rotation en $\mathbb{R}^2$. It is a rotation by the angle $\arctan(-3)$.

If we define

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} x+3y \\ 3x-y \end{pmatrix},$$

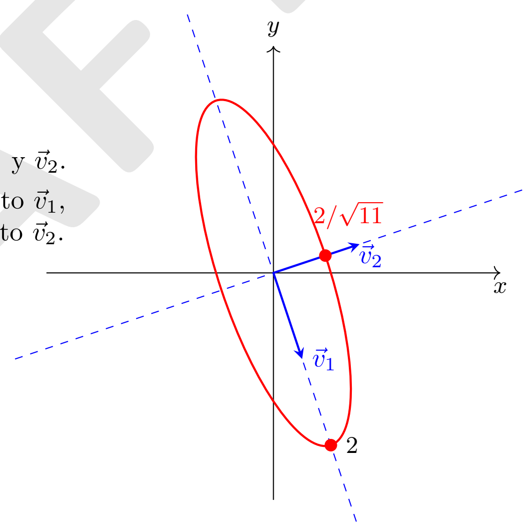
then (8.20) gives

$$4 = \left\langle G\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \left\langle DQ^t\begin{pmatrix} x \\ y \end{pmatrix}, Q^t\begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \left\langle D\begin{pmatrix} x' \\ y' \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right\rangle,$$

and therefore

$$\boxed{4 = x'^2 + 11y'^2 = \frac{1}{10}(x-3y)^2 + \frac{11}{10}(3x+y)^2.}$$

(iii) The solution of (8.19) is an ellipse whose principal axes are parallel to the vectors $\vec{v}_1$ y $\vec{v}_2$. x' is the coordinate along the axis parallel to $\vec{v}_1$, y' is the coordinate along the axis parallel to $\vec{v}_2$.



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Example 8.65. Consider the equation

$$-\frac{47}{17}x^2 - \frac{32}{17}xy + \frac{13}{17}13y^2 = 4. \tag{8.21}$$

(i) Write the equation in matrix form.

(ii) Make a change of coordinates so that the quadratic equation (8.21) has no mixed term.

(iii) Describe the solution of (8.21) in geometrical terms and sketch it. Indicate the principal axes and important intersections.

Solution. (i) First we write (8.21) in the form $\langle G\vec{x}, \vec{x}\rangle$ with symmetric matrix G. Let us define $G = \frac{1}{17}\begin{pmatrix} -47 & -16 \\ -16 & 13 \end{pmatrix}$. Then (8.21) is equivalent to

$$\left\langle G\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = 17. \tag{8.22}$$

(ii) Now we calculate the **eigenvalues** of G. They are the roots of the characteristic polynomial

$0 = \det(G-\lambda) = (-\frac{47}{17}-\lambda)(\frac{13}{17}-\lambda) - \frac{128}{17^2} = \lambda^2 + \frac{34}{17}\lambda - \frac{611}{17^2} - \frac{256}{17^2} = \lambda^2 + 2\lambda - 3 = (\lambda-1)(\lambda+3).$

Hence the eigenvalues of G are

$$\lambda_1 = -3, \quad \lambda_2 = 1.$$

Next we need the normalised **eigenvectors**. To this end, we calculate $\ker(G-\lambda_j)$ Gauß elimination:

- $G - \lambda_1 = \dfrac{1}{17}\begin{pmatrix} 4 & -16 \\ -16 & 64 \end{pmatrix} \longrightarrow \dfrac{1}{17}\begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \vec{v}_1 = \dfrac{1}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix},$
- $G - \lambda_2 = \dfrac{1}{17}\begin{pmatrix} -64 & -16 \\ -16 & -4 \end{pmatrix} \longrightarrow \dfrac{1}{17}\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \vec{v}_2 = \dfrac{1}{\sqrt{17}}\begin{pmatrix} -1 \\ 4 \end{pmatrix}.$

Observation. With the information obtained so far, we already can sketch the solution.

- The solution are hyperbola because the eigenvalues have opposite signs.
- The principal axes (symmetry axes) are parallel to the vectors $\vec{v}_1$ and $\vec{v}_2$. The intersections of the hyperbola with the axis parallel to $\vec{v}_2$ are $\pm\sqrt{2}$.

Set

$$Q = (\vec{v}_1|\vec{v}_2) = \frac{1}{\sqrt{17}}\begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$Q^{-1} = Q^t \quad \text{y} \quad D = Q^{-1}GQ = Q^tGQ.$$

Observe that $\det Q = 1$, hence Q is a rotation of $\mathbb{R}^2$. It is a rotation by the angle $\arctan(1/4)$.

If we define

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} 4x - y \\ x + 4y \end{pmatrix},$$

then (8.22) gives

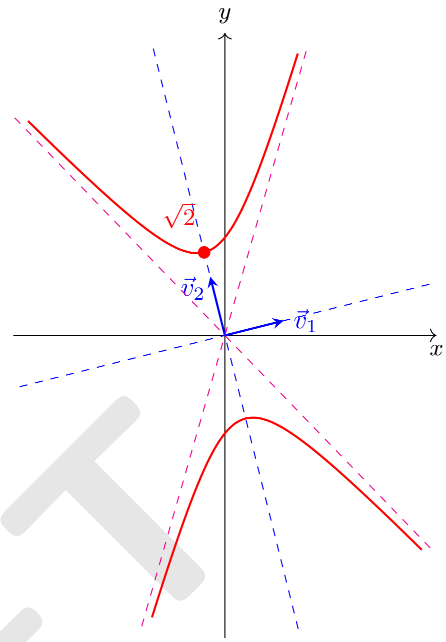
$$\frac{1}{4} = \left\langle G\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \left\langle DQ^t\begin{pmatrix} x \\ y \end{pmatrix}, Q^t\begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \left\langle D\begin{pmatrix} x' \\ y' \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right\rangle,$$

hence

$$\boxed{2 = -3x'^2 + y'^2 = -\frac{3}{17}(4x-y)^2 + \frac{1}{17}(x+4y)^2.}$$

(iii) The solution of equation (8.19) are hyperbola whose principal axes are parallel to the vectors $\vec{v}_1$ y $\vec{v}_2$.

x' is the coordinate along the axis parallel to $\vec{v}_1$,
y' is the coordinate along the axis parallel to $\vec{v}_2$.
The angle between the x- and the x'-axis is $\arctan(1/4)$.



Asymptotes of the hyperbola. In order to calculate the slopes of the asymptotes of the hyperbola, we first calculate in the x'-y'-coordinate system. Our starting point is the equation $2 = -3x'^2 + y'^2$.

$$2 = -3x'^2 + y'^2 \quad \iff \quad \frac{y'^2}{x'^2} = 3 + \frac{1}{2x'^2} \quad \iff \quad \frac{y'}{x'} = \pm\sqrt{3 + \frac{1}{2x'^2}}.$$

We see that $|y'| \to \infty$ if and only if $|x'| \to \infty$ and that $\frac{y'}{x'} \approx \pm\sqrt{3}$. So the slopes of the asymptotes in $x' - y'$-coordinates are $\pm\sqrt{3}$.

How do we find the slope in $x - y$-coordinates?

- **Method 1: Use Q.** We know that if we rotate our hyperbola by the linear transformation $Q^{-}1$ (i.e. if we rotate by $\arctan(1/4)$), then we obtain hyperbola whose symmetry axes are the x- and y-axes and whose asymptotes have slopes ± 3. Hence, in order to obtain the asymptotes of our parabola, we only need to apply Q to the vectors $\vec{w}_1$ y $\vec{w}_2$ which are parallel to the new asymptotes. The resulting vectors are then parallel to our original hyperbola. In our case $\vec{w}_1 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$. Hence

$$\vec{w}_1' = Q\vec{w}_1 = \frac{1}{\sqrt{17}}\begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} 4+\sqrt{3} \\ -1+4\sqrt{3} \end{pmatrix},$$

$$\vec{w}_2' = Q\vec{w}_2 = \frac{1}{\sqrt{17}}\begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} 4-\sqrt{3} \\ -1-4\sqrt{3} \end{pmatrix}.$$

Therefore the slopes of the asymptotes of our hyperbola are

$$\frac{-1+4\sqrt{3}}{4+\sqrt{3}} \qquad \text{and} \qquad \frac{-1-4\sqrt{3}}{4-\sqrt{3}}.$$

- **Method 2: Insert in the formulas.** The asymptotes are lines which satisfy $\frac{y'}{x'} = \pm\sqrt{3}$. Using $x' = \frac{1}{\sqrt{17}}(4x-y)$ y $y' = \frac{1}{\sqrt{17}}(x+4y)$, we obtain

$$\begin{aligned} & \pm\sqrt{3} = \frac{y'}{x'} = \frac{\frac{1}{\sqrt{17}}(x+4y)}{\frac{1}{\sqrt{17}}(4x-y)} = \frac{x+4y}{4x-y} \\ \iff \quad & \pm\sqrt{3}(4x-y) = x+4y \\ \iff \quad & (\pm 4\sqrt{3}-1)x = (4\pm\sqrt{3})y \\ \iff \quad & \frac{y}{x} = \frac{-1\pm 4\sqrt{3}}{4\pm\sqrt{3}}. \end{aligned}$$

- **Method 3: Adding angles.** We know that the angle between the x'-axis and an asymptote is $\arctan\sqrt{3}$ and the angle between the x'-axis and the x-axis is $\arctan(1/4)$. Therefore the angel between the asymptote and the x-axis is $\arctan\sqrt{3} + \arctan(1/4)$ (see Figure 8.3.)

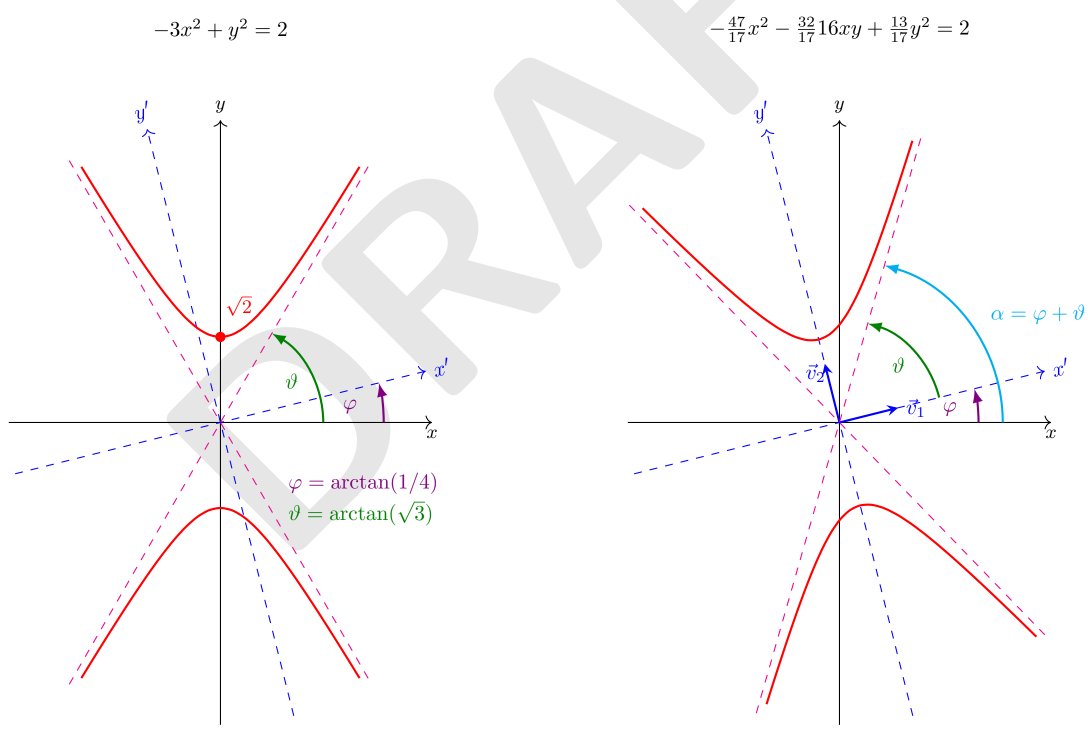


FIGURE 8.3: The figure on the right (our hyperbola) is obtained from the figure on the left by apply the transformation Q to it (that is, by rotating it by $\arctan(1/4)$).

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Example 8.66. Consider the equation

$$9x^2 - 6xy + y^2 = 25. \tag{8.23}$$

(i) Write the equation in matrix form.

(ii) Make a change of coordinates so that the quadratic equation (8.23) has no mixed term.

(iii) Describe the solution of (8.23) in geometrical terms and sketch it. Indicate the principal axes and important intersections.

Solution 1.
- First we write (8.21) in the form $\langle G\vec{x}, \vec{x}\rangle$ with symmetric matrix G. Let us define $G = \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}$. Then (8.23) is equivalent to

$$\left\langle G\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = 9. \tag{8.24}$$

- Now we calculate the **eigenvalue's** of G. They are the roots of the characteristic polynomial

$$0 = \det(G - \lambda) = (9 - \lambda)(1 - \lambda) - 9 = \lambda^2 - 10\lambda = \lambda(\lambda - 10).$$

Hence the eigenvalues of G are

$$\lambda_1 = 0, \quad \lambda_2 = 10.$$

Next we need the normalised **eigenvectors**. To this end, we calculate $\ker(G - \lambda_j)$ Gauß elimination:

- $G - \lambda_1 = \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \implies \vec{v}_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix},$
- $G - \lambda_2 = \begin{pmatrix} -1 & -3 \\ -3 & -9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \implies \vec{v}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$

Observation. With the information obtained so far, we already can sketch the solution.

– The solution are two parallel lines because one of the eigenvalues is zero and the other is positive.

– The lines are parallel to $\vec{v}_1$ and their intersections with the axis parallel to $\vec{v}_1$ are $\pm\sqrt{25/10} = \pm\sqrt{5/2}$.

Set

$$Q = (\vec{v}_1|\vec{v}_2) = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix},$$

then

$$Q^{-1} = Q^t \quad \text{y} \quad D = Q^{-1}GQ = Q^tGQ.$$

Observe that $\det Q = 1$, hence Q is a rotation in $\mathbb{R}^2$. It is a rotation by the angle $\arctan(3)$.

If we define

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} x+3y \\ -3x+y \end{pmatrix},$$

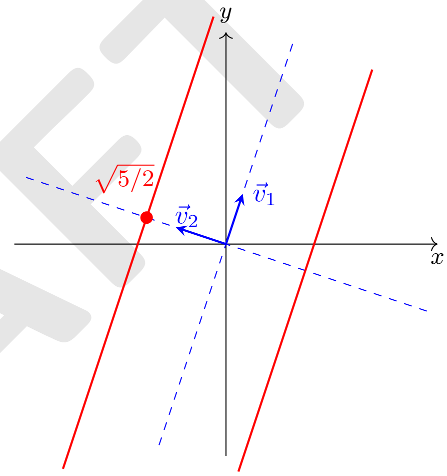
then (8.24) gives

$$25 = \left\langle G\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \left\langle DQ^t\begin{pmatrix} x \\ y \end{pmatrix}, Q^t\begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \left\langle D\begin{pmatrix} x' \\ y' \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right\rangle,$$

therefore

$$\boxed{25 = 10y'^2 = (-3x+y)^2.}$$

- The solution of (8.19) are two lines parallel to the vector $\vec{v}_1$ which intersect the y'-axis at $\pm\sqrt{25/10} = \pm\sqrt{5/2}$.
  x' is the coordinate along the axis parallel to $\vec{v}_1$,
  y' is the coordinate along the axis parallel to $\vec{v}_2$.
  The angle between the x- and the x'-axis is $\arctan(3)$.



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Solution 2. Note that

$$25 = 9x^2 - 6xy + y^2 = (3x-y)^2 \qquad \Longleftrightarrow \qquad 5 = |3x-y|.$$

Therefore the solution are two parallel lines given by

$$y = 3x \pm 5$$

which coincides with the result above. ◇

8.6.1 Solutions of $ax^2 + bxy + cy^2 = d$ as conic sections

The reason why the title of this section is "conic section" is because most of the solution sets of the quadratic equations can be obtained as the intersection of a double cone with a planes.

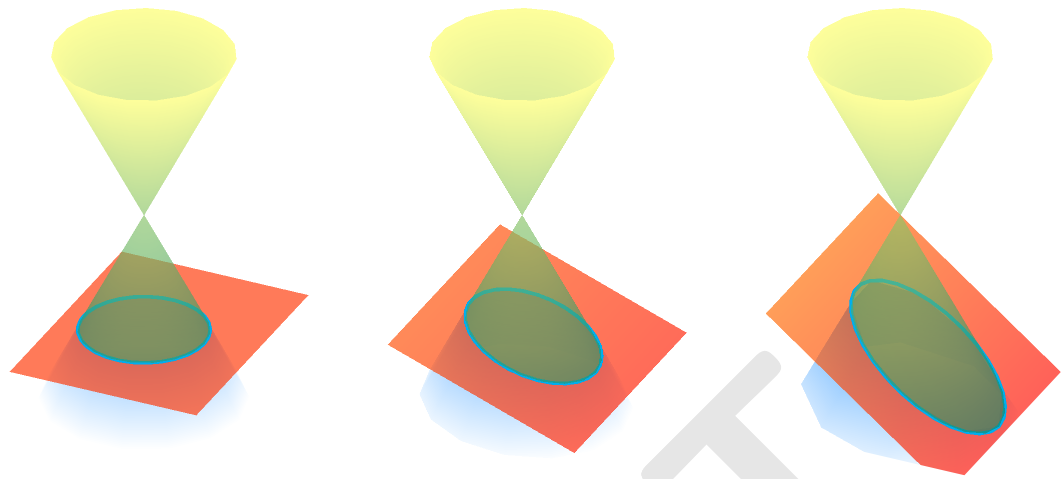

FIGURE 8.4: **Ellipses.** The plane in the picture on the left is parallel to the xy-plane. Therefore the intersection with the cone is a circle. If the plane starts to incline, the intersection becomes an ellipse. The more inclined the plane is, the more prolonged is the ellipse. As long as the plane is not yet parallel to the surface of the cone, the intersects only either the upper or the lower part of the cone and the intersection is an ellipse.

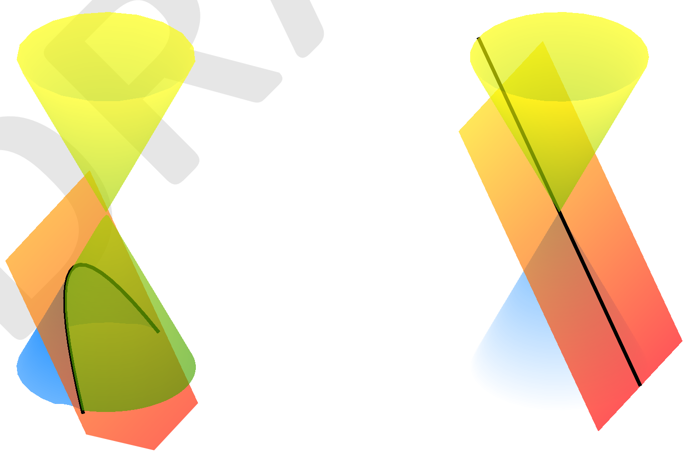

FIGURE 8.5: **Parabola.** If the plane is parallel to the surface of the cone and does not pass through the origin, then the intersection with the cone is a parabola (this is **not** a possible solution of (8.14)). If the plane is parallel to the surface of the cone and passes through the origin, then the plane is tangential to the cone and the intersection is one line.

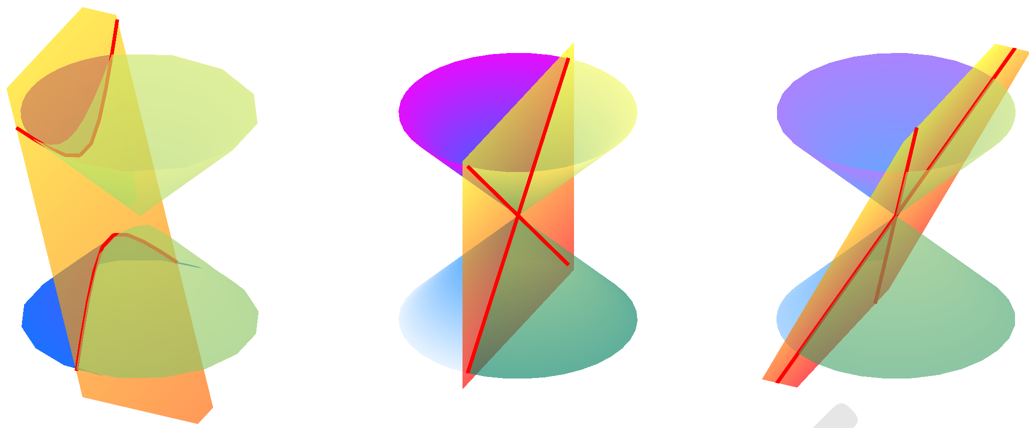

FIGURE 8.6: **Hyperbola.** If the plane is steeper than the cone, then it intersects both the upper and the lower part of the cone. The intersection are hyperbola. If the plane passes through the origin, then the hyperbola degenerate to two intersecting lines. The plane in the picture in the middle is parallel to the yz-plane. Therefore the intersection with the cone is a circle.

8.6.2 Solutions of $ax^2 + bxy + cy^2 + rx + sy = d$

Let us briefly discuss the case then the quadratic equation (8.14) contains linear terms:

$$\boxed{ax^2 + bxy + cy^2 + rx + sy = d} \tag{8.25}$$

We want to find a transformation so that (8.25) can be written without the linear terms rx and sy. Let $G = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ and let λ_1, λ_2 be its eigenvalues. Moreover, let $D = \operatorname{diag}(\lambda_1, \lambda_2)$ and Q an orthogonal matrix with $\det Q = 1$ and $D = Q^{-1}GQ$.
In the following we assume that G is invertible.

Method 1. First eliminate the mixed term bxy.
If we set $\vec{x}\,' = Q^{-1}\vec{x}$, then $ax^2 + bxy + cy^2 = \lambda_1 x'^2 + \lambda_2 y'^2$. Since x' and y' are linear in x and y, equation (8.25) becomes

$$\lambda_1 x'^2 + \lambda_2 y'^2 + r'x' + s'y' = d'.$$

Now we only need to complete the squares on the left hand sides to obtain

$$\lambda_1(x' + r'/2)^2 + \lambda_2(y' + s'/2)^2 - (r'/2)^2 - (s'/2)^2 = d'.$$

Note that this can always be done if λ_1 and λ_2 are not 0 (here we use that G is invertible).
If we set $d'' = d' + (r'/2)^2 + (s'/2)^2$, $x'' = x' + r'/2$ $y'' = y' + s'/2$, then

$$\lambda_1 x''^2 + \lambda_2 y''^2 = d''. \tag{8.26}$$

Since $\vec{x}\,'' = \begin{pmatrix} r'/2 \\ s'/2 \end{pmatrix} + \vec{x}\,' = \begin{pmatrix} r'/2 \\ s'/2 \end{pmatrix} + Q^{-1}\vec{x}$ we see that the solution is the solution of $\lambda_1 x^2 + \lambda_2 y^2 = d''$ but rotated by Q and shifted by the vector $\begin{pmatrix} r'/2 \\ s'/2 \end{pmatrix}$.

Method 2. First eliminate the linear term rx and sy.
Let us make the ansatz $x = x_0 + \widetilde{x}$ and $y = y_0 + \widetilde{y}$. Inserting in (8.25) gives

$$\begin{aligned} d &= a(x_0+\widetilde{x})^2 + b(x_0+\widetilde{x})(y_0+\widetilde{y}) + c(y_0+\widetilde{y})^2 + r(x_0+\widetilde{x}) + s(y_0+\widetilde{y})^2 \\ &= a\widetilde{x}^2 + b\widetilde{x}\widetilde{y} + c\widetilde{y}^2 + \big[2ax_0 + by_0 + r\big]\widetilde{x} + \big[2cy_0 + bx_0 + s\big]\widetilde{y} + ax_0^2 + bx_0y_0 + cy_0^2 \end{aligned} \tag{8.27}$$

We want the linear terms in $\widetilde{x}$ and $\widetilde{y}$ disappear, so we need $2ax_0+by_0+r = 0$ and $2cy_0+bx_0+s = 0$. In matrix form this is

$$-\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 2G\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Assume that G is invertible. Then we can solve for x_0 and y_0 and obtain $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -\frac{1}{2}G^{-1}\begin{pmatrix} r \\ s \end{pmatrix}$.
Now if we set $\widetilde{d} = d - ax_0^2 - bx_0y_0 - cy_0^2$, then (8.27) becomes

$$\widetilde{d} = a\widetilde{x}^2 + b\widetilde{x}\widetilde{y} + c\widetilde{y}^2 \tag{8.28}$$

which is now in the form of (8.14) (if $\widetilde{d}$ is negative, then we must multiply both sides of (8.28) by -1. In this case, the eigenvalues of G change its sign, hence D also changes sign, but Q does not). Hence if we set $\vec{x}\,' = Q^{-1}\vec{\widetilde{x}}$, then

$$\widetilde{d} = \lambda_1 x'^2 + \lambda_2 y'^2$$

and $\vec{x}\,' = Q^{-1}\vec{\widetilde{x}} = Q^{-1}(\vec{x} - \vec{x}_0) = Q^{-1}\vec{x} - Q^{-1}\vec{x}_0 = Q^{-1}\vec{x} + \frac{1}{2}Q^{-1}G^{-1}\begin{pmatrix} r \\ s \end{pmatrix}$. So again we see that the solution is the solution of $\lambda_1 x^2 + \lambda_2 y^2 = \widetilde{d}$ but rotated by Q and shifted by the vector $\frac{1}{2}Q^{-1}G^{-1}\begin{pmatrix} r \\ s \end{pmatrix}$.

Example 8.67. Find the solutions of

$$10x^2 + 6xy + 2y^2 + 8x - 2y = 4. \tag{8.19'}$$

Solution. We know from Example 8.64 that

$$G = \begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}, \quad Q = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$$

. and that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} x+3y \\ 3x-y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = Q\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} x'-3y' \\ 3x'-y' \end{pmatrix}.$$

Method 1. With the notation above, we know from Example 8.64 that(8.19') is

$$\begin{aligned}4 &= 10x^2+6xy+2y^2+8x-2y = x'^2+11y'^2+8x-2y\\ &= x'^2+11y'^2+\frac{8}{\sqrt{10}}(x'-3y')-\frac{2}{\sqrt{10}}(3x'-y')\\ &= x'^2+\frac{2}{\sqrt{10}}x'+y'^2-\frac{22}{\sqrt{10}}y'\\ &= \left(x'+\frac{1}{\sqrt{10}}\right)^2+11\left(y'-\frac{1}{\sqrt{10}}\right)^2+\frac{2}{10},\end{aligned}$$

hence

$$\left(x'+\frac{1}{\sqrt{10}}\right)^2+11\left(y'-\frac{1}{\sqrt{10}}\right)^2 = 4-\frac{2}{10} = \frac{19}{5}.$$

This is an ellipse oriented as the one from Example 8.64 but shifted by $-1/\sqrt{10}$ in x'-direction and $1/\sqrt{10}$ in y'-direction. The length of the semiaxes are $\frac{1}{2}\sqrt{\frac{19}{2}}$ and $\frac{1}{2}\sqrt{\frac{19}{22}}$.

Method 2. Note that

$$\begin{pmatrix}x_0\\ y_0\end{pmatrix} = -\frac{1}{2}G^{-1}\begin{pmatrix}r\\ s\end{pmatrix} = -\frac{1}{2}\cdot\frac{1}{11}\begin{pmatrix}2 & -3\\ -3 & 10\end{pmatrix}\begin{pmatrix}8\\ -2\end{pmatrix} = -\frac{1}{22}\begin{pmatrix}22\\ -44\end{pmatrix} = \begin{pmatrix}-1\\ 2\end{pmatrix}.$$

Set $\widetilde{x} = x-x_0 = x+1$ and $\widetilde{y} = y-y_0 = y-2$. Then

$$\begin{aligned}4 &= 10x^2+6xy+2y^2+8x-2y = 10(\widetilde{x}-1)^2+6(\widetilde{x}-1)(\widetilde{y}+2)+2(\widetilde{y}+2)^2+8(\widetilde{x}-1)-2(\widetilde{y}+2)\\ &= 10\widetilde{x}^2-20\widetilde{x}+1+6\widetilde{x}\widetilde{y}+12\widetilde{x}-6\widetilde{y}-12+2\widetilde{y}^2+8\widetilde{y}+8+8\widetilde{x}-8-2\widetilde{y}-4\\ &= 10\widetilde{x}^2+6\widetilde{x}\widetilde{y}+2\widetilde{y}^2-15\end{aligned}$$

hence

$$19 = 10\widetilde{x}^2+6\widetilde{x}\widetilde{y}+2\widetilde{y}^2 = \widetilde{x}'^2+11\widetilde{y}'^2$$

with

$$\begin{pmatrix}\widetilde{x}'\\ \widetilde{y}'\end{pmatrix} = Q^{-1}\begin{pmatrix}\widetilde{x}\\ \widetilde{y}\end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix}\widetilde{x}+3\widetilde{y}\\ 3\widetilde{x}-\widetilde{y}\end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix}(x+1)+3(y-2)\\ 3(x+1)-(y-2)\end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix}x+3y-5\\ 3x-y+5\end{pmatrix}.$$

◇

You should now have understood

- that a symmetric 2×2 matrix which is not a multiple of the identity marks two distinguished directions in $\mathbb{R}^2$, namely the ones parallel to its eigenvectors,
- why a change of variables is helpful to find solutions of a quadratic equation in two variables,
- . . . ,

You should now be able to

- find the solutions of quadratic equations in two variables,
- make a change of coordinates such that the transformed equation has no mixed term,
- sketch the solution in the xy-plane,
- ...

8.7 Summary

$\mathbb{C}^n$ as an inner product space

$\mathbb{C}^n$ is an inner product space if we set

$$\langle \vec{z}\,, \vec{w}\rangle = \sum_{j=1}^{n} z_j \overline{w_j}.$$

We have for all $\vec{v}\,\vec{w}$, $\vec{z} \in \mathbb{C}^n$ and $c \in \mathbb{C}$:

- $\langle \vec{v}\,, \vec{z}\rangle = \overline{\langle \vec{z}\,, \vec{w}\rangle}$,
- $\langle \vec{v} + c\vec{w}\,, \vec{z}\rangle = \langle \vec{v}\,, \vec{z}\rangle + c\langle \vec{w}\,, \vec{z}\rangle, \quad \langle \vec{z}\,, \vec{v} + c\vec{w}\rangle = \langle \vec{z}\,, \vec{v}\rangle + \bar{c}\langle \vec{z}\,, \vec{w}\rangle$,
- $\langle \vec{z}\,, \vec{z}\rangle = \|\vec{z}\|^2$,
- $\langle \vec{v}\,, \vec{z}\rangle \leq \|\vec{v}\|\,\|\vec{z}\|$,
- $\|\vec{v} + \vec{z}\|^2 \leq \|\vec{v}\|^2 + \|\vec{z}\|^2$,

The adjoint of a matrix $A \in M_{\mathbb{C}}(n\times n)$ is $A^* = \overline{(A^t)} = (\overline{A})^t$ (= transposed and complex conjugated). The matrix A is called *hermitian* if $A^* = A$. The matrix Q is called *unitary* if it is invertible and $Q^* = Q^{-1}$.
Note that $\det A^* = \overline{\det A}$.

Eigenvalues and eigenvectors

Definition. Let $A \in M(n \times n)$. Then λ is called an *eigenvalue* of A with *eigenvector* $\vec{v}$ if $\vec{v} \neq \vec{0}$ and $A\vec{v} = \lambda\vec{v}$. The set of all solutions of $A\vec{v} = \lambda\vec{v}$ for an eigenvalue λ is called the eigenspace of A for λ. It is denoted by $\mathrm{Eig}_\lambda(A)$.

The eigenvalues of A are exactly the zeros of the *characteristic polynomial*

$$p_A(\lambda) = \det(A - \lambda).$$

It is a polynomial of degree n. Since every polynomial of degree ≥ 1 has at least one complex root, every complex matrix has at least one eigenvalue (but there are real matrices without eigenvalues.) Moreover, a $n \times n$-matrix has at most n eigenvalues. If we factorise p_A, we obtain

$$p_A(\lambda) = (\lambda - \mu_1)^{m_1} \cdots (\lambda - \mu_k)^{m_k}$$

where $\mu_1, \ldots, \mu)k$ are the *different* eigenvalues of A. The exponent m_j is called *algebraic multiplicity of* μ_j. The *geometric multiplicity of* μ_j is $\dim(\mathrm{Eig}_{\mu_j}(A)$. Note that

- geometric multiplicity $\leq$ algebraic multiplicity,
- the sum of all algebraic multiplicities is $m_1 + \cdots + m_k = n$.

Similar matrices.

- Two matrices $A, B \in M(n \times n)$ are called *similar* if there exists an invertible matrix C such that $A = C^{-1}BC$.
- A matrix A is called *diagonalisable* if it is similar to a diagonal matrix.

Characterisation of diagonalisability. Let $A \in M_{\mathbb{C}}(n \times n)$ and let $\mu_1, \ldots, \mu_k$ be the different eigenvalues of A. We set $n_j = \dim(\mathrm{Eig}_{\mu_j}(A)$ = geometric multiplicity of μ_j and m_j = algebraic multiplicity of μ_j. Then the following is equivalent:

(i) A is diagonalisable.

(ii) $\mathbb{C}^n$ has a basis consisting of eigenvectors of A.

(iii) $\mathbb{C}^n = \mathrm{Eig}_{\mu_1}(A) \oplus \cdots \oplus \mathrm{Eig}_{\mu_k}(A)$.

(iv) $n_j = m_j$ for every $j = 1, \ldots, k$.

(v) $n_1 + \cdots + n_k = n$.

The same is true for symmetric matrices with $\mathbb{C}^n$ replaced by $\mathbb{R}^n$.

Properties of unitary matrices. Let Q be a unitary $n \times n$ matrix. Then:

- $|\det Q| = 1$,
- If $\lambda \in \mathbb{C}$ is an eigenvalue of Q, then $|\lambda| = 1$.
- Q is unitarily diagonalisable (we did not prove this fact), hence $\mathbb{C}^n$ has a basis consisting of eigenvectors of Q. They can be chosen to be mutually orthogonal.

Moreover, Q is unitary if and only if $\|Q\vec{z}\| = \|\vec{z}\|$ for all $\vec{z} \in \mathbb{C}^n$.

Properties of hermitian matrices. Let $A \in M_{\mathbb{C}}(n \times n)$ be a hermitian $n \times n$ matrix. Then:

- $\det A \in \mathbb{R}$,
- If λ is an eigenvalue of Q, then $\lambda \in \mathbb{R}$.
- A is unitarily diagonalisable hence $\mathbb{C}^n$ has a basis consisting of eigenvectors of A. They can be chosen to be mutually orthogonal.

Moreover, A is hermitian if and only if $\langle A\vec{v}\,, \vec{z}\rangle = \langle \vec{v}\,, A\vec{z}\rangle$ for all $\vec{v}$, $\vec{z} \in \mathbb{C}^n$.

Properties of symmetric matrices. Let $A \in M_{\mathbb{R}}(n \times n)$ be a symmetric $n \times n$ matrix. Then:

- A is orthogonally diagonalisable. hence $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. They can be chosen to be mutually orthogonal.

Moreover, A is symmetric if and only if $\langle A\vec{v}\,, \vec{z}\rangle = \langle \vec{v}\,, A\vec{z}\rangle$ for all $\vec{v}$, $\vec{z} \in \mathbb{R}^n$.

Solution of $ax^2+bxy+cy^2 = d$. The equation can be rewritten as $\langle G\vec{x}\,, \vec{x}\rangle = d$ with the symmetric matrix

$$G = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

Let λ_1, λ_2 be the eigenvalues of G and let us assume that $d \geq 0$. Then the solutions are:

- an **ellipse** if $\det G > 0$, more precisely,
  - an ellipse with length of its axes $\sqrt{d/\lambda_1}$ and $\sqrt{d/\lambda_2}$ if λ_1, $\lambda_2 > 0$ and $d > 0$,
  - the point $(0,0)$ if $d = 0$,
  - the empty set if λ_1, $\lambda_2 < 0$ and $d > 0$,
- **hyperbola** if $\det G < 0$, more precisely,
  - hyperbola $d > 0$,
  - two lines crossing at the origin if $d = 0$,
- two parallel lines, one line or $\mathbb{R}^2$ if $\det G = 0$.

8.8 Exercises

1. Sea Q una matriz unitaria. Demuestre que todos sus autovalores tienen norma 1.

2. Sea A una matriz con autovalores $\mu_1, \dots, \mu_k$ y sea c una constante.
   (a) ¿Qué se puede decir sobre los autovalores de cA? ¿Qué se puede decir sobre los autovalores de $A + c\,\mathrm{id}$?

3. Dados la matriz A y los vectores u y w:
$$A = \begin{pmatrix} 25 & 15 & -18 \\ -30 & -20 & 36 \\ -6 & -6 & 16 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad w = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$
   (a) Diga si los vectores u y w son autovectores de A. Si lo son, cuáles son los vectores propios correspondientes?
   (b) Puede usar que $\det(A-\lambda) = -\lambda^3 + 21\lambda^2 - 138\lambda + 280$. Calcule todos los autovalores de A.

4. Para las siguientes matrices, encuentre los vectorios propios, los espacios propios, una matriz invertible C y una matrix diagonal D tal que $C^{-1}AC = D$.
$$A_1 = \begin{pmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ 9 & 0 & 6 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

5. We consider a string of lenth L which is fixed on both end points. It is excited then its vertical elongation satisfies the partial differential equation $\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x)$. If we make the ansatz $u(,x) = \mathrm{e}^{\mathrm{i}\omega t}\, v(x)$ for some number ω and a function v which depends only on x, we obtain $-\omega^2 v = v''$. If we set $\lambda = -\omega^2$, we see that we have to solve the following eigenvalue problem:
$$T : V \to V, \quad Tv = v''$$
with
$$V = \{f : [0, L] \to \mathbb{R},\ f \text{ is twice differentiable and } f(0) = f(L) = 0\}.$$
   (i) Show that V is a vector space.
   (ii) Show that T is a well-defined linear opertor.
   (iii) Find the eigenvalues and eigenspaces of T.

6. Para cada una de las siguientes matrices, determine si son diagonalizables. Si lo es, encuentre una D que es semejante. $D = CAC^{-1}$.
$$A_1 = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 4 & 2 & -7 \\ 0 & 5 & -3 & 6 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 11 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 3 & 2 & 5 & 1 \\ 2 & 0 & 2 & 6 \\ 5 & 2 & 7 & -1 \\ 1 & 6 & -1 & 3 \end{pmatrix}.$$

7. Encuentre una substitción ortogonal que diagonalice las formas cuadráticas dadas y encuentre la forma diagonal. Haga un bosquejo de las soluciones. Si es un elipse, calcule las longitudes de las ejes principales y el ángulo que tienen con el eje x. Si es una hipérbola, calcule en ángulo que tiene las asíntotas con el eje x.
   (a) $10x^2 - 6xy + 2y^2 = 4$,
   (b) $x^2 - 9y^2 = 2$,
   (c) $x^2 - 9y^2 = 20$ (compare la solución con la del literal anterior!)
   (d) $11x^2 - 16xy - y^2 = 30$.
   (e) $x^2 + 4xy + 4^2 = 4$.

8. Encuentre los valores propios y los espacios propios de las siguientes matrices $n \times n$:
$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}.$$
   Compare con el Ejercicio 9.

9. Sea $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Calcule $\mathrm{e}^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

   Hint. Encuentre una matriz invertible C y una matriz diagonal D tal que $A = C^{-1}DC$ y use esto para calcular A^n.

10. Sea $A \in M(n \times n, \mathbb{C})$ una matriz hermitiana tal que todos sus autovalores son estrictamente mayores a 0. Sea $\langle \cdot\,, \cdot \rangle$ el producto interno estandar en $\mathbb{C}^n$. Demuestre que A induce un producto interno en $\mathbb{C}^n$ a través de

$$\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \qquad (x, y) := \langle Ax\,, y \rangle.$$

11. (a) Sea $\Phi : M(2 \times 2, \mathbb{R}) \to M(2 \times 2, \mathbb{R})$, $\Phi(A) = A^t$. Encuentre los valores propios y los espacios propios de Φ.

    (b) Sea P_2 el espacio vectorial de polinomios de grade menor o igual a 0 con coeficientes reales. Encuentre los valores propios y los espacios propios de $T : P_2 \to P_2, \quad Tp = p' + 3p$.

    (c) Sea R la reflexión en el plano $P : x + 2y + 3z = 0$ en $\mathbb{R}^3$. Calcule los valores propios y los espacios propios de R.

Appendix A

Complex Numbers

A *complex number* is an expression of the form

$$a + \mathrm{i}b$$

where $a, b \in \mathbb{R}$ and i is called the *imaginary unit*. The number a is called the *real part of* z, denoted by $\mathrm{Re}(z)$ and b is called the *imaginary part of* z, denoted by $\mathrm{Im}(z)$.
The set of all complex numbers is sometimes called the *complex plane* and it is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + \mathrm{i}b : a, b \in \mathbb{R}\}.$$

A complex number can be visualised as a point in plane where a is the coordinate on the *real axis* and b is the coordinate on the *imaginary axis*.
Let $a, b, x, y \in \mathbb{R}$. We define the algebraic operations sum and product for complex numbers $z = a + \mathrm{i}b$, $w = x + \mathrm{i}y$:

$$\begin{aligned} z + w &= (a + \mathrm{i}b) + (x + \mathrm{i}y) := a + x + \mathrm{i}(b + y), \\ zw &= (a + \mathrm{i}b)(x + \mathrm{i}y) := ax - by + \mathrm{i}(ay + bx). \end{aligned}$$

Exercise A.1. Show that if we identify the complex number $z = a+\mathrm{i}b$ with the vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, then the addition of complex planes is the same as the addition of vectors in $\mathbb{R}^n$.

We will give a geometric interpretation of the multiplication of complex numbers later after formula (A.5).
It follows from the definition above that $\mathrm{i}^2 = -1$. Moreover, we can view the real numbers $\mathbb{R}$ as a subset of $\mathbb{C}$ if we identify a real number x with the complex number $x + 0\mathrm{i}$.
Let $a, b \in \mathbb{R}$ and $z = a + \mathrm{i}b$. Then the *complex conjugate* of xz is

$$\overline{z} = a - \mathrm{i}b$$

and its *modulus* or *norm* is

$$|z| = \sqrt{a^2 + b^2}.$$

Geometrically, the complex conjugate is obtained from the z by an reflection on the x-axis and its norm is the distance of the point represented by z from the origin of the complex plane.

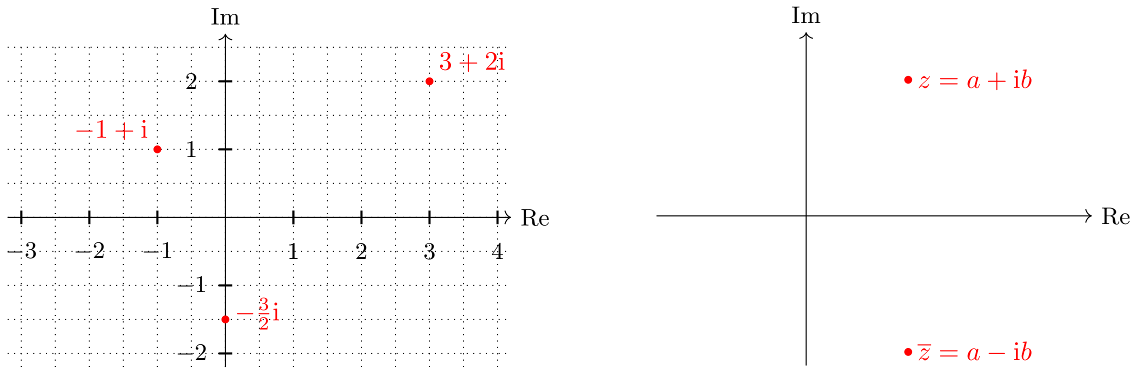

FIGURE A.1: Complex plane.

Properties A.2. *Let* $a, b, x, y \in \mathbb{R}$ *and let* $z = a + \mathrm{i}b$, $w = x + \mathrm{i}y$. *Then:*

(i) $z = \operatorname{Re} z + \mathrm{i} \operatorname{Im} z$.

(ii) $\operatorname{Re}(z+w) = \operatorname{Re}(z) + \operatorname{Re}(w)$, $\operatorname{Im}(z+w) = \operatorname{Re}(z) + \operatorname{Im}(w)$.

(iii) $\overline{(\overline{z})} = z$, $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$.

(iv) $z\overline{z} = |z|^2$.

(v) $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$, $\operatorname{Re} z = \frac{1}{2\mathrm{i}}(z - \overline{z})$.

Proof. (i) and (ii) should be clear. For (iii) not that $\overline{\overline{z}} = \overline{a - \mathrm{i}b} = a + \mathrm{i}b$,

$$\begin{aligned} \overline{z+w} &= \overline{a + x + \mathrm{i}(b+y)} = a + x - \mathrm{i}(b+y) = a - \mathrm{i}b + x - \mathrm{i}y = \overline{a + \mathrm{i}b} + \overline{x + \mathrm{i}y} = \overline{z} + \overline{w}, \\ \overline{zw} &= \overline{ax - by + \mathrm{i}(ay + bx)} = ax - by + \mathrm{i}(ay+bx) = (a - \mathrm{i}b)(x - \mathrm{i}y) = (\overline{a + \mathrm{i}b})(\overline{x + \mathrm{i}y})\overline{z}\,\overline{w}. \end{aligned}$$

(iv) follows from

$$z\overline{z} = (a + \mathrm{i}b)(\overline{a + \mathrm{i}b}) = (a + \mathrm{i}b)(a - \mathrm{i}b) = a^2 + b^2 + \mathrm{i}(ab - ba) = a^2 + b^2 = |z|^2$$

and (v) follows from

$$\begin{aligned} z + \overline{z} &= a + \mathrm{i}b + (\overline{a + \mathrm{i}b}) = a + \mathrm{i}b + a - \mathrm{i}b = 2a = 2\operatorname{Re}(z), \\ z + \overline{z} &= a + \mathrm{i}b - (\overline{a + \mathrm{i}b}) = a + \mathrm{i}b - (a - \mathrm{i}b) = 2\mathrm{i}b = 2\mathrm{i}\operatorname{Im}(z). \end{aligned}$$ □

We call a complex number *real* if it is of the form $z = a + \mathrm{i}0$ for some $a \in \mathbb{R}$ and we call it *purely imaginary* if it is of the form $z = 0 + \mathrm{i}b$ for some $b \in \mathbb{R}$. Hence

$$\begin{array}{rcccl} z \ \text{ is real} & \iff & z = \overline{z} & \iff & z = \operatorname{Re}(z) \\ z \ \text{ is purely imaginary} & \iff & z = -\overline{z} & \iff & z = \operatorname{Im}(z). \end{array}$$

It turns out that $\mathbb{C}$ is a *field*, that is, it satisfies

(a) **Associativity of addition:** $(u+v)+w = u+(v+w)$ for every $u, v, w \in \mathbb{C}$.

(b) **Commutativity of addition:** $v+w = w+v$ for every $u, v \in \mathbb{C}$.

(c) **Identity element of addition:** There exists an element 0, called the *additive identity* such that for every $v \in \mathbb{C}$, we have $0+v = v+0 = v$.

(d) **Additive inverse:** For all $z \in \mathbb{C}$, we have an inverse element $-z$ such that $z+(-z) = 0$.

(e) **Associativity of multiplication** $(uv)w = u(vw)$ for every $u, v, w \in \mathbb{C}$.

(f) **Commutativity of multiplication** $vw = wv$ for every $u, v \in \mathbb{C}$.

(g) **Identity element of addition:** There exists an element 1, called the *multiplicative identity* such that for every $v \in \mathbb{C}$, we have $1 \cdot v = v + \cdot 1 = v$.

(h) **Multiplicative inverse:** For all $z \in \mathbb{C} \setminus \{0\}$, we have an inverse element z^{-1} such that $z \cdot z^{-1} = 1$.

(i) **Distributivity laws:** For all $u, v, w \in \mathbb{C}$ we have

$$u(w+v) = uw + uv.$$

It is easy to check that commutativity, associativity and distributivity hold. Clearly,the additive identity is $0+\mathrm{i}0$ and the multiplicative identity is $1+0\mathrm{i}$. If $z = a+\mathrm{i}b$, then its additive inverse is $-a-\mathrm{i}b$. If $z \in \mathbb{C} \setminus \{0\}$, then $z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{a-\mathrm{i}b}{a^2+b^2}$. This can be seen easily if we recall that $|z|^2 = z\overline{z}$. The proof of the next theorem is beyond the scope of these lecture notes.

Theorem A.3 (Fundamental theorem of algebra). *Every non-constant complex polynomial has at least one complex root.*

We obtain immediately the following corollary.

Corollary A.4. *Every complex polynomial p can be written in the form*

$$p(z) = c(z-\lambda_1)^{n_1} \cdot (z-\lambda_2)^{n_2} \cdot \dots \cdot (z-\lambda_k)^{n_1} \tag{A.1}$$

where $\lambda_1, \dots, \lambda_k$ are the different roots of p. Note that $n_1 + \dots + n_k = \deg(p)$.

The integers $n_1, \dots, n_k$ are called the *multiplicity* of the corresponding root.

Proof. Let $n = \deg(p)$. If $n = 0$, then p is constant and it clearly of the form (**??**). If $n > 0$, then, by Theorem **??** there exists $\mu_1 \in \mathbb{C}$ such that $p(\mu) = 0$. Hence there exists some polynomial q_1 such that $p(z) = (z-\mu)q_1(z)$. Clearly, $\deg(q) = n-1$. If q_1 is constant, we are done. If q_1 is not constant, then it must have a zero μ_2. Hence $q_1(z) = (z-\mu_2)q_2(z)$ with some polynomial q_2 with $\deg(q_2) = n-2$. If we repeat this process n times, we finally obtain that

$$p(z) = c(z-\mu_1)(z-\mu_2)\cdots(z-\mu_n).$$

Now we only have to group all μ_j which are equal and we obtain the form (**??**). $\square$

Functions of complex numbers

It is more or less obvious how to form a complex polynomial. We can also extend functions which admit a power series representation to the complex numbers. To this end, we recall (from some calculus course) that a power series is an expression of the form

$$\sum_{n=0}^{\infty} c_n(z-a)^n \tag{A.2}$$

where the c_n are the coefficients and a is where the power series is centred. In our case, they are complex numbers and z is a complex number. Recall that a series $\sum_{n=0}^{\infty} c_n$ is called *absolutely convergent* if and only if $\sum_{n=0}^{\infty} |c_n|$ is convergent. It can be shown that every absolutely convergent series of complex numbers is convergent. Moreover, for every power series of the form (A.2) there exists a number $R > 0$ or $R = \infty$, called the *radius of convergence* such that the series converges absolutely for every $z \in \mathbb{C}$ with $|z-a| < R$ and it diverges for z with $|z-a| > R$. That means that the series converges absolutely for all z in the open disc with radius R centred in a, and it diverges outside the closed disc with radius R centred in a. For z on the boundary the series may converge or diverge. Note that $R = 0$ and $R = \infty$ are allowed. If $R = 0$, then the series converges only for $z = a$ and if $R = \infty$, then the series converges for all $z \in \mathbb{C}$.

Important functions that we know from the real numbers and have a power series are sine, cosine and the exponential function. We can use their power series representation to define them also for complex functions.

Definition A.5. Let $z \in \mathbb{C}$. Then we define

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \qquad \mathrm{e}^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n. \tag{A.3}$$

Note that for every z the series in (A.3) is absolutely convergent because, for instance $\sum_{n=0}^{\infty} |\frac{(-1)^n}{(2n+1)!} z^{2n+1}| = \sum_{n=0}^{\infty} |\frac{(-1)^n}{(2n+1)!}| |z|^{2n+1}$ is convergent because $|z|$ is a real number and we know that the cosine series is absolutely convergent for every real argument. Hence the sine series is absolutely convergent for any $z \in \mathbb{C}$, hence converges. The same argument shows that the series for the cosine and for the exponential are convergent for every $z \in \mathbb{C}$.

Remark A.6. Since the series for the sine function contains only odd powers of z, it is an odd function and cosine is an even function because it contains only even powers of z. In formulas:

$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z.$$

Next we show the relation between the trigonometric and the exponential function.

Theorem A.7 (Euler formulas). *For every $z \in \mathbb{C}$ we have that*

$$\begin{aligned} \mathrm{e}^{\mathrm{i}z} &= \cos z + \mathrm{i} \sin z, \\ \cos(z) &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}), \\ \sin(z) &= \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}). \end{aligned}$$

Proof. Let us show the formula for $\mathrm{e}^{\mathrm{i}z}$. In the calculation we will use that $\mathrm{i}^{2n} = (\mathrm{i}^2)^n = (-1)^n$ and $\mathrm{i}^{2n+1} = (\mathrm{i}^2)^n \mathrm{i} = (-1)^n \mathrm{i}$ and

$$\begin{aligned} \mathrm{e}^{\mathrm{i}z} &= \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{i}z)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{i}^n z^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\mathrm{i}^{(2n)} z^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\mathrm{i}^{(2n+1)} z^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n z^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\mathrm{i}(-1)^n z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + \mathrm{i} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= \cos z + \mathrm{i} \sin z. \end{aligned}$$

For the proof of the formula for $\cos z$ we note that from what we just proved, it follows that

$$\begin{aligned} \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) &= \frac{1}{2}(\cos(z) + \mathrm{i}\sin(z) + \cos(-z) + \mathrm{i}\sin(-z)) = \frac{1}{2}(\cos(z) + \mathrm{i}\sin(z) + \cos(z) - \mathrm{i}\sin(z)) \\ &= \cos(z). \end{aligned}$$

The formula for the sine function follows analogously. □

Exercise. Let $z, w \in \mathbb{C}$. Show the following.

(i) $\mathrm{e}^z \, \mathrm{e}^w = \mathrm{e}^{z+w}$. *Hint.* Use Cauchy product.

(ii) Use the Euler formulas to prove $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$, $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$, $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$.

(iii) $(\cos z)^2 + (\sin z)^2 = 1$.

(iv) $\cosh(z) = \cos(\mathrm{i}z)$, $\sinh(z) = -\mathrm{i}\sin(\mathrm{i}z)$. In particular, sin and cos are not bounded functions in $\mathbb{C}$.

(v) Show that the exponential functions is $2\pi\mathrm{i}$ periodic.

Polar representation of complex numbers

Let $z \in \mathbb{C}$ with $|z| = 1$ and let φ be the angle between the positive real axis and the line connecting the origin and z. φ is called the *argument* of z. It is denoted by $\arg(z)$. Observe that the argument is only determined modulo 2π. That means, if we add or subtract any integer multiple of 2π to the argument, we obtain another valid argument.

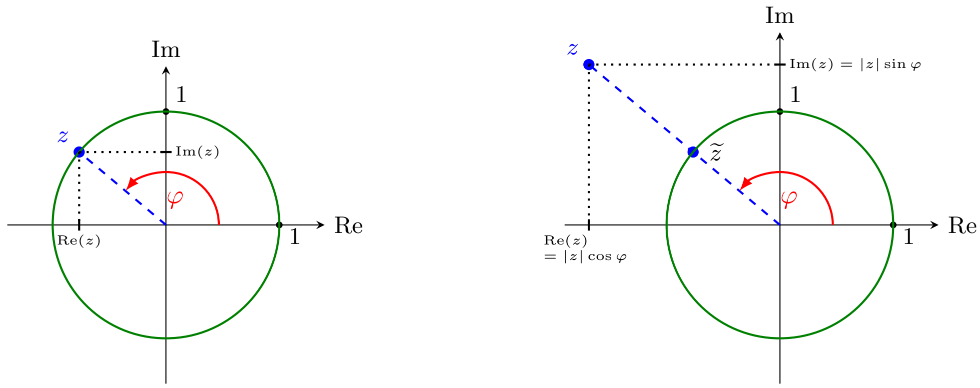

FIGURE A.2: Left picture: If $|z| = 1$, then $z = \cos\varphi + \mathrm{i}\sin\varphi = \mathrm{e}^{\mathrm{i}\varphi}$.
Right picture: If $z \neq 0$, then $z = |z|\cos\varphi + \mathrm{i}|z|\sin\varphi = |z|\,\mathrm{e}^{\mathrm{i}\varphi}$.

Then the real and imaginary part of z are $\mathrm{Re}(z) = \cos\varphi$ and $\mathrm{Im}(z) = \mathrm{i}\sin\varphi$, and therefore $z = \cos\varphi + \mathrm{i}\varphi = \mathrm{e}^{\mathrm{i}\varphi}$. We saw in Remark 2.3 how we can calculate the argument of a complex number.

Now let $z \in \mathbb{C} \setminus \{0\}$ and again let φ be the angel between the positive real axis and the line connecting the origin with z. Let $\widetilde{z} = \frac{z}{|z|}$. Then $|\widetilde{z}| = 1$ and therefore $\widetilde{z} = \mathrm{e}^{\mathrm{i}\varphi}$. It follows that

$$z = |z|\,\mathrm{e}^{\mathrm{i}\varphi}\,. \tag{A.4}$$

(A.4) is called de polar representation of z.

Now we can give a geometric interpretation of the product of two complex numbers. Let $z, w \in \mathbb{C} \setminus \{0\}$ and let $\alpha = \arg z$ and $\beta = \arg w$. Then

$$zw = |z|\,\mathrm{e}^{\mathrm{i}\alpha}\;|w|\,\mathrm{e}^{\mathrm{i}\beta} = |z|\,|w|\,\mathrm{e}^{\mathrm{i}(\alpha+\beta}\,. \tag{A.5}$$

This shows that the product zw is the complex number whose norm is the product of the norms of z and w and whose argument is the sum of the arguments of z and w.

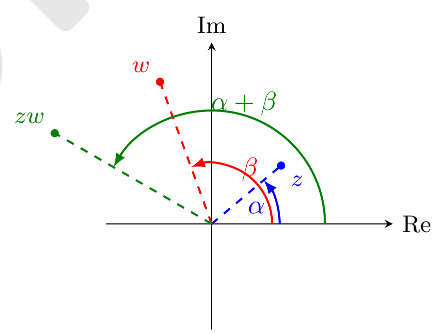

FIGURE A.3: Left picture: If $|z| = 1$, then $z = \cos\varphi + \mathrm{i}\sin\varphi = \mathrm{e}^{\mathrm{i}\varphi}$.
Right picture: If $z \neq 0$, then $z = |z|\cos\varphi + \mathrm{i}|z|\sin\varphi = |z|\,\mathrm{e}^{\mathrm{i}\varphi}$.

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