Work in progress. Use at your own risk.

Contents

Chapter 1

Systems of Linear Equations

Bla bla bla

1.1 Examples of systems of linear equations

Let us start with a few examples of linear systems of linear equations.

Example 1.1. Assume that a zoo has birds and cats. All of their animals combined, they have 60 heads and 200 legs. How many birds and cats are in the zoo?

Solution. First, we give names to the quantities we want to calculate. So let B = number of birds, C = number of cats in the zoo. If we write the information given in the exercise in formulas, we obtain

$$\begin{array}{lrl} \textcircled{1} & b + c = 60, & \text{(total number of heads)} \\ \textcircled{2} & 2b + 4c = 200, & \text{(total number of legs)} \end{array}$$

since each bird has 1 head and 2 legs and each cat has 1 head and legs. Equation $\textcircled{1}$ tells us that $B = 60 - C$. If we insert this into equation $\textcircled{2}$, we find

$$200 = 2(60 - C) + 4C = 120 - 2C + 4C = 120 + 2C \quad \Longrightarrow \quad 2c = 80 \quad \Longrightarrow \quad c = 40.$$

This implies that $B = 60 - C = 60 = 40 = 20$. Note that in our calculations and arguments, all the arrow all go "from left to right", so we found that the only possible solution is $B = 40, C = 20$. Inserting this in the original equation shows that this is indeed a solution. So there are 40 birds and 20 cats. ◇

Let us put one more equation into the zoo.

Example 1.2. Assume that a zoo has birds and cats. All of their animals combined, they have 60 heads and 200 legs. Moreover, there are 140 cage and in every cage there are either 2 birds or 3 cats. How many birds and cats are in the zoo?

Solution. Again, let B = number of birds, C = number of cats in the zoo. The information of the exercise gives the following equations:

$$
\begin{array}{lrl}
① & B+\quad C = 60, & \text{(total number of heads)} \\
② & 2B + 4C = 200, & \text{(total number of legs)} \\
③ & 2B + 3C = 140. & \text{(total number of cages)}
\end{array}
$$

As in the previous exercise, we obtain from that $B = 40, C = 20$. Clearly, this also satisfies equation ③. ◇

Example 1.3. Assume that a zoo has birds and cats. All of their animals combined, they have 60 heads and 200 legs. Moreover, there are 100 cage and in every cage there are either 2 birds or 3 cats. How many birds and cats are in the zoo?

Solution. Again, let B = number of birds, C = number of cats in the zoo. The information of the exercise gives the following equations:

$$
\begin{array}{lrl}
① & B+\quad C = 60, & \text{(total number of heads)} \\
② & 2B + 4C = 200, & \text{(total number of legs)} \\
③ & 2B + 3C = 100. & \text{(total number of cages)}
\end{array}
$$

As in the previous exercise, we obtain from that $B = 40, C = 20$. However, this does not satisfy equation ③; so there is no way to choose B and C such that all three equations are satisfied simultaneously. Therefore, a zoo as in this example does not exist. ◇

We give a few more examples.

Example 1.4. Find a polynomial P of degree at most 3 with

$$P(0) = 1, \quad P(1) = 7, \quad P'(0) = 3, \quad P'(2) = 23. \tag{1.1}$$

Solution. A polynomial of degree at most 3 is known, if we know its 4 coefficients. In this exercise, the unknowns are the coefficients of the polynomial P. We can write $P(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$ and we have to find $\alpha, \beta, \gamma, \delta$ such that (1.1) is satisfied. Note that $P'(x) = 3\alpha x^2 + 2\beta x + \gamma$. Hence (1.1) is equivalent to the following system of equations:

$$
\left.\begin{array}{r}
P(0) = 1, \\
P(1) = 7, \\
P'(0) = 3, \\
P'(2) = 23.
\end{array}\right\}
\iff
\left\{\begin{array}{lr}
① & \delta = 1, \\
② & \alpha + \quad \beta + \quad \gamma + \delta = 7, \\
③ & \gamma \quad = 3, \\
④ & 24\alpha + 8\beta + 2\gamma + \delta = 23.
\end{array}\right.
$$

Clearly, $\delta = 1$ and $\gamma = 3$. If we insert this in the remaining equations, we obtain a system of two equations for the two unknowns α, β:

$$
\begin{array}{lr}
②' & \alpha + \quad \beta = 3, \\
④' & 24\alpha + 8\beta = 16.
\end{array}
$$

From ②' we obtain $\beta = 4 - \alpha$. If we insert this into ④', we get that $16 = 24\alpha + 8(4 - \alpha) = 16\alpha + 32$, that is, $\alpha = (32 - 16)/16 = 1$. So the only possible solution is

$$\alpha = 1, \quad \beta = 2, \quad \gamma = 3, \quad \delta = 1.$$

It is easy to verify that the polynomial $P(x) = x^3 + 2x^2 + 3x + 1$ has all the desired properties. $\diamond$

Example 1.5. A pole is 5 metres long and shall be coated with varnish. There are two types of varnish available: The green one adds 3 g per 50 cm to the pole, the red one adds 6 g per meter to the pole. Is it possible to coat the pole in a combination of the varnishes so that the total weight added is

(a) 35 g? (b) 30 g?

Solution. (a) We call g the length of the pole which will be covered in green and r the length of the pole which will be covered in red. Then we obtain the system of equations

$$\begin{array}{lrl} ① & g + r = 5 & \text{(total length)} \\ ② & 6g + 6r = 35 & \text{(total weight)} \end{array}$$

The first equation gives $r = 5 - g$. Inserting into the second equation yields $35 = 6g + 6(5 - g) = 30$ which is a contradiction. This shows that there is no solution.
(b) As in (a), we obtain the system of equations

$$\begin{array}{lrl} ① & g + r = 5 & \text{(total length)} \\ ② & 6g + 6r = 30 & \text{(total weight)} \end{array}$$

Again, the first equation gives $r = 5 - g$. Inserting into the second equation yields $30 = 6g + 6(5 - g) = 30$ which is always true, independently of how we choose g and r as long as ① is satisfied. This means that in order to solve the system of equations, it is sufficient to solve only the first equation since then the second one is automatically satisfied. So we have infinitely many solutions. Any pair g, r such that $g + r = 5$ gives a solution. So for any g that we choose, we only have to set $r = 5 - g$ and we have a solution of the problem. Of course, we could also fix r and then choose $g = 5 - r$ to obtain a solution.
For example, we could choose $g = 1$, then $r = 4$, or $g = 0.00001$, then $r = 4.99999$, or $r = -2$ then $g = 7$. Clearly, the last example does not make sense for the problem at hand, but it still does satisfy our system of equations. $\diamond$

All the examples were so-called linear systems of linear equations. Let us define what we mean by this,

Definition 1.6. A $m \times n$ *system of linear equations* is a system of m linear equations for n unknowns of the form

$$\begin{array}{llll} a_{11}x_1 & + a_{12}x_2 & + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots + a_{2n}x_n & = b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + \cdots + a_{mn}x_n & = b_m. \end{array}$$

The unknowns are $x_1, \dots, x_n$. The numbers a_{ij} and b_i ($i = 1, \dots, m$, $j = 1, \dots, n$) are given. The numbers a_{ij} are called the *coefficients of the linear system* and numbers $b_1, \dots, b_n$ are called the *right side of the linear system*.

In the special case when all b_i are equal to 0, the system is called a *homogeneous*; otherwise it is called *inhomogeneous*.

The *coefficient matrix* A of the system is the collection of all coefficients a_{ij}

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The coefficient matrix is nothing else than the collection of the coefficients a_{ij} ordered in some sort of table or rectangle such that the place of the coefficient a_{ij} is in the ith row of the jth column.

Let us come back to our examples.

Example 1.1: This is a 2×2 system with coefficients $a_{11} = 1$, $a_{11} = 1$, $a_{21} = 2$, $a_{22} = 4$ and right hand side $b_1 = 60, b_2 = 200$. The system has a unique solution. The coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}.$$

Example 1.2: This is a 3×2 system with coefficients $a_{11} = 1$, $a_{11} = 1$, $a_{21} = 2$, $a_{22} = 4$, $a_{31} = 2$, $a_{32} = 3$, and right hand side $b_1 = 60, b_2 = 200$, $b_3 = 140$. The system has a unique solution. The coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 2 & 3 \end{pmatrix}.$$

Example 1.3: This is a 3×2 system with coefficients $a_{11} = 1$, $a_{11} = 1$, $a_{21} = 2$, $a_{22} = 4$, $a_{31} = 2$, $a_{32} = 3$, and right hand side $b_1 = 60, b_2 = 200$, $b_3 = 100$. The system has no solution. The coefficient matrix is the same as in Example 1.2.

Example 1.4: This is a 4×4 system with coefficients $a_{11} = 0$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = 1$, $a_{21} = 1$, $a_{22} = 1$, $a_{23} = 1$, $a_{24} = 1$, $a_{31} = 0$, $a_{32} = 0$, $a_{33} = 1$, $a_{34} = 0$, $a_{41} = 24$, $a_{42} = 8$, $a_{43} = 2$, $a_{44} = 1$, and right hand side $b_1 = 1, b_2 = 7$, $b_3 = 3$, $b_4 = 23$. The system has a unique solution. The coefficient matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 24 & 8 & 2 & 1 \end{pmatrix}.$$

Example 1.5: This is a 2×2 system with coefficients $a_{11} = 1$, $a_{11} = 6$, $a_{21} = 1$, $a_{22} = 6$. In case (a) the right hand side is $b_1 = 5, b_2 = 35$ and the system has no solution.
In case (b) the right hand side is $b_1 = 5, b_2 = 30$ and the system has infinite solutions.
In both cases, the coefficient matrix is

$$A = \begin{pmatrix} 1 & 6 \\ 1 & 6 \end{pmatrix}.$$

Given an $m \times n$ system of linear equations, two important solutions arise:

- *Existence*: Does the system have a solution?
- *Uniqueness*: If the system has a solution, is it unique?

As we saw, in Examples 1.1, 1.2, 1.4, 1.5 (b) solutions do exist. In Example 1.5 (b) the solution is not unique (on the contrary: it has infinite solutions!). Examples 1.3 and 1.5(a) do not admit solutions.

More generally, we would like to be able so say something about the structure of solutions of linear systems. For example, is it possible that there is only one solution? That there are exactly two solutions? That there are infinite solutions? That there is is no solution? Can we give criteria for existence and/or uniqueness of solutions? Can we give criteria for existence of infinite solutions?

(Spoiler alert: *A system of linear equations has either no or exactly one or infinite solutions. It is not possible that it has, e.g., exactly* 7 *solutions.*)

Before answering these questions for general $m \times n$ systems, we will have a closer look at 2×2 systems in the next section.

1.2 Linear 2×2 systems of equations

Let us come back to the equation from Example 1.1. For convenience, we write now x instead of B and y instead of C. Recall that the system of equations that we are interested in solving is

$$\begin{array}{cr} \textcircled{1} & x + y = 60, \\ \textcircled{2} & 2x + 4y = 200. \end{array} \tag{1.2}$$

We want to give a geometric meaning to this system of equations. To this end we think of pairs x, y as points (x, y) in the plane. Let's forget about equation ② for a moment and concentrate only on ①. Clearly, there are infinitely many solutions. If we choose an arbitrary x, we can always find y such that ① satisfied (just take $y = 60 - x$). Similarly, if we choose any y, then we only have to take $x = 60 - y$ and we obtain a solution of ①.
Now, where in the xy-plane lie *all* solutions of ①? Clearly, ① is equivalent to $y = 60 - x$ which we easily identify of the equation of the line L_1 in the xy-plane which passes through $(0, 60)$ and has slope -1. In summary, a pair (x, y) is a solution of ① if and only if it lies on the line L_1.

If we apply the same reasoning to ②, we find that a pair (x, y) satisfies ② if and only if (x, y) lies on the line L_2 in the xy-plane given by $y = \frac{1}{4}(200 - 2x)$ (this is the line in the xy-plane passing through $(9, 50)$ with slope $-\frac{1}{2}$).

Now it is clear that a pair (x, y) satisfies both ① and ② if and only if it lies both on L_1 and L_2. So finding the solution of our system (1.2) is the same as finding the intersection of the two lines L_1 and L_2. From elementary geometry we know that there are exactly three possibilities:

(i) L_1 and L_2 are not parallel. Then they intersect in exactly one point.

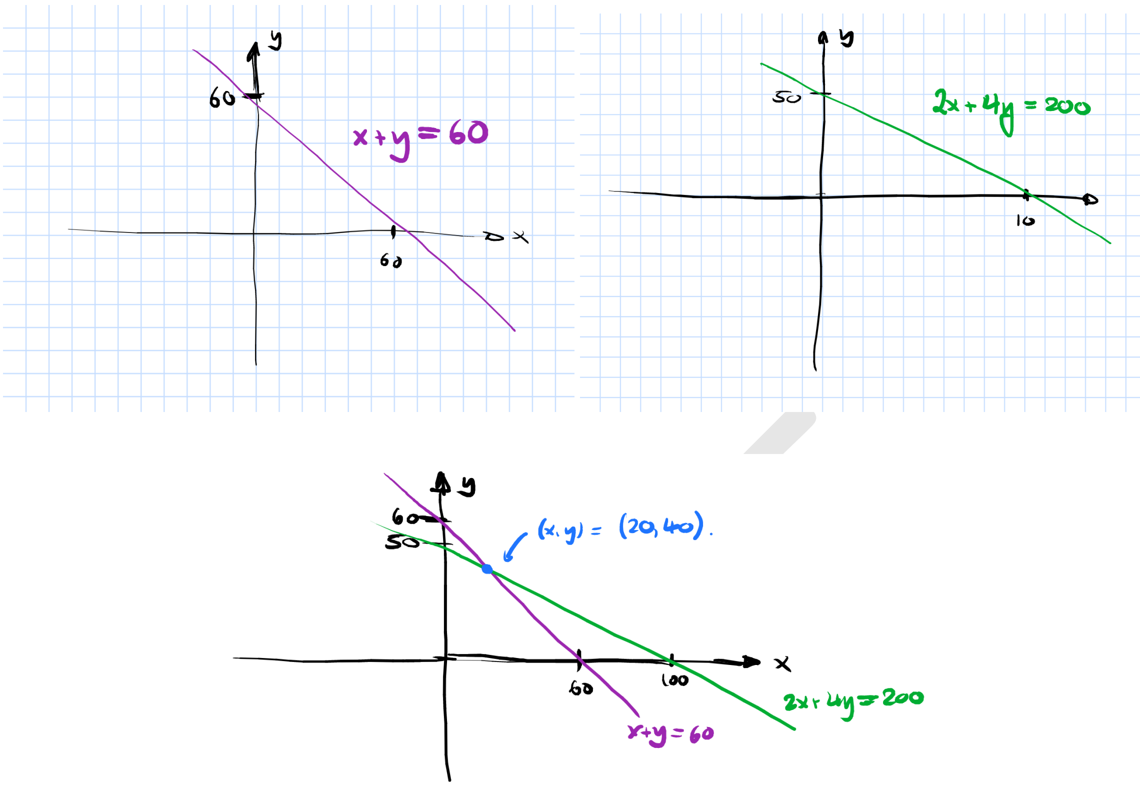

FIGURE 1.1: Example 1.1. Graphs of L_1, L_2 and their intersection.

(ii) L_1 and L_2 are parallel and not equal. Then they do not intersect.

(iii) L_1 and L_2 are parallel and equal. Then $L_1 = L_2$ and they intersect in infinite points (they intersect in every point of $L_1 = L_2$).

In our example we know that the slope of L_1 is -1 and that the slope of L_2 is $-\frac{1}{2}$, so they are not parallel and therefore intersect in exactly one point. Consequently, the system (1.2) has exactly one solution, see Figure 1.1

If we look again at Example 1.5, we see that in Case (a) we look for the intersection of the lines

$$L_1 : y = 5 - x, \qquad L_2 : y = \frac{35}{6} - x.$$

Both lines have slope -1 so they are parallel. Since the constant terms in both lines are not equal, they never intersect, showing that the system of equations has no solution, see Figure 1.2.
In Case (b), the two lines that we have to intersect are

$$G_1 : y = 5 - x, \qquad G_2 : y = 5 - x.$$

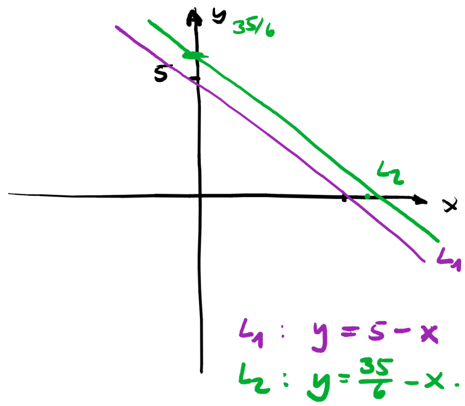

FIGURE 1.2: Example 1.5. Graphs of G_1, G_2.

We see that $G_1 = G_2$, so every point on G_1 (or G_2) is solution of the system and therefore we have infinite solutions.

Now let us consider the general case.

One linear equation with two unknowns

The general form of one linear equation with two unknowns is

$$\alpha x + \beta y = \gamma. \tag{1.3}$$

For the set of solutions, there are three possibilities:

(i) *The set of solutions forms a line.* This happens if at least one of the coefficients α or β is different from 0. If $\beta \neq 0$, then set of all solutions is equal to the line $L : y = -\frac{\alpha}{\beta}\,x + \frac{\gamma}{\beta}$ which is a line with slope $-\frac{\alpha}{\gamma}$. If $\beta = 0$ and $\alpha \neq 0$, then the set of solutions of (1.3) is a line parallel to the y-axis passing through $(\frac{\gamma}{\alpha})$.

(ii) *The set of solutions is all of the plane.* This happens if $\alpha = \beta = \gamma = 0$. In this case, clearly every pair (x, y) is a solution of (1.3).

(iii) *The set of solutions is empty.* This happens if $\alpha = \beta = 0$ and $\gamma \neq 0$. In this case, no pair (x, y) can be a solution of (1.3) since the left hand side is always 0.

Two linear equations with two unknowns

The general form of one linear equation with two unknowns is

$$\begin{array}{ll} ① & Ax + By = U \\ ② & Cx + Dy = V. \end{array} \tag{1.4}$$

We are using the letters A, B, C, D instead of $a_{11}, a_{12}, a_{21}, a_{22}$ in order to make the calculations more readable. If we interprete the system of equations as intersection of two geometrical objects, we already know how the possible solutions will be:

- *A point* if ① and ② describe two non-parallel lines.
- *A line* if ① and ② describe the same line; or if one of the equations is a plane and the other one is a line.
- *A plane* if both equations describe a plane.
- *The empty set* if the two equations describe parallel but different lines; or if one of the equations has no solution.

In summary, we have:

Remark 1.7. The system (1.4) has either exactly 1 solution or infinite solutions or no solution.

Exercise. How is the situation if we had a system of 3 linear equations for 2 unknowns?

Proof of Remark 1.7. Now we want proof the Remark 1.7 algebraically and we want to find a criteria on a, b, c, d which allows us to decide easily how many solutions there are. Let's look at the different cases.

Case 1. $B \neq 0$. In this case we can solve ① for y and obtain $y = \frac{1}{B}(U - Ax)$. In ② this gives $Cx + \frac{D}{B}(U - Ax) = V$. If we put all terms with x on one side and all other terms on the other side, we obtain

$$②' \quad (AD - BC)x = DU - BV$$

(i) If $AD - BC \neq 0$ then there is at most one solution, namely $x = \frac{DU-BV}{AD-BC}$ and consequently $y = \frac{1}{B}(U - Ax) = \frac{AV-CU}{AD-BC}$. Inserting these expressions for x and y in our system of equations, we see that they indeed solve the system (1.4), so that we have exactly one solution.

(ii) If $AD - BC = 0$, then equation ②' reduces to $0 = DU - BV$. This equation has either no solution (if $DU - BV \neq 0$) or infinite solutions (if $DU - BV = 0$). Since ① has infinite solutions, it follows that the system (1.4) has either no solution or infinite solutions.

Case 2. $D \neq 0$. In this case we can solve ② for y and obtain $y = \frac{1}{D}(V - Cx)$. In ② this gives $Ax + \frac{B}{D}(V - Cx) = U$. If we put all terms with x on one side and all other terms on the other side, we obtain

$$②' \quad (AD - BC)x = DU - BV$$

We have the same subcases as before:

(i) If $AD - BC \neq 0$ then there is exactly one solution, namely $x = \frac{DU-BV}{AD-BC}$ and consequently $y = \frac{1}{B}(U - Ax) = \frac{AV-CU}{AD-BC}$.

(ii) If $AD - BC = 0$, then equation $\textcircled{2'}$ reduces to $0 = DU - BV$. This equation has either no solution (if $DU - BV \neq 0$) or infinite solutions (if $DU - BV = 0$). Since $\textcircled{2}$ has infinite solutions, it follows that the system (1.4) has either no solution or infinite solutions.

Case 3. $B = 0$ and $D = 0$. Observe that in this case $AD - BC = 0$. In this case the system (1.4) reduces to

$$Ax = U, \quad Cx = V. \tag{1.5}$$

We see that the system no longer depends on y. So, if the system (1.5) has at least one solution, then we automatically have infinite solutions since we can choose y freely. If the system (1.5) has no solution, then the original system (1.4) cannot have a solution either.

Note that there are no other cases for the coefficients than these three cases. □

Summing up, we find the following theorem:

Theorem 1.8. *The system of linear equations*

$$\begin{aligned} &\textcircled{1} \quad Ax + By = U \\ &\textcircled{2} \quad Cx + Dy = V. \end{aligned} \tag{1.6}$$

has

(i) *exactly one solution if and only if* $AD - BC \neq 0$. *In this case, the solution is*

$$x = \frac{DU - BV}{AD - BC}, \quad y = \frac{AV - CU}{AD - BC}. \tag{1.7}$$

(ii) *no solution or infinite solutions if* $AB - BC = 0$.

Definition 1.9. The number $d := AD - BC$ is called the *determinant* of the system (1.6).

Later we will generalise this concept to systems with more equations and more variables.

Remark 1.10. Let us see how this connects to our geometric interpretation of the system of equations. Assume that $B \neq 0$ and $D \neq 0$. Then we can solve $\textcircled{1}$ and $\textcircled{2}$ for y obtain equations for lines

$$L_1: \quad y = -\frac{A}{B}x + \frac{1}{B}U, \qquad L_2: \quad y = -\frac{C}{D}x + \frac{1}{D}V.$$

The two lines intersect in exactly one point if and only if they have different slopes, i.e., if $-\frac{A}{B} \neq -\frac{C}{D}$. After multiplication by $-BD$ we see that this is the same as $AD \neq BC$, or $AD - BC \neq 0$.
On the other hand, the lines are parallel (and hence have either no intersection or are equal) if $-\frac{A}{B} \neq -\frac{C}{D}$. This is the case if and only if $AD = BC$, or in other word, if $AD - BC = 0$.

Exercise. Consider the cases when $B = 0$ or $D = 0$ and make the connection between Theorem 1.8 and the geometric interpretation of the system of equations.

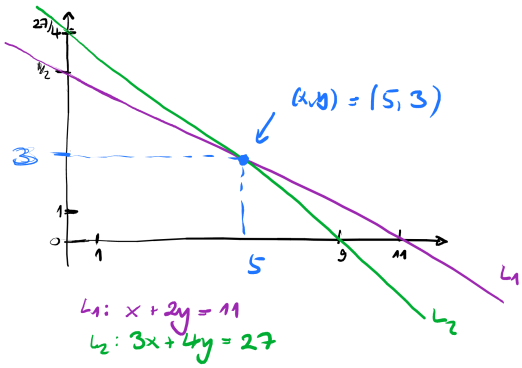

FIGURE 1.3: Example 1.11(a). Graphs of L_1, L_2 and their intersection $(5, 3)$.

Let us consider same examples.

Examples 1.11. (a)

$$\begin{aligned} ① \quad x + 2y &= 11 \\ ② \quad 3x + 4y &= 27. \end{aligned}$$

Clearly, the determinant is $d = 4 - 6 = -2 \neq 0$. So we expect *exactly one solution*.

We can check this easily: The first equation gives $x = 11 - 2y$. Inserting this into the second equations leads to

$$3(11 - 2y) + 4y = 27 \quad \Longrightarrow \quad -2y = -6 \quad \Longrightarrow \quad y = 3 \quad \Longrightarrow \quad x = 11 - 2 \cdot 3 = 5.$$

So the solution is $x = 5, y = 3$. (If we did not have Theorem 1.8, we would have to check that this is not only a candidate for a solution, but indeed is one.)

Exercise. Check that the formula (1.7) is satisfied.

(b)

$$\begin{aligned} ① \quad x + 2y &= 1 \\ ② \quad 2x + 4y &= 5. \end{aligned}$$

Here, the determinant is $d = 4 - 4 = 0$, so we expect *either no solution or infinite solutions*. The first equations gives $x = 1 - 2y$. Inserting into the second equations gives $2(1 - 2y) + 4y = 5$. We see that the terms with y cancel and we obtain $2 = 5$ which is a contradiction. Therefore, the system of equations has *no solution*.

(c)

$$\begin{aligned} ① \quad x + 2y &= 1 \\ ② \quad 3x + 6y &= 3. \end{aligned}$$

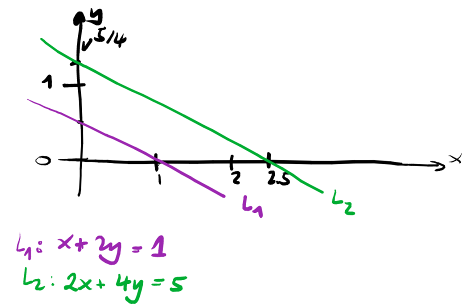

FIGURE 1.4: Example 1.11(b). The lines L_1, L_2 are parallel and do not intersect.

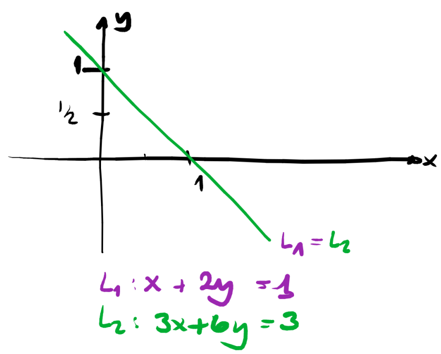

FIGURE 1.5: Example 1.11(c). The lines L_1, L_2 are equal.

The determinant is $d = 6 - 6 = 0$, so again we expect *either no solution or infinite solutions*. The first equations gives $x = 1-2y$. Inserting into the second equations gives $3(1-2y)+6y = 3$. We see that the terms with y cancel and we obtain $3 = 3$ which is true. Therefore, the system of equations has *infinite solutions* given by $x = 1 - 2y$.

Remark. This was somewhat clear since we can obtain the second equation from the first one by multiplying both sides by 3 which shows that both equations carry the same information and we loose nothing if we simply forget about one of them.

Example 1.12. Find all $k \in \mathbb{R}$ such that the system

$$\begin{aligned} &(1) \quad kx + (15/2 - k)y = 1 \\ &(2) \quad 4x + \qquad\quad 2ky = 3 \end{aligned}$$

has exactly one solution.

Solution. We only need to calculate the determinant and find all k such that it is different from zero. So let's start by calculating

$$d = k \cdot 2k - (15/2 - k) \cdot 4 = 2k^2 + 4k - 30 = 2(k^2 + 2k - 15) = 2[(k+1)^2 - 16].$$

So we see that there are exactly two values for k where $d = 0$, namely $k = -1 \pm 4$, that is $k_1 = 3$, $k_2 = -5$. For all other k, we have that $d \neq 0$.
So the answer is: The system has exactly one solution if and only if $k \in \mathbb{R} \setminus \{-5, 3\}$. ◇

Remark 1.13. 1. Note that the answer does not depend on the right hand side of the system of the equation. Only the coefficients on the left hand side determine if there is exactly one solution or not.

2. If we wanted, we could also calculate the solution x, y in the case $k \in \mathbb{R} \setminus \{-3, 1\}$. We could do it by hand or use (1.7). Either way, we find

$$x = \frac{1}{d}[2k - 3(15/2 - k)] = \frac{5k - 45/2}{2k^2 + 4k - 30}, \qquad y = \frac{1}{d}[6k - 4] = \frac{6k - 4}{2k^2 + 4k - 30}.$$

Note that the denominators would become 0 if $k = -5$ or $k = 3$.

3. What happens if $k = -3$ or $k = 1$? In both cases, $d = 0$, so we will either have no solution or infinite solutions.

If $k = -3$, then the system becomes

$$3x + 9/2y = 1, \qquad 4x + 6y = 3.$$

Multiplying the first equation by 4/3, we obtain

$$4x - 6y = \frac{4}{9}, \qquad 4x - 6y = 3$$

which clearly cannot be satisfied simultaneously.

If $k = 5$, then the system becomes

$$5x + 5/2y = 1, \qquad 4x + 10y = 3.$$

Multiplying the first equation by 4/5, we obtain

$$4x - +10 = \frac{4}{5}, \qquad 4x + 10y = 3$$

which clearly cannot be satisfied simultaneously.

1.3 Summary

1.4 Exercises

Chapter 2

$\mathbb{R}^2$ and $\mathbb{R}^3$

2.1 Vectors in $\mathbb{R}^2$

Recall that the xy-plane is the set of all pairs (x, y) with $x, y \in \mathbb{R}$. We will denote it by $\mathbb{R}^2$.

Maybe you already encountered vectors in a physics lecture. For instance velocities and forces are described by vectors. The velocity of a particle says how fast and in which direction the particle moves. Usually, a velocity are represented by an arrow which points in the direction in which the particle moves and whose length is proportional to the magnitude of the velocity.

A force has strength and a direction so it is represented by an arrow which point in the direction in which it acts and with length proportional to its strength.

Observe that it is not important where in the space $\mathbb{R}^2$ or $\mathbb{R}^3$ we put the arrow. As long it points in the same direction and has the same length, it is considered the same vector. We call two arrows *equivalent* if they have the same direction and the same length. A *vector* is the set of all arrows which are equivalent to a given arrow. Each specific arrow in this set is called a *representation* of the vector. A special representation is the arrow that starts in the origin $(0, 0)$.

Given two points P, Q in the xy-plane, we write $\overrightarrow{PQ}$ for the vector which is represented by the arrow that starts in P and ends in Q.

Example 2.1.

Let $P(1,1)$ and $Q(3,4)$ be points in the xy-plane. The arrow from P to Q is $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

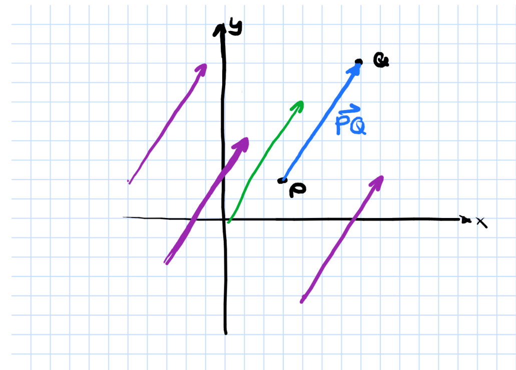

FIGURE 2.1: The vector $\overrightarrow{PQ}$ and several of its representations. The green arrow is the special representation whose initial point in is in the origin.

We can identify a point $P(p_1, p_2)$ in the xy-plane with the vector starting in $(0,0)$ and ending in P. We denote this vector by $\overrightarrow{0P}$ or $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ or sometimes by $(p_1,\, p_2)^t$ in order to save space (the subscript ^{t} stands for "transposed"). p_1 is called the *x-coordinate* or the *x-component* of $\vec{v}$ and p_2 is called the *y-coordinate* or the *y-component* of $\vec{v}$.
On the other hand, given a vector (a,b), then it describes a unique point in the xy-plane, namely the tip of the arrow which represents the given vector and starts in the origin.
So we can identify the set of all vectors in $\mathbb{R}^2$ with $\mathbb{R}^2$ itself.
Observe that the slope of the arrow $\vec{v} = (a,b)$ is $\frac{b}{a}$ if $a \neq 0$. If $a = 0$, then we obtain a vector which is parallel to the y-axis. Vectors are usually denoted by a small letter with an arrow on top.
If a vector is given, e.g., as $\vec{v} = (2,5)^t$, then this is an arrow whose tip would be at the point $(2,5)$ if its initial point is in the origin. If it is anywhere else, then we find the tip if we move 2 units to the right parallel to the x-axis and 5 units up parallel to the y-axis.

A very special vector is the zero vector $(0,0)^t$. Is is usually denoted by $\vec{0}$.

In order to distinguish numbers in $\mathbb{R}$ from vectors, we call them *scalars*.

Now we want to do algebra with vectors. If we think of a force and we double its strength then the corresponding vector should be twice as long. If we multiply the force by 5, then the length of the corresponding vector should be 5 times as long, that is, if for instance a force $\vec{F} = (3,4)$ is given, then $5\vec{F}$ should be $(5\cdot 3,\, 5\cdot 4) = (15,20)$. In general, if a vector $\vec{v} = (a,b)$ is given, then $c\vec{v} = (ca, cb)$. Note that the resulting vector is always parallel to the original one. If $c > 0$, then the resulting vector points in the same direction as the original one, if $c < 0$, then it points in the opposite direction, see Figure 2.2

How should we sum two vectors? Again, let us think of forces. Assume we have two forces $\vec{F}_1$ and $\vec{F}_2$ both acting on the same particle. Then we get the resulting force by drawing the arrow representing $\vec{F}_1$ and at its tip put the initial point of the arrow representing $\vec{F}_2$. The total force is then represented by the arrow starting in the initial point of $\vec{F}_1$ and ending in the tip of $\vec{F}_2$.

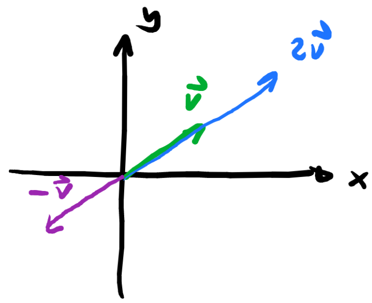

FIGURE 2.2: Multiplication of a vector by a scalar.

Exercise. Convince yourself that we obtain the same result if we start with $\vec{F}_2$ and put the initial point of $\vec{F}_1$ at the tip of $\vec{F}_2$.

We could also think of the sum of velocities. For example, if the have a train with velocity $\vec{v}_t$ and on the train a passenger is moving with relative velocity $\vec{v}_p$, then the total velocity is the vector sum of the two.
Now assume that $\vec{F}_1 = (a, b)^t$ and $\vec{F}_2 = (p, q)^t$. Algebraically, we obtain the components of their sum by summing the components: $\vec{F}_1 + \vec{F}_2 = (a + p, b + q)$, see Figure 2.3. When you do vector sums, you should always think in triangles (or polygons if you sum more than two vectors).

Exercise. Given two points $P(p_1, p_2)$, $Q(q_1, q_2)$ in the xy-plane. Convince yourself that $\overrightarrow{0P} + \overrightarrow{PQ} = \overrightarrow{0Q}$ and consequently $\overrightarrow{PQ} = \overrightarrow{0Q} - \overrightarrow{0P}$.
How could you write $\overrightarrow{QP}$ in terms of $\overrightarrow{0P}$ and $\overrightarrow{0Q}$? What is its relation with $\overrightarrow{PQ}$?

We sum up:

Definition 2.2. Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{w} = \begin{pmatrix} p \\ q \end{pmatrix}$, $c \in \mathbb{R}$. Then:

Vector sum: $$\vec{v} + \vec{w} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a + p \\ b + q \end{pmatrix},$$

Product with a scalar: $$c\vec{v} = c \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix},$$

With this definition, it is easy to see that for arbitrary vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and scalars $\alpha, \beta \in \mathbb{R}$ the so-called *vector space axioms* hold:

Vector Space Axioms.

(a) **Associativity:** $\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

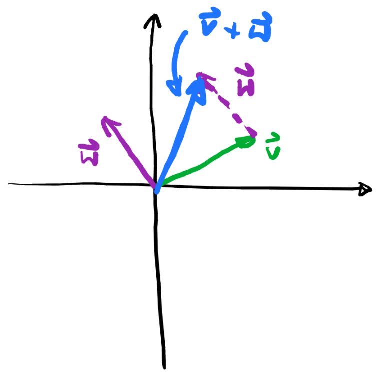

FIGURE 2.3: Sum of two vectors.

(b) **Commutativity:** $\vec{v} + \vec{w} = \vec{v} + \vec{w}$.

(c) **Identity element of addition:** For every $\vec{v} \in \mathbb{R}^2$, we have $\vec{0} + vecv = \vec{v} + \vec{0} = \vec{v}$.

(d) **Inverse element:** For every $\vec{v} \in \mathbb{R}^2$, we have an inverse element $\overrightarrow{v'}$ such that $\vec{v} + \overrightarrow{v'}) = \vec{0}$, namely $\overrightarrow{v'} = -\vec{v}$.

(e) **Identity element of multiplication by scalar:** For every $\vec{v} \in \mathbb{R}^2$, we have that $1\vec{v} = \vec{v}$.

(f) **Compatibility:** For every $\vec{v} \in \mathbb{R}^2$ and $a, b \in \mathbb{R},$, we have that $(ab)\vec{v} = a(b\vec{v})$.

(g) **Distributivity laws:** For all $\vec{v}, \vec{w} \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$, we have

$$(a + b)\vec{v} = a\vec{v} + b\vec{v} \quad \text{and} \quad a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}.$$

These axioms are fundamental for linear algebra. We will come back to them later when we deal with abstract vector spaces in XXX.

Let us look at some more geometric properties of vectors. Clearly a vector is known if we know its length and its angle with the x-axis.
From the Pythagoras theorem it is clear that the length of a vector $\vec{v} = (a, b)^t$ is $\sqrt{a^2 + b^2}$.

Definition 2.3 (Norm of a vector in $\mathbb{R}^2$). The *length* $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ is denoted by $\|\vec{\ }\|$. It is given by

$$\|\vec{v}\| = \sqrt{a^2 + b^2}\ .$$

Other names for the length of $\vec{v}$ are *magnitude of* $\vec{v}$ or *norm of* $\vec{v}$.

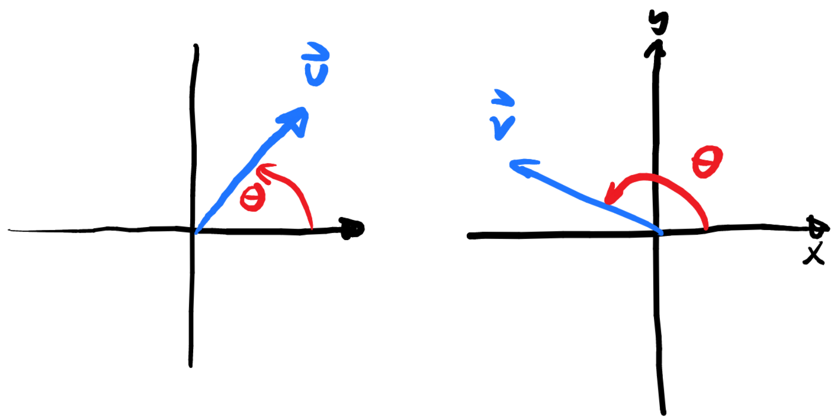

FIGURE 2.4: Length and angle of a vector.

As already mentioned earlier, the slope of vector $\vec{v}$ is $\frac{b}{a}$ if $a \neq 0$. If φ is the angle of the vector $\vec{v}$ with the x-axis then $\tan\varphi = \frac{b}{a}$ if $a \neq 0$. If $a = 0$, then $\varphi = 0$ or $\varphi = \pi$. Recall that the range of arctan is $(-\pi/2,\ \pi/2)$, so we cannot simply take arctan of the fraction $\frac{a}{b}$ in order to obtain φ. Observe that $\arctan\frac{b}{a} = \arctan -b-a$, however the angles of the vectors $(a,b)^t$ and $(-a,-b)^t$ are parallel but point in opposite directions, so they do not have the same angle with the x-axis. From geometry, we find

$$\varphi = \begin{cases} \arctan\frac{b}{a} & \text{if } a > 0, \\ \pi - \arctan\frac{b}{a} & \text{if } a < 0, \\ \pi/2 & \text{if } a = 0,\ b > 0, \\ -\pi/2 & \text{if } a = 0,\ b < 0. \end{cases}$$

Note that this formula gives angles with values $[-\pi/2,\ 3\pi/2)$.

Proposition 2.4 (Properties of the norm). *Let* $\lambda \in \mathbb{R}$ *and* $\vec{v}$, $\vec{w} \in \mathbb{R}^2$. *Then the following is true:*

(i) $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$,

(ii) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$,

(iii) $\|\vec{v}\| = 0$ *if and only if* $\vec{v} = \vec{0}$.

Proof. Let $\vec{v} = (a,b)^t$, $\vec{w} = (c,d)^t \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

(i)
$$\begin{aligned} \|\lambda\vec{v}\| = \|\lambda(a,b)^t\| = \|(\lambda a,\ \lambda b)^t\| = \sqrt{(\lambda a)^2 + (\lambda b)^2} = \sqrt{\lambda^2(a^2+b^2)} = |\lambda|\sqrt{a^2+b^2} \\ = |\lambda|\|\vec{v}\|. \end{aligned}$$

(ii) This will be shown later in XXX.

(iii) Since $\|\vec{v}\| = \sqrt{a^2 + b^2}$ it follows that $\|\vec{\,}\| = 0$ if and only if $a = 0$ and $b = 0$. This is the case if and only if $\vec{v} = \vec{0}$.

□

Definition 2.5. A vector $\vec{v} \in \mathbb{R}^2$ is called a *unit vector* if $\|\vec{v}\| = 1$.

Note that every vector $\vec{v} \neq \vec{0}$ defines a unit vector pointing in the same direction as itself by $\|\vec{v}\|^{-1}\vec{v}$.

Remark 2.6. (i) The tip of every unit vector lies on the unit circle, and every vector whose initial point is the origin and whose tip lies on the unit circle is a unit vector.

(ii) Every unit vector is of the from $\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$ where φ is its angle with the positive x-axis.

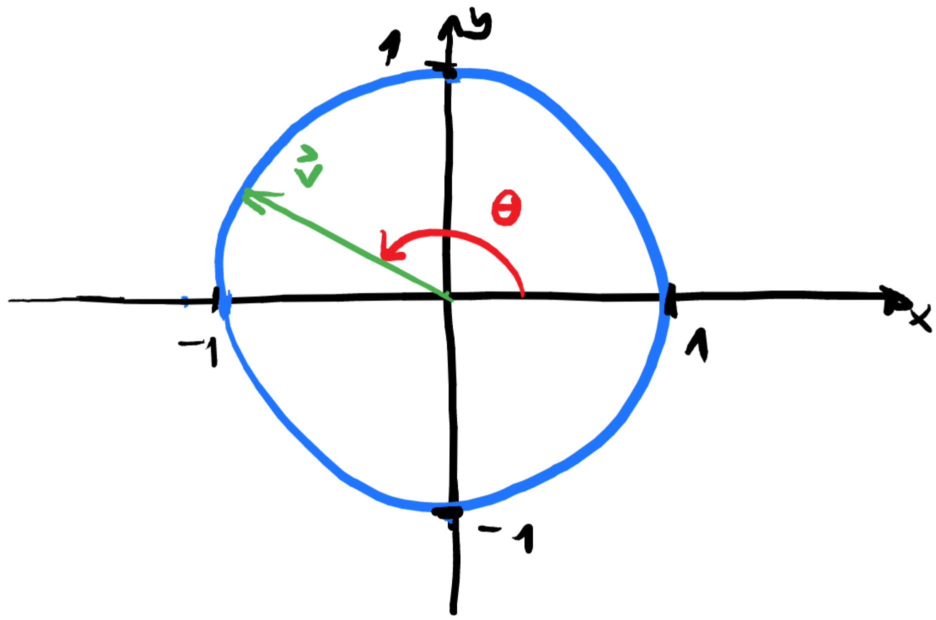

FIGURE 2.5: Unit vectors.

Finally, we define two very special unit vectors:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Clearly, $\vec{e_1}$ is parallel to the x-axis, $\vec{e_2}$ is parallel to the y-axis and $\|\vec{e}_1\| = \|\vec{e}_2\| = 1$.

Remark 2.7. Every vector $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ can be written as

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = a\vec{e}_1 + b\vec{e}_2.$$

Remark 2.8. Another notation for $\vec{e}_1$ and $\vec{e}_2$ is $\hat{\imath}$ and $\hat{\jmath}$.

2.2 Inner product and orthogonal projections

Let us start with a definition.

Definition 2.9. Sean $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ vectors in $\mathbb{R}^2$. The *inner product* of $\vec{v}$ and $\vec{w}$ is

$$\langle \vec{v}\,, \vec{w} \rangle := v_1 w_1 + v_2 w_2.$$

The inner product is also called *scalar product* or *dot product* and it can also be denoted by $\vec{v} \cdot \vec{w}$.

We usually prefer the notation $\langle \vec{v}\,, \vec{w} \rangle$ since this notation is used frequently in physics and extends naturally to abstract vector spaces with an inner product. Moreover, the the notation with the dot seems to suggest that the dot product behaves like a usual product, but it does not, see Remark 2.12.

Before we give properties of the inner product, we want to calculate a few examples.

Examples 2.10.

(i) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \end{pmatrix} \right\rangle = 2 \cdot (-1) + 3 \cdot 5 = -2 + 15 = 13.$

(ii) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\rangle = 2^2 + 3^2 = 4 + 9 = 13.$ Note that this is equal to $\left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\|^2$.

(iii) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 2, \quad \left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 3,$

(iv) $\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\rangle = 0.$

Proposition 2.11 (Properties of the inner product). *Let $\vec{u},\ vecv, \vec{w} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Then the following holds.*

(i) $\langle \vec{v}\,, \vec{v} \rangle = \|\vec{v}\|^2$. *In dot notation:* $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

(ii) $\langle \vec{u}\,, \vec{v} \rangle = \langle \vec{v}\,, \vec{u} \rangle$. *In dot notation:* $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

(iii) $\langle \vec{u}\,, \vec{v} + \vec{w} \rangle = \langle \vec{u}\,, \vec{v} \rangle + \langle \vec{u}\,, \vec{w} \rangle$. *In dot notation:* $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdots \vec{w})$.

(iv) $\langle \lambda\vec{u}\,, \vec{v} \rangle = \lambda \langle \vec{u}\,, \vec{v} \rangle$. *In dot notation:* $(\lambda\vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v})$.

Proof. Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$,$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

(i) $\langle \vec{v}\,, \vec{v} \rangle = v_1^1 + v_2^2 = \|\vec{v}\|^2$.

(ii) $\langle \vec{u}\,, \vec{v} \rangle = u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2 = \langle \vec{v}\,, \vec{u} \rangle$.

(iii)

$$\begin{aligned} \langle \vec{u}\,, \vec{v} + \vec{w} \rangle &= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \right\rangle = u_1(v_1 + w_1) + u_2(v_2 + w_2) = u_1 v_1 + u_2 v_2 + u_1 w_1 + u_2 w_2 \\ &= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle = \langle \vec{u}\,, \vec{v} \rangle + \langle \vec{u}\,, \vec{w} \rangle. \end{aligned}$$

(iv) $\langle \lambda\vec{u}\,, \vec{v} \rangle = \langle \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rangle = \lambda u_1 v_1 + \lambda u_2 v_2 = \lambda(u_1 v_1 + u_2 v_2) = \lambda \langle \vec{u}\,, \vec{v} \rangle$. □

Remark 2.12. Observe that the proposition says that the inner product is commutative and distributive, so has some properties of "usual multiplication" that we are used to from the product in $\mathbb{R}$ or $\mathbb{C}$, but there are some properties that show that the inner product is NOT a product:

(a) The inner products takes to vectors and gives back a number, so it gives back an object which is not of the same type as the two things we put in.

(b) In Example 2.10(iv) we saw that it may happen that $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$ but still $\langle \vec{v}\,, \vec{w} \rangle = 0$, something that is impossible for a "decent" product.

(c) Given a vector $\vec{v} \neq 0$ and a number $c \in \mathbb{R}$, there are many solutions of the equation $\langle \vec{v}\,, \vec{x} \rangle = c$ for the vector $\vec{x}$, in stark contrast to the usual product in $\mathbb{R}$ or $\mathbb{C}$. As an example, look at Example 2.10(i) and (ii). Therefore it makes NO sense to write something like $\vec{v}^{-1}$.

(d) There is no such thing as a neutral element for scalar multiplication.

Now let us see what the inner product is good for. We will see that inner product between two vectors is connected to the angle between them and it will help us to define orthogonal projections of one vector onto another.
Let us start with a definition.

Definition 2.13. Let $\vec{v}, \vec{w}$ be vectors in $\mathbb{R}^2$. The *angle between* $\vec{v}$ *and* $\vec{w}$ is the smallest nonnegative angle between them, see Figure 2.6. It is denoted by $\sphericalangle(\vec{v}, \vec{w})$.

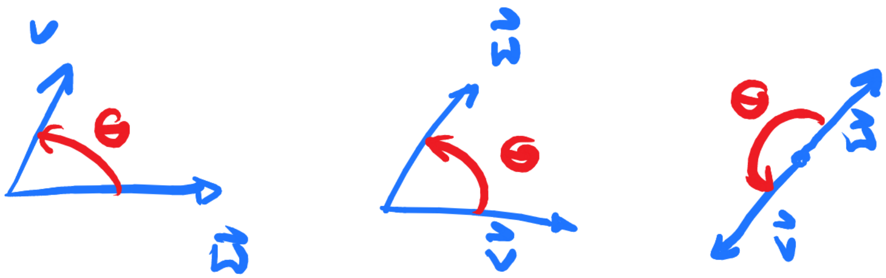

FIGURE 2.6: Angle between two vectors. XXXXXX Faltan π y 0.

The following properties of the angle are easy to see.

Proposition 2.14. (i) *Note that by definition,* $\sphericalangle(\vec{v}, \vec{w}) \in [0, \pi]$.

(ii) $\sphericalangle(\vec{v}, \vec{w}) = \sphericalangle(\vec{w}, \vec{v})$.

(iii) *If* $\lambda > 0$, *then* $\sphericalangle(\lambda\vec{v}, \vec{w}) = \sphericalangle(\vec{v}, \vec{w})$.

(iv) *If* $\lambda < 0$, *then* $\sphericalangle(\lambda\vec{v}, \vec{w}) = \pi - \sphericalangle(\vec{v}, \vec{w})$.

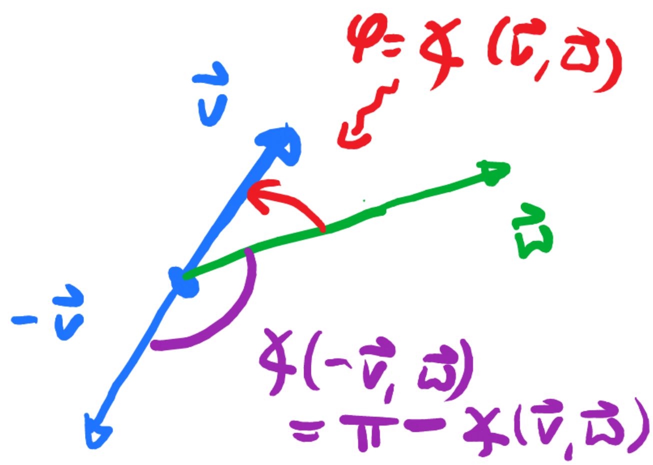

FIGURE 2.7: Angle between vectors $\vec{v}$ and $\vec{w}$.

Definition 2.15. (a) Two vectors $\vec{v}$ and $\vec{w}$ are called *parallel* if $\sphericalangle(\vec{v},\vec{w}) = 0$ or π. In this case we use the notation $\vec{v} \parallel \vec{w}$.

(b) Two vectors $\vec{v}$ and $\vec{w}$ are called *orthogonal* or *perpendicular* if $\sphericalangle(\vec{v},\vec{w}) = \pi/2$. In this case we use the notation $\vec{v} \perp \vec{w}$.

The following properties should be known from geometry. We will proof them after Theorem 2.19.

Proposition 2.16. *Let $\vec{v}, \vec{w}$ be vectors in $\mathbb{R}^2$. Then:*

(i) *$\vec{v} \parallel \vec{w}$ and $\vec{v} \neq \vec{0}$, then there exists $\lambda \in \mathbb{R}$ such that $\vec{w} = \lambda\vec{v}$.*

(ii) *If $\vec{v} \parallel \vec{w}$ and $\lambda, \mu \in \mathbb{R}$, then also $\lambda\vec{v} \parallel \mu\vec{w}$.*

(iii) *If $\vec{v} \perp \vec{w}$ and $\lambda, \mu \in \mathbb{R}$, then also $\lambda\vec{v} \perp \mu\vec{w}$.*

Remark 2.17. Observe that (i) is wrong if we do not assume that $\vec{v} \neq \vec{0}$ because if $\vec{v} = \vec{0}$, then it is parallel to every vector $\vec{w}$ in $\mathbb{R}^2$, but there is no $\lambda \in \mathbb{R}$ such that $\lambda\vec{v}$ could ever become different from $\vec{0}$.

Further observe that the reverse direction in (ii) is true only if $\lambda \neq 0$ and $\mu \neq 0$.

Without proof, we state the following theorem which should be known.

Theorem 2.18 (Cosine Theorem). *Let a, b, c be the sides or a triangle and let φ be the angle between the sides a and b. Then*

$$c^2 = a^2 + b^2 - 2ab\cos\varphi. \tag{2.1}$$

Theorem 2.19. *Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ and let $\varphi = \sphericalangle(\vec{v},\vec{w})$. Then*

$$\boxed{\langle \vec{v}\,, \vec{w} \rangle = \|\vec{v}\|\|\vec{w}\| \cos\varphi.}$$

Proof. The vectors $\vec{v}$ and $\vec{w}$ define a triangle in $\mathbb{R}^2$, see Figure 2.8

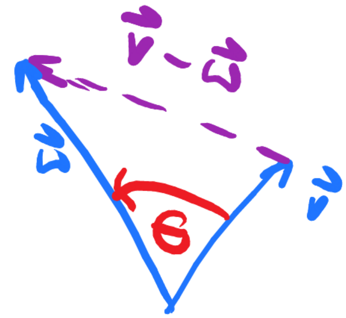

FIGURE 2.8: Triangle given by $\vec{v}$ and $\vec{w}$.

Now we apply the cosine theorem with $a = \|\vec{v}\|$, $b = \|\vec{w}\|$, $c = \|\vec{v} - w\|$. We obtain

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\varphi. \tag{2.2}$$

On the other hand,

$$\begin{aligned} \|\vec{v} - \vec{w}\|^2 &= \langle \vec{v} - \vec{w}\,, \vec{v} - \vec{w}\rangle = \langle \vec{v}\,, \vec{v}\rangle - \langle \vec{v}\,, \vec{w}\rangle - \langle \vec{w}\,, \vec{v}\rangle + \langle \vec{w}\,, \vec{w}\rangle = \langle \vec{v}\,, \vec{v}\rangle - 2\langle \vec{v}\,, \vec{w}\rangle + \langle \vec{w}\,, \vec{w}\rangle \\ &= \|\vec{v}\|^2 - 2\langle \vec{v}\,, \vec{w}\rangle + \|\vec{w}\|^2. \end{aligned} \tag{2.3}$$

Comparison of (2.2) and (2.3) show that

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\varphi = \|\vec{v}\|^2 - 2\langle \vec{v}\,, \vec{w}\rangle + \|\vec{w}\|^2,$$

which gives the claimed formula. □

A very important consequence of this theorem is that we can now determine if two vectors ara parallel or perpendicular to each other by simply calculating their inner product as can be seen from the following corollary.

Corollary 2.20. *Let* $\vec{v}, \vec{w} \in \mathbb{R}^2$ *and* $\varphi = \sphericalangle(\vec{v}, \vec{w})$. *Then:*

(i) $|\langle \vec{v}\,, \vec{w}\rangle| \le \|\vec{v}\|\,\|\vec{w}\|$.

(ii) $\vec{v} \parallel \vec{w} \iff \|\vec{v}\|\,\|\vec{w}\| = |\langle \vec{v}\,, \vec{w}\rangle|$.

(iii) $\vec{v} \perp \vec{w} \iff \langle \vec{v}\,, \vec{w}\rangle = 0$.

Proof. (i) From Theorem **??** we have that $|\langle \vec{v}\,, \vec{w}\rangle| = \|\vec{v}\|\,\|\vec{w}\| \cos\varphi \le \|\vec{v}\|\,\|\vec{w}\|$ since $0 \le \cos\varphi \le 1$. The claims in (ii) and (iii) are clear if one of the vectors is equal to $\vec{0}$ since the zero vector is parallel and orthogonal to every vector in $'R^2$. So let us assume now that $\vec{v} \ne \vec{0}$ and $\vec{w} \ne \vec{0}$.

(ii) From Theorem **??** we have that $|\langle \vec{v}\,, \vec{w}\rangle| = \|\vec{v}\|\,\|\vec{w}\|$ if and only if $\cos\varphi = 1$. This is the case if and only if $\varphi = 0$ or π, that is, if and only if $\vec{v}$ and $\vec{w}$ are parallel.

(iii) From Theorem **??** we have that $|\langle \vec{v}\,, \vec{w}\rangle| = 0$ if and only if $\cos\varphi = 0$. This is the case if and only if $\varphi = \pi/2$, that is, if and only if $\vec{v}$ and $\vec{w}$ are perpendicular. □

With this corollary, the proof of Proposition 2.16(ii) and (iii) is now easy and left to the reader.

Example 2.21. Theorem **??** lets us calculate the angle of a given vector with the x-axis easily (see Figure 2.9):

$$\cos\varphi_x = \frac{\langle \vec{v}\,, \vec{e}_1\rangle}{\|\vec{v}\|\|\vec{e}_1\|}, \qquad \cos\varphi_y = \frac{\langle \vec{v}\,, \vec{e}_2\rangle}{\|\vec{v}\|\|\vec{e}_2\|}.$$

If we now use that $\|\vec{e}_1\| = \|\vec{e}_2\| = 1$ and that $\langle \vec{v}\,, \vec{e}_1\rangle = v_1$ and $\langle \vec{v}\,, \vec{e}_2\rangle = v_2$, then

$$\cos\varphi_x = \frac{v_1}{\|\vec{v}\|}, \qquad \cos\varphi_y = \frac{v_2}{\|\vec{v}\|}.$$

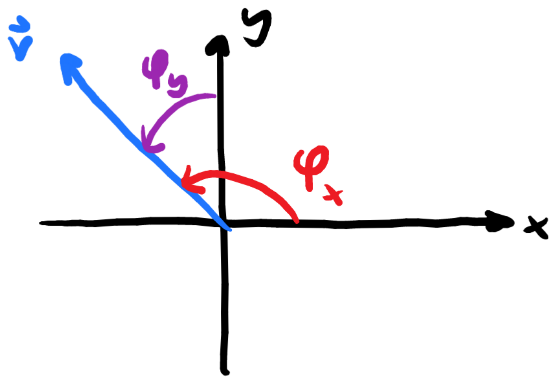

FIGURE 2.9: Angle of $\vec{v}$ with the axes.

Orthogonal Projections in $\mathbb{R}^2$.

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ and $\vec{w} \neq \vec{0}$. We want to find the orthogonal projection of $\vec{v}$ onto $\vec{w}$. Geometrically, we find it as follows: We move $\vec{v}$ such that its initial point coincides with that of $\vec{w}$. Then we extend $\vec{w}$ to a line and construct a line that passes through the tip of $\vec{v}$. The vector from the initial point to the intersection of the two lines is the see Figure 2.10

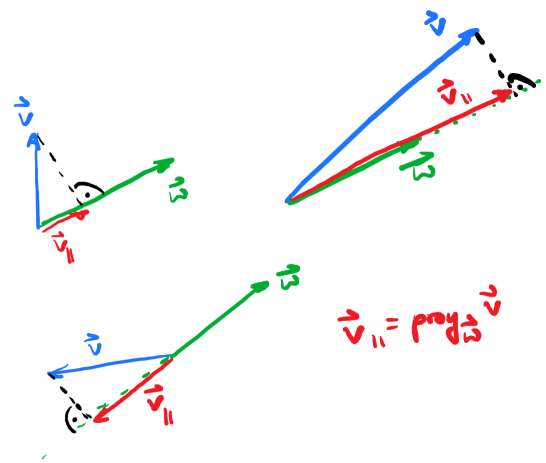

FIGURE 2.10: Orthogonal projections in $\mathbb{R}^2$.

We denote the orthogonal projection of $\vec{v}$ onto $\vec{w}$ by $\text{proj}_{\vec{w}}\,\vec{v}$, or sometimes by $\vec{v}_{\parallel}$ it is clear on which vector we are projecting. By construction of $\text{proj}_{\vec{w}}\,\vec{v}$ it is clear that

- $\text{proj}_{\vec{w}}\,\vec{v}$ is parallel to $\vec{w}$,
- $\vec{v} - \text{proj}_{\vec{w}}\,\vec{v}$ is orthogonal to $\vec{w}$. Therefore, we sometimes write $\vec{v}_{\perp} = \vec{v} - \text{proj}_{\vec{w}}\,\vec{v}$.

This procedure allows us to write $\vec{v}$ as sum of a vector parallel to $\vec{w}$ and one orthogonal to $\vec{w}$. How we can calculate these two vectors, is the content of the next theorem.

Theorem 2.22. *Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ and $\vec{w} \neq \vec{0}$. Then*

$$\text{proj}_{\vec{w}}\,\vec{v} = \frac{\langle \vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2}\,\vec{w}. \tag{2.4}$$

Before we prove the formula, note that it seems to make sense. The right hand side is a multiple of $\vec{w}$, so it is parallel to $\vec{w}$ as it should be. Moreover, it does not depend on $\|w\|$ as it should be because it should not matter if we project on $\vec{w}$ or on $5\vec{w}$ or on $-0.4\vec{w}$; only the direction of $\vec{w}$ matters, not its length.

Proof. Let $\vec{v}_{\parallel} = \text{proj}_{\vec{w}}\,\vec{v}$ and $\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel}$. Then $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$. Since $\vec{v}_{\parallel} \parallel \vec{w}$, there exists a $\lambda \in \mathbb{R}$ such that $\vec{v}_{\parallel} = \lambda\vec{w}$, so we only need to determine λ. For this, we write

$$\begin{aligned} & \vec{v} = \lambda\vec{w} + \vec{v}_{\perp} \\ \implies\quad & \langle \vec{v}\,, \vec{w} \rangle = \langle \lambda\vec{w} + \vec{v}_{\perp}\,, \vec{w} \rangle = \langle \lambda\vec{w}\,, \vec{w} \rangle + \underbrace{\langle \vec{v}_{\perp}\,, \vec{w} \rangle}_{=0 \text{ since } \vec{v}_{\perp} \perp \vec{w}} = \langle \lambda\vec{w}\,, \vec{w} \rangle = \lambda\langle \vec{w}\,, \vec{w} \rangle = \lambda\|\vec{w}\|^2 \\ \implies\quad & \lambda = \frac{\langle \vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2} \end{aligned}$$

So it follows that

$$\operatorname{proj}_{\vec{w}} \vec{v} = \vec{v}_{\|} = \lambda \vec{w} = \frac{\langle \vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w}. \qquad \square$$

Remark 2.23. (i) $\operatorname{proj}_{\vec{w}} \vec{v}$ depends only of the direction of $\vec{w}$. It does not depend on its length.

Proof. By our geometric intuition, this should be clear. But we can see this also from the formula. Suppose we want to project on $c\vec{w}$ for some $c \in \mathbb{R} \setminus \{0\}$. Then

$$\operatorname{proj}_{c\vec{w}} \vec{v} = \frac{\langle \vec{v}\,, c\vec{w} \rangle}{\|c\vec{w}\|^2}(c\vec{w}) = \frac{c\langle \vec{v}\,, \vec{w} \rangle}{c^2\|\vec{w}\|^2}(c\vec{w}) = \frac{\langle \vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2}\vec{w} = \operatorname{proj}_{\vec{w}} \vec{v}. \qquad \square$$

(ii) For every $c \in \mathbb{R}$, we have that $\operatorname{proj}_{\vec{w}}(c\vec{v}) = c \operatorname{proj}_{\vec{w}} \vec{v}$.

Proof. Again, by geometric considerations, this should be clear. The corresponding calculus is

$$\operatorname{proj}_{\vec{w}}(c\vec{v}) = \frac{\langle c\vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2}\vec{w} = \frac{c\langle \vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2}\vec{w} = c \operatorname{proj}_{\vec{w}} \vec{v}. \qquad \square$$

(iii) As special cases of the above, we find $\operatorname{proj}_{\vec{w}}(-\vec{v}) = \operatorname{proj}_{\vec{w}} \vec{v}$ and $\operatorname{proj}_{-\vec{w}} \vec{v} = -\operatorname{proj}_{\vec{w}} \vec{v}$.

(iv) $\vec{v} \parallel \vec{w} \quad \Longrightarrow \quad \operatorname{proj}_{\vec{w}} \vec{v} = \vec{v}$.

(v) $\vec{v} \perp \vec{w} \quad \Longrightarrow \quad \operatorname{proj}_{\vec{w}} \vec{v} = \vec{0}$.

(vi) $\operatorname{proj}_{\vec{w}} \vec{v}$ is the unique vector in $\mathbb{R}^2$ such that

$$\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v} \perp \vec{v} \quad \text{and} \quad \operatorname{proj}_{\vec{w}} \vec{v} \parallel \vec{w}.$$

We end this section with some examples.

Example 2.24. Let $\vec{u} = 2\vec{e}_1 + 3\vec{e}_2$, $\vec{v} = 4\vec{e}_1 - \vec{e}_2$.

(i) $\operatorname{proj}_{\vec{e}_1} \vec{u} = \frac{\langle \vec{u}\,, \vec{e}_1 \rangle}{\|\vec{e}_1\|^2}\vec{e}_1 = \frac{2}{1^2}\vec{e}_1 = 2\vec{e}_1$.

(ii) $\operatorname{proj}_{\vec{e}_2} \vec{u} = \frac{\langle \vec{u}\,, \vec{e}_2 \rangle}{\|\vec{e}_2\|^2}\vec{e}_2 = \frac{3}{1^2}\vec{e}_2 = 3\vec{e}_2$.

(iii) Similarly, we can calculate $\operatorname{proj}_{\vec{e}_1} \vec{v} = 4\vec{e}_1$, $\operatorname{proj}_{\vec{e}_2} \vec{v} = -\vec{e}_2$.

(iv) $\operatorname{proj}_{\vec{u}} \vec{v} = \frac{\langle \vec{u}\,, \vec{v} \rangle}{\|\vec{u}\|^2}\vec{u} = \dfrac{\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \end{pmatrix} \right\rangle}{\|\vec{u}\|^2}\vec{u} = \frac{8-3}{2^2+3^2}\vec{u} = \frac{5}{13}\vec{u} = \frac{5}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(v) $\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\langle \vec{v}\,, \vec{u} \rangle}{\|\vec{v}\|^2}\vec{v} = \dfrac{\left\langle \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\rangle}{\|\vec{u}\|^2}\vec{u} = \frac{8-3}{4^2+(-1)^2}\vec{v} = \frac{5}{17}\vec{v} = \frac{5}{17}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Example 2.25 (Angle with coordinate axes). Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$.

Then $\cos\sphericalangle(\vec{v}, \vec{e}_1) = \frac{a}{\|\vec{v}\|}$, $\cos\sphericalangle(\vec{v}, \vec{e}_2) = \frac{b}{\|\vec{v}\|}$, hence

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \|\vec{v}\| \begin{pmatrix} \cos\sphericalangle(\vec{v}, \vec{e}_1) \\ \cos\sphericalangle(\vec{v}, \vec{e}_2) \end{pmatrix}.$$

2.3 Vectors in $\mathbb{R}^3$

In this section we extend our calculations from $\mathbb{R}^2$ to $\mathbb{R}^3$. Recall that $\mathbb{R}^3$ is the space of all points $P(a, b, c)$ with $a, b, c \in \mathbb{R}$. This is a model for our usual physical everyday space. Recall that the distance between two points $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ is $\overline{PQ} = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$.

As in $\mathbb{R}^2$, we can identify every point in $\mathbb{R}^3$ with the arrow that starts in the origin of coordinate system and ends in the given point. The set of all arrows with the same length and the same direction is called a vector in $\mathbb{R}^3$. Again, we denote a vector in $\mathbb{R}^3$ as a column

$$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

In order to save space, we will also use the notation $(a, b, c)^t$, where, as in $\mathbb{R}^2$, the superscript t stands for *transposed*.

Definition 2.26. Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$ and $c \in \mathbb{R}$. We define the sum of $\vec{v}$ and $\vec{w}$ and the product of the scalar c with the vector $\vec{v}$ as follows:

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}, \qquad c\vec{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ cv_3 \end{pmatrix}.$$

It is easy to see that $\mathbb{R}^3$ with this sum and product satisfies the vector space axioms on page 19.

As in $\mathbb{R}^2$, we define an *inner product*

$$\langle \vec{v}\,, \vec{w} \rangle = \left\langle \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\rangle = v_1 w_1 + v_2 w_2 + v_3 + w_3$$

and a *norm*

$$\|\vec{v}\| = \left\| \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\| := \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

We also use the words *magnitude* or *length* of $\vec{w}$. .

Two vectors in $\mathbb{R}^3$ which are not parallel generate a plane. Then we can measure the angle between the two vectors in this plane as if it was $\mathbb{R}^2$ and we call it the *angle between the two vectors.*
As in $\mathbb{R}^2$, we have the following properties:

(i) *Symmetry of the inner product:* For all vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$, we have that $\langle \vec{v}\,, \vec{w} \rangle = \langle \vec{w}\,, \vec{v} \rangle$.

(ii) *Bilinearity of the inner product:* For all vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and all $c \in \mathbb{R}$, we have that $\langle \vec{u}\,, \vec{v} + \vec{w} \rangle = \langle \vec{u}\,, \vec{v} \rangle + c\langle \vec{u}\,, \vec{w} \rangle$.

(iii) *Relation of the inner product with the angle between vectors:* Let $\vec{v}, \vec{w} \in \mathbb{R}^3$ and let $\varphi = \sphericalangle(\vec{v}, \vec{w})$. Then

$$\langle \vec{v}\,, \vec{w} \rangle = \|\vec{v}\|\, \|\vec{w}\| \, \cos\varphi.$$

Remark 2.27. Actually, the inner product usually is used to *define* the angle between two vectors by the formula above.

In particular, we have (cf. Proposition 2.16):

$$\begin{array}{llllll} \text{(a)} & \vec{v} \parallel \vec{w} & \iff & \sphericalangle(\vec{v}, \vec{w}) \in \{0, \pi\} & \iff & |\langle \vec{v}\,, \vec{w} \rangle| = \|\vec{v}\|\, \|\vec{w}\| \\ \text{(b)} & \vec{v} \perp \vec{w} & \iff & \sphericalangle(\vec{v}, \vec{w}) = \pi/2 & \iff & \langle \vec{v}\,, \vec{w} \rangle = 0. \end{array}$$

(iv) *Relation of norm and inner product:* For all vectors $\vec{v} \in \mathbb{R}^3$, we have that $\|\vec{v}\|^2 = \langle \vec{v}\,, \vec{v} \rangle$.

(v) *Properties of the norm:* For all vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ and scalars $c \in \mathbb{R}$, we have that $\|c\vec{v}\| = |c|\|\vec{v}\|$ and $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

(vi) *Orthogonal projections of one vector onto another:* For all vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ the *orthogonal projection of $\vec{v}$ onto $\vec{w}$* is

$$\operatorname{proj}_{\vec{w}} \vec{v} = \frac{\langle \vec{v}\,, \vec{w} \rangle}{\|\vec{w}\|^2}\, \vec{w}.$$

As in $\mathbb{R}^3$, we have three sort of special vectors which are parallel to the coordinate system:

$$\vec{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Another notation for them is $\hat{\imath}, \hat{\jmath}, \hat{k}$.
For a given vector $\vec{v} \neq \vec{0}$, we can now easily determine its angle with the coordinate axes:

$$\begin{aligned} \varphi_x = \sphericalangle(\vec{v}, \vec{e}_1) \quad &\Longrightarrow \quad \cos\varphi_x = \frac{\langle \vec{v}\,, \vec{e}_1 \rangle}{\|\vec{v}\|\, \|\vec{e}_1\|} = \frac{v_1}{\|\vec{v}\|}, \\ \varphi_y = \sphericalangle(\vec{v}, \vec{e}_2) \quad &\Longrightarrow \quad \cos\varphi_x = \frac{\langle \vec{v}\,, \vec{e}_2 \rangle}{\|\vec{v}\|\, \|\vec{e}_2\|} = \frac{v_2}{\|\vec{v}\|}, \\ \varphi_z = \sphericalangle(\vec{v}, \vec{e}_3) \quad &\Longrightarrow \quad \cos\varphi_x = \frac{\langle \vec{v}\,, \vec{e}_3 \rangle}{\|\vec{v}\|\, \|\vec{e}_3\|} = \frac{v_3}{\|\vec{v}\|}. \end{aligned}$$

Esto nos dice que

$$\vec{v} = \|\vec{v}\| \begin{pmatrix} \cos\varphi_x \\ \cos\varphi_y \\ \cos\varphi_z \end{pmatrix}.$$

If we take the norm both sides of the equation, we find

$$(\cos\varphi_x)^2 + (\cos\varphi_y)^2 + (\cos\varphi_z)^2 = 1.$$

2.4 Cross product

In this section we define the so-called *cross product*. Another name for it its *vector product*. It takes two vectors and gives back two vectors. It does have several properties which makes it look like a product, however we will see that it is NOT a product. Here is the definition.

Definition 2.28 (Cross product). Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$. Their *cross product* or *vector product* is

$$\vec{v} \times \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} := \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}.$$

Remark 2.29. The cross product exists only in $\mathbb{R}^3$!

Before we collect some easy properties of the cross product, let us calculate a few examples.

Examples 2.30. Let $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$.

- $\vec{u} \times \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 2\cdot 7 - 3\cdot 6 \\ 3\cdot 5 - 1\cdot 7 \\ 1\cdot 6 - 2\cdot 5 \end{pmatrix} = \begin{pmatrix} 14 - 18 \\ 15 - 7 \\ 6 - 10 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -4 \end{pmatrix}.$
- $\vec{v} \times \vec{u} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6\cdot 3 - 7\cdot 2 \\ 7\cdot 1 - 3\cdot 5 \\ 5\cdot 2 - 6\cdot 1 \end{pmatrix} = \begin{pmatrix} 18 - 14 \\ 7 - 15 \\ 10 - 6 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}.$
- $\vec{v} \times \vec{e}_1 = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6\cdot 0 - 7\cdot 0 \\ 7\cdot 0 - 7\cdot 1 \\ 5\cdot 0 - 6\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \\ -6 \end{pmatrix}.$

Proposition 2.31 (Properties of the cross product). *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and let $c \in \mathbb{R}$. Then:*

(i) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$.

(ii) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

(iii) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$.

(iv) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$.

(v) $\vec{u} \parallel \vec{v} \quad \Longrightarrow \quad \vec{u} \times \vec{v} = \vec{0}$. *In particular,* $\vec{v} \times \vec{v} = \vec{0}$.

(vi) $\langle \vec{u}\,, \vec{v} \times \vec{w} \rangle = \langle \vec{u} \times \vec{v}\,, \vec{w} \rangle$.

(vii) $\langle \vec{u}\,, \vec{u} \times \vec{v} \rangle = 0$ *and* $\langle \vec{v}\,, \vec{u} \times \vec{v} \rangle = 0$, *in particular*

$$\boxed{\vec{v} \perp \vec{v} \times \vec{u}, \quad \vec{u} \perp \vec{v} \times \vec{u}}$$

that means that the vector $\vec{v} \times \vec{w}$ *is orthogonal to both* $\vec{v}$ *and* $\vec{w}$.

Proof. The proofs of the formulas (i) to (v) are easy calculations (you should do them!).

(vi) The proof is a long but straightforward calculation:

$$\begin{aligned}
\langle \vec{u}\,, \vec{v} \times \vec{w} \rangle &= \left\langle \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - w_3v_1 \\ v_1w_2 - v_2w_1 \end{pmatrix} \right\rangle \\
&= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) \\
&= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1 \\
&= u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3 \\
&= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 \\
&= \langle \vec{u} \times \vec{v}\,, \vec{w} \rangle.
\end{aligned}$$

(vii) It follows from (vi) and (v) that

$$\langle \vec{u}\,, \vec{u} \times \vec{v} \rangle = \langle \vec{u} \times \vec{u}\,, \vec{v} \rangle = \langle \vec{0}\,, \vec{v} \rangle = 0.$$ □

Note that the cross product is distributive but it is not commutative nor associative.

Recall that for the inner product we proved the formula $\langle \vec{v}\,, \vec{w} \rangle = \|\vec{v}\|\,\|\vec{w}\| \cos\varphi$ where φ is the angle between the two vectors, see Theorem 2.19. In the next theorem we will prove a similar relation for the cross product.

Theorem 2.32. *Let* $\vec{v}, \vec{w}$ *be vectors in* $\mathbb{R}^3$ *and let* φ *be the angle between them. Then*

$$\boxed{\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\,\|\vec{w}\| \sin\varphi}$$

Proof. A long, but straightforward calculations shows that $\|\vec{v} \times \vec{w}\|^2 = \|\vec{u}\|^2\|\vec{w}\|^2 - \langle \vec{v}\,, \vec{w} \rangle^2$. Now it follows from Theorem 2.19 that

$$\begin{aligned}
\|\vec{v} \times \vec{w}\|^2 &= \|\vec{u}\|^2\|\vec{w}\|^2 - \langle \vec{v}\,, \vec{w} \rangle^2 = \|\vec{u}\|^2\|\vec{w}\|^2 - \|\vec{v}\|^2\|\vec{w}\|^2(\cos\varphi)^2 \\
&= \|\vec{u}\|^2\|\vec{w}\|^2(1 - (\cos\varphi)^2) = \|\vec{u}\|^2\|\vec{w}\|^2(\sin\varphi)^2.
\end{aligned}$$

Observe that $\sin\varphi \geq 0$ because $\varphi \in [0, \pi]$. So if we take the square root we we do not need to take the absolute value and we arrive at the claimed formula. □

Application: Area of a parallelogram and volume of a parelellepiped

Area of a parallelogram

Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{R}^3$. Then they define a parallelogram (if the vectors are parallel or one of them is equal to $\vec{0}$, it is a *degenerate parallelogram*).

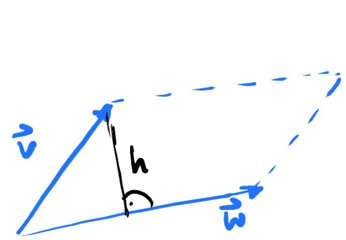

FIGURE 2.11: Parallelogram spanned by $\vec{v}$ and $\vec{w}$.

Proposition 2.33 (Area of a parallelogram). *The area of the parallelogram spanned by the vectors $\vec{v}$ and $\vec{w}$ is*

$$A = \|\vec{v} \times \vec{w}\|. \tag{2.5}$$

Proof. The area of a parallelogram is the product of the length of its base with the height. We can take $\vec{w}$ as base. Let φ be the angle between $\vec{w}$ and $\vec{v}$. Then we obtain that $h = \|\vec{v}\| \sin\varphi$ and therefore, with the help of Theorem 2.32

$$A = \|\vec{w}\|h = \|\vec{w}\|\|\vec{v}\| \sin\varphi = \|\vec{v} \times \vec{w}\|. \qquad \square$$

Note that in the case when $\vec{v}$ and $\vec{w}$ are parallel, this gives the right answer $A = 0$.

Any three vectors in $\mathbb{R}^3$ define a parallelepiped.

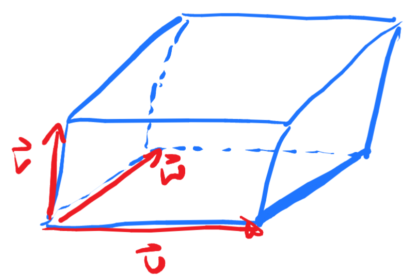

Figure 2.12: Parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

Proposition 2.34 (Volume of a parallelepiped). *The volume of the parallelepiped spanned by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is*

$$V = \|\vec{u}(\vec{v} \times \vec{w})\|. \tag{2.6}$$

Proof. The volume of a parallelepiped is the product of the area of its base with the height. Let us take the parallelogram spanned by $\vec{v}, \vec{w}$ as base. If $\vec{v}$ and $\vec{w}$ are parallel or one or them is equal to $\vec{0}$, then (2.6) is true because $V = 0$ and $\vec{v} \times \vec{w} = \vec{0}$ in this case.
Now let us assume that they are not parallel. By Proposition 2.33 we already know that its base has area $A = \|\vec{v} \times \vec{w}\|$. The height is the length of the orthogonal projection of $\vec{u}$ onto the normal vector of the plane spanned by $\vec{v}$ and $\vec{w}$. We already know that $\vec{v} \times \vec{w}$ is such a normal vector. Hence we obtain that

$$h = \|\operatorname{proj}_{\vec{v}\times\vec{w}} \vec{u}\| = \left\| \frac{\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle}{\|\vec{v} \times \vec{w}\|^2} \vec{v} \times \vec{w} \right\| = \frac{|\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|}{\|\vec{v} \times \vec{w}\|^2} \|\vec{v} \times \vec{w}\| = \frac{|\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|}{\|\vec{v} \times \vec{w}\|}.$$

We can take $\vec{w}$ as base. Let φ be the angle between $\vec{w}$ and $\vec{v}$. Then we obtain that $h = \|\vec{v}\| \sin\varphi$ and therefore, with the help of Theorem 2.32

$$A = \|\vec{w}\| h = \|\vec{w}\| \|\vec{v}\| \sin\varphi = \|\vec{v} \times \vec{w}\|.$$

Therefore, the volume of the parallelepiped is

$$V = Ah = \|\vec{v} \times \vec{w}\| \frac{|\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|}{\|\vec{v} \times \vec{w}\|} = |\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle|. \qquad \square$$

Corollary 2.35. *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$. Then*

$$|\langle \vec{u}\,, \vec{v} \times \vec{w}\rangle| = |\langle \vec{v}\,, \vec{w} \times \vec{u}\rangle| = |\langle \vec{w}\,, \vec{u} \times \vec{v}\rangle|.$$

Proof. The formula holds because each of the expressions describes the volume of the parallelepiped spanned by the three given vectors since we can take any of the sides of the parallelogram as its base. □

2.5 Lines and planes in $\mathbb{R}^3$

Lines

In order to know a line in $\mathbb{R}^3$ completely, it is not necessary to know all its points. It is sufficient to know either

(a) two different points P, Q on the line

or

(b) one point P on the line and the direction of the line.

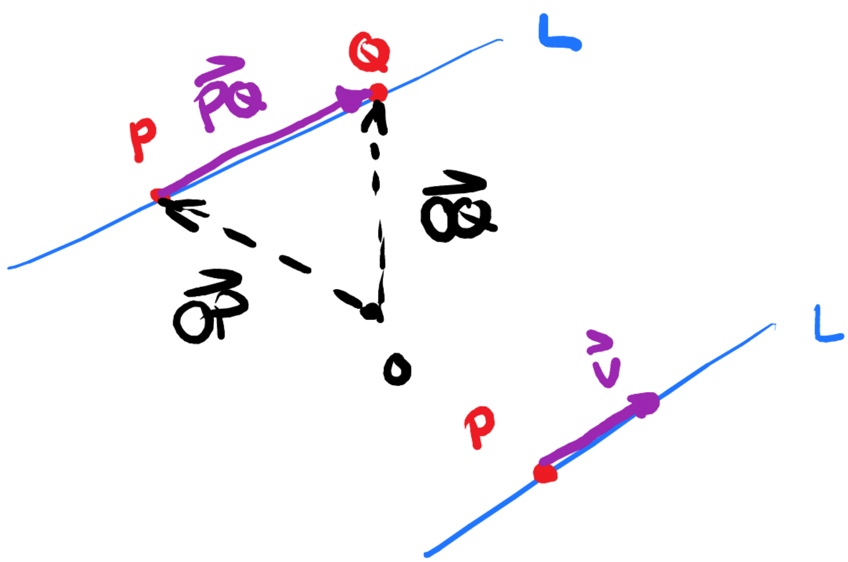

FIGURE 2.13: Line L given (a) by two points P, Q on L, (b) by a point P on L and the direction of L.

Clearly, both descriptions are equivalent. If we have two different points P, Q on the line L, then its direction is given by the vector $\overrightarrow{PQ}$. If on the other hand we are given a point P on L and a vector $\vec{v}$ which is parallel to L, then we easily get another point Q on L by $\overrightarrow{OQ} = \overrightarrow{0P} + \vec{v}$.

Now we want to give formulas for the line.

Vector equation

Given two points $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ with $P \neq Q$, there is exactly one line L which passes through both points. In formulas, this line is described as

$$L = \left\{ \overrightarrow{0P} + t\overrightarrow{PQ} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} p_1 + (q_1 - p_1)t \\ p_2 + (q_2 - p_2)t \\ p_3 + (q_3 - p_3)t \end{pmatrix} : t \in \mathbb{R} \right\} \tag{2.7}$$

If we are given a point $P(p_1, p_2, p_3)$ on L and a vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \neq \vec{0}$ parallel to L, then

$$L = \left\{ \overrightarrow{0P} + t\vec{v} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} p_1 + v_1 t \\ p_2 + v_2 t \\ p_3 + v_3 t \end{pmatrix} : t \in \mathbb{R} \right\} \tag{2.8}$$

The formulas (2.7) and (2.8) are called *vector equation* for the line L. Note that they are the same if we set $v_1 = q_1 - p_1$, $v_2 = q_2 - p_2$, $v_3 = q_3 - p_3$. We will mostly use the notation with the v's since it is shorter. The vector $\vec{v}$ is called *directional vector* of the line L. Observe that if $\vec{v}$ is a directional vector for L, then $c\vec{v}$ is so too for every $c \in \mathbb{R} \setminus \{0\}$.

Parametric equation

From the formula (2.8) it is clear that a point (x, y, z) belongs to L if and only if there exists $t \in \mathbb{R}$ such that

$$\begin{aligned} x &= p_1 + tv_1, \\ y &= p_2 + tv_2, \\ z &= p_3 + tv_3. \end{aligned} \tag{2.9}$$

If we had started with (2.7), then had obtained

$$\begin{aligned} x &= p_1 + t(q_1 - p_1), \\ y &= p_2 + t(q_2 - p_2), \\ z &= p_3 + t(q_3 - p_3) \end{aligned} \tag{2.10}$$

The system of equations (2.9) or (2.10) are called the *parametric equations* of L. Here, t is the parameter.

Symmetric equation

Observe that for $(x, y, z) \in L$, the three equations in (2.9) must hold for the same t. So if we assume that $v_1, v_2, v_3 \neq 0$, then we can solve for t and we obtain that

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3} \tag{2.11}$$

If we use (2.10) then we obtain

$$\frac{x-p_1}{q_1-p_1} = \frac{y-p_2}{q_2-p_2} = \frac{z-p_3}{q_3-p_3}. \tag{2.12}$$

The system of equations (2.11) or (2.12) is called the *symmetric equation* of L.
If for instance, $v_1 = 0$ and $v_2, v_3 \neq 0$, then the symmetric equation would be

$$x = p_1, \quad \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}.$$

This is a line which is parallel to the yz-plane.
If $v_1 = v_2 = 0$ and $v_3 \neq 0$, then the symmetric equation would be

$$x = p_1, \quad y = p_2, z \in \mathbb{R}.$$

This is a line which is parallel to the z-axis.

Remark 2.36. It is important to observe that a given line has many different parametrizations. For example, the vector equation that we write down depends on the points we choose on L. Clearly, we have infinitely many possibilities to do so.

Example 2.37. The following equations describe the same line:

$$L = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Two lines G and L in $'R^3$ are parallel if and only if their directional vectors are parallel.

Planes

In order to know a plane in $\mathbb{R}^3$ completely, it is sufficient to

(a) three points P, Q on the plane that do not lie on a line,

or

(b) one point P on the plane and two non-parallel vectors $\vec{v}, \vec{w}$ which are both parallel the plane,

or

(c) one point P on the plane and a vector $\vec{n}$ which is perpendicular to the plane,

Figure 2.14: Plane π given (a) by three points P, Q, R on π, (b) by a point P on L and two vectors $\vec{v}, \vec{w}$ parallel to π. (c) by a point P on L and a vector $\vec{n}$ perpendicular to π.

First, let us see how we can pass from one description to another. Clearly, the descriptions ((a)) and ((b)) are equivalent because given three points P, Q, R on π which do not lie on a line, we can form the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Theses vectors are then parallel to the plane π but are not parallel with each other. (Of course, we also could have taken $\overrightarrow{QR}$ and $\overrightarrow{QP}$ or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$.) If, on the other hand, we have one point P on π and two vectors $\vec{v}$ and $\vec{w}$, parallel to π and $\vec{v} \not\parallel \vec{w}$, then we can easily get two other points on π, for instance by $\overrightarrow{0Q} = \overrightarrow{0P} + \vec{v}$ and $\overrightarrow{0R} = \overrightarrow{0P} + \vec{w}$. Then the three points P, Q, R lie on π and do not lie on a plane.
In formulas, we can now describe our plane π as

$$\pi = \left\{ (x, y, z) : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overrightarrow{0P} + s\vec{v} + t\vec{w} \quad \text{for some } s, t \in \mathbb{R} \right\}$$

Now we want to use the normal vector of the plane to describe it. Assume that we are given a point P on π and a normal vector $\vec{n}$ perpendicular to the plane. This means that every vector which is parallel to the plane π must be perpendicular to $\vec{n}$. If we take an arbitrary point $Q(x, y, z) \in \mathbb{R}^3$, then $Q \in \pi$ if and only if $\overrightarrow{PQ}$ is parallel to π, that means that $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. Recall that two vectors are perpendicular if and only if their inner product is 0, so $Q \in \pi$ if and only if

$$\begin{aligned} 0 = \langle n \,, \overrightarrow{PQ} \rangle &= \left\langle \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} , \begin{pmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{pmatrix} \right\rangle = n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) \\ &= n_1 x + n_2 y + n_3 z - (n_1 p_1 + n_2 p_2 + n_3 - p_3) \end{aligned}$$

If we set $d = n_1 p_1 + n_2 p_2 + n_3 - p_3$, then it follows that a point $Q(x, y, z)$ belongs to π if and only if its coordinates satisfy

$$n_1 x + n_2 y + n_3 z = d. \tag{2.13}$$

Equation (2.13) is called the *normal equation* for the plane π.

Remark 2.38. As before, note that the normal equation for a plane is not unique. For instance,

$$x + 2y + 3z = 5 \qquad \text{and} \qquad 2x + 4y + 6z = 10$$

describe the same plane. The reason is that "the" normal vector of a plane is not unique. Given one normal vector $\vec{n}$, than every $c\vec{n}$ with $c \in \mathbb{R} \setminus \{0\}$ is also a normal vector to the plane.

Definition 2.39. The *angle between two planes* is the angle between their normal vectors.

Note that this definition is consistent with the fact that two planes are parallel if and only if their normal vectors are parallel.

Remark 2.40.
- Assume a plane is given as in ((b)) (that is, we know a point P on π and two vectors $\vec{v}$ and $\vec{w}$ parallel to π but with $\vec{v} \not\parallel \vec{w}$). In order to have description as in ((c)) (that is one point on ı and a normal vector), we only have to find a vector $\vec{n}$ that is perpendicular to both $\vec{v}$ and $\vec{w}$. Proposition 2.31(vii) tells us how to do this: we only need to calculate $\vec{v} \times \vec{w}$.

- Assume a plane is given as in ((c)) (that is, we know a point P on π and its normal vector). In order to find vectors $\vec{v}$ and $\vec{w}$ as in ((b)), we can guess either find two solutions of $\vec{x} \times \vec{n} = 0$ which are not parallel. Or we find only one solution $\vec{v}$ which usually is easy to guess and then calculate $\vec{w} = \vec{v} \times \vec{n}$. This vector is perpendicular to $\vec{n}$ and therefore it is parallel to the plane. It is also perpendicular to $\vec{v}$ and therefore it is not parallel to $\vec{v}$. In total, this vector $\vec{w}$ does what we need.

2.6 Intersections of lines and planes in $\mathbb{R}^3$

Intersection of lines

Given two lines G and L in $\mathbb{R}^3$, there are three possibilities:

(a) The lines intersect in exactly one point. In this case, they cannot be parallel.

(b) The lines intersect in infinitely many points. In this case, the lines have to be equal. In particular the have to be parallel.

(c) The lines do not intersect. Not that in contrast to the case in $\mathbb{R}^2$, the lines do not have to be parallel for this to happen. For example, the line $L : x = y = 1$ is a line parallel to the z-axis passing through $(1, 1, 0)$, and $G : x = z = 0$ is a line parallel to the y-axis passing through $(0, 0, 0)$, The lines do not intersect and they are not parallel.

Example 2.41. We consider four lines $L_j = \{\vec{p}_j + t\vec{v}_j : t \in \mathbb{R}\}$ with

$$\text{(i)} \quad \vec{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \vec{p}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \text{(ii)} \quad \vec{v}_2 = \begin{pmatrix}2\\4\\6\end{pmatrix}, \vec{p}_2 = \begin{pmatrix}2\\4\\7\end{pmatrix},$$

$$\text{(iii)} \quad \vec{v}_3 = \begin{pmatrix}1\\1\\2\end{pmatrix}, \vec{p}_3 = \begin{pmatrix}-1\\0\\0\end{pmatrix}, \qquad \text{(iv)} \quad \vec{v}_4 = \begin{pmatrix}1\\1\\2\end{pmatrix}, \vec{p}_4 = \begin{pmatrix}3\\0\\5\end{pmatrix}.$$

We will calculate their mutual intersections.

$\boxed{L_1 \cap L_2 = L_1}$

Proof. A point $Q(x, y, z)$ belongs to $L_1 \cap L_2$ if and only if it belongs both to L_1 and L_2. This means that there must exist an $s \in \mathbb{R}$ such that $\overrightarrow{0Q} = \vec{p}_1 + s\vec{v}_1$ and there must exist a $t \in \mathbb{R}$ such that $\overrightarrow{0Q} = \vec{p}_2 + t\vec{v}_2$. Note the s and t are different parameters. So we are looking for s and t such that

$$\vec{p}_1 + s\vec{v}_1 = \vec{p}_2 + t\vec{v}_2, \qquad \text{that is} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} + s\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\4\\7\end{pmatrix} + t\begin{pmatrix}2\\4\\6\end{pmatrix} \tag{2.14}$$

Once we have solved this for s and t, we insert the into the equation for L_1 and L_2 respectively, and obtain Q. Note that (2.14) in reality is a system of three equations: one equation for each component of the vector equation. Writing it out, and solving each equation for s, we obtain

$$\begin{aligned} 0 + s &= 2 + 2t \\ 0 + 2s &= 4 + 4t \\ 1 + 3s &= 7 + 6t \end{aligned} \qquad \iff \qquad \begin{aligned} s &= 2 + 2t \\ s &= 2 + 2t \\ s &= 2 + 2t. \end{aligned}$$

This means that we have infinitely many solutions: Given any point R on L_1, there is a corresponding $s \in \mathbb{R}$ such that $\overrightarrow{0R} = \vec{p}_1 + s\vec{v}_1$. Now if we choose $t = (s-2)/2$, then $\overrightarrow{0R} = \vec{p}_2 + t\vec{v}_2$ holds, hence $R \in L_2$ too. If on the other hand we have a point $R' \in L_2$, then there is a corresponding $t \in \mathbb{R}$ such that $\overrightarrow{0R'} = \vec{p}_2 + t\vec{v}_2$. Now if we choose $s = 2 + 2t$, then $\overrightarrow{0R'} = \vec{p}_1 + t\vec{v}_1$ holds, hence $R' \in L_2$ too. In summary, we showed that $L_1 = L_2$. □

Remark 2.42. We could also have seen that the directional vectors of L_1 and L_2 are parallel. In fact, $\vec{v}_2 = 2\vec{v}_1$. It then suffices to show that L_1 and L_2 have at least one point in common in order to conclude that the lines are equal.

$\boxed{L_1 \cap L_3 = \{(1, 2, 4)\}}$

Proof. As before, we need to find $s, t \in \mathbb{R}$ such that

$$\vec{p}_1 + s\vec{v}_1 = \vec{p}_3 + t\vec{v}_3, \qquad \text{that is} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{2.15}$$

We write this as a system of equations, we get

$$\begin{aligned} &\textcircled{1} & 0 + s &= -1 + t \\ &\textcircled{2} & 0 + 2s &= 0 + t \\ &\textcircled{3} & 1 + 3s &= 0 + 2t \end{aligned} \qquad \iff \qquad \begin{aligned} &\textcircled{1} & s - t &= -1 \\ &\textcircled{2} & 2s - t &= 0 \\ &\textcircled{3} & 3s - 2t &= -1 \end{aligned}$$

From ① it follows that $s = t - 1$. Inserting in ② gives $0 = 2(t-1) - t = t - 2$, hence $t = 2$. From ① we then obtain that $s = 2 - 1 = 1$. Observe that so far we used only equations ① and ②. In order to see if we really found a solution, we must check if it is consistent with ③. Inserting our candidates for s and t, we find that $3 \cdot 1 - 2 \cdot 2 = -1$ which is consistent with ③.
So we have exactly one point of intersection. In order to find it, we put $s = 1$ in the equation for L_1:

$$\overrightarrow{0Q} = \vec{p}_1 + 1 \cdot \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

hence the intersection point is $Q(1, 2, 4)$.
In order to check if this result is correct, we can put $t = 2$ in the equation for L_3. The result must be the same. The corresponding calculation is:

$$\overrightarrow{0Q} = \vec{p}_3 + 2 \cdot \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

which confirms that the intersection point is $Q(1, 2, 4)$. □

$\boxed{L_1 \cap L_4 = \varnothing}$

Proof. As before, we need to find $s, t \in \mathbb{R}$ such that

$$\vec{p}_1 + s\vec{v}_1 = \vec{p}_4 + t\vec{v}_4, \qquad \text{that is} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{2.16}$$

We write this as a system of equations, we get

$$\begin{array}{rrcl} ① & s &=& 3 + t \\ ② & 2s &=& t \\ ③ & 1 + 3s &=& 5 + 2t \end{array} \qquad \Longleftrightarrow \qquad \begin{array}{rrcl} ① & s - t &=& 3 \\ ② & 2s - t &=& 0 \\ ③ & 3s - 2t &=& 5 \end{array}$$

From ① it follows that $s = t + 3$. Inserting in ② gives $0 = 2(t + 3) - t = t + 6$, hence $t = -6$. From ① we then obtain that $s = -6 + 3 = -3$. Observe that so far we used only equations ① and ②. In order to see if we really found a solution, we must check if it is consistent with ③. Inserting our candidates for s and t, we find that $3 \cdot (-3) - 2 \cdot (-6) = 3$ which is inconsistent with ③. Therefore we conclude that there is no pair of real numbers s, t which satisfies all three equations ①–③ simultaneously, so the two lines do not intersect. □

Exercise. Show that $L_3 \cap L_4 = \varnothing$.

Intersection of planes

Given two planes π_1 and π_2 in $\mathbb{R}^3$, there are two possibilities:

(a) The planes intersect. In this case, they necessarily intersect in infinitely many points. The intersection is either a line. In this case π_1 and π_2 are not parallel. Or the intersection is a plane. In this case $\pi_1 = \pi_2$.

(b) The planes do not intersect. In this case, the planes must be parallel and not equal.

Example 2.43. We consider the following four planes:

$$\pi_1 : x + y + 2z = 3, \quad \pi_2 : 2x + 2y + 4z = 3, \quad \pi_3 : 2x + 2y + 4z = 6, \quad \pi_4 : x + y - 2z = 5.$$

We will calculate their mutual intersections.

$\boxed{\pi_1 \cap \pi_2 = \varnothing}$

Proof. The set of all points $Q(x, y, z)$ which belong both to π_1 and π_2 is the set of all x, y, z which simultaneously satisfy

$$\begin{array}{rl} ① & x + y + 2z = 3, \\ ② & 2x + 2y + 4z = 3. \end{array}$$

Now clearly, if x, y, z satisfies ①, then it cannot satisfy ② (the right side would be 6). We can see this more formally if we solve ①, e.g., for x and then insert into ②. We obtain from ①: $x = 3 - y - 2z$. Inserting into ② leads to

$$3 = 2(3 - y - 2z) + 2y + 4z = 6,$$

which is absurd.

Geometrically, this was to be expected. The normal vectors of the planes are $\vec{n}_1 = \begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\vec{n}_2 = \begin{pmatrix}2\\2\\4\end{pmatrix}$ respectively. Since they are parallel, the planes are parallel and therefore they either are equal or they have empty intersection. Now we see that for instance $(3,0,0) \in \pi_1$ but $(3,0,0) \notin \pi_2$, so the planes cannot be equal. Therefore they have empty intersection. ☐

$\boxed{\pi_1 \cap \pi_3 = \pi_1}$

Proof. The set of all points $Q(x,y,z)$ which belong both to π_1 and π_3 is the set of all x,y,z which simultaneously satisfy

$$\begin{array}{ll} ① & x + y + 2z = 3, \\ ② & 2x + 2y + 4z = 6. \end{array}$$

Clearly, both equations are equivalent: if x,y,z satisfies ①, then it also satisfies ② and vice versa. Therefore, $\pi_1 = \pi_3$. ☐

$$\boxed{\pi_1 \cap \pi_4 = \left\{ \begin{pmatrix}4\\0\\-\frac{1}{2}\end{pmatrix} + t\begin{pmatrix}-1\\1\\0\end{pmatrix} : t \in \mathbb{R} \right\}.}$$

Proof. First, we notice that the normal vectors $\vec{n}_1 = \begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\vec{n}_4 = \begin{pmatrix}1\\1\\-2\end{pmatrix}$ are not parallel, so we expect that the solution is a line in $\mathbb{R}^3$.

The set of all points $Q(x,y,z)$ which belong both to π_1 and π_4 is the set of all x,y,z which simultaneously satisfy

$$\begin{array}{ll} ① & x + y + 2z = 3, \\ ② & x + y - 2z = 5. \end{array}$$

Equation ① shows that $x = 3 - y - 2z$. Inserting into ② leads to $5 = 3 - y - 2z + y - 2z = 3 - 4z$, hence $z = -\frac{1}{2}$. Putting this into ①, we find that $x + y = 3 - 2z = 4$. So in summary, the intersection consists of all points (x,y,z) which satisfy

$$z = -\frac{1}{2}, \quad x = 4 - y \quad \text{with} \quad y \in \mathbb{R} \ \text{ arbitrary},$$

in other words,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4-y\\y\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}4\\0\\-\frac{1}{2}\end{pmatrix} + \begin{pmatrix}-y\\y\\0\end{pmatrix} = \begin{pmatrix}4\\0\\-\frac{1}{2}\end{pmatrix} + y\begin{pmatrix}-1\\1\\0\end{pmatrix} \quad \text{with} \quad y \in \mathbb{R} \ \text{ arbitrary.}$$ ☐

Intersection of several lines and planes

If we wanted to intersect for instance, 5 planes in $\mathbb{R}^3$, then we would have to solve a system of 5 equations for 3 unknowns. Or if we wanted to intersect 7 lines in $\mathbb{R}^3$, then we had to solve a system of 3 equations for 7 unknowns. If we do it like here, this could become quite messy. So the next chapter is devoted to find a systematic way how to solve a system of m linear equations for n unknowns.

2.7 Summary

$$\begin{aligned} x - 2y - 4z &= 1 \\ 3x - y - z &= -1 \\ x - 11y + 22z &= 110 \end{aligned}$$

Faltan Figures 11, 12.

Chapter 3

Linear Systems and Matrices