Sistemas de ecuaciones lineales

Example 1. A zoo has birds and cats. In total, there are 60 heads and 200 legs.
How many birds and cats are in the zoo?

Solution. b = number of birds
c = number of cats

We know: $\left.\begin{array}{l} ① \ b + c = 60 \\ ② \ 2b + 4c = 200 \end{array}\right\}$ $\xrightarrow{①}$ $b = 60 - c$ $(*)$

$\xrightarrow{\text{in } ②}$ $2(60-c) + 4c = 200$

$\Rightarrow 2c = 200 - 120 = 80$

$\Rightarrow \boxed{c = 40}$

$\xrightarrow{(*)} \boxed{b = 20}$

Example 2. Find a polynomial P of degree ≤ 3 with:

$P(0) = 1$, $P(1) = 7$, $P'(0) = 3$, $P'(2) = 23$.

Solution: Write $P(X) = \alpha X^3 + \beta X^2 + \gamma X + \delta$.

We have to find $\alpha, \beta, \gamma, \delta$.

Observe: $P'(X) = 3\alpha X^2 + 2\beta X + \gamma$.

We obtain the set of equations:

$$\left.\begin{array}{lrcl} ① & \delta &=& 1 \\ ② & \alpha + \beta + \gamma + \delta &=& 7 \\ ③ & \gamma &=& 3 \\ ④ & 12\alpha + 4\beta + \gamma &=& 23 \end{array}\right\}$$

① and ③ $\xrightarrow{\text{in } ② \text{ and } ④}$ $\begin{cases} \alpha + \beta = 3 & ②' \\ 12\alpha + 4\beta = 20 & ④' \end{cases}$

$②' \Rightarrow \alpha = 3 - \beta \xrightarrow{\text{in } ④'} 20 = 12(3-\beta) + 4\beta = 36 - 8\beta$

$\Rightarrow \beta = \frac{1}{8}(36 - 20) = 2$

$\Rightarrow \alpha = 3 - \beta = 1$.

$\Rightarrow \boxed{P(X) = X^3 + 2X^2 + 3X + 1}$

Important question:

Existence and uniqueness of solutions of systems of linear equations?

Examples:

(A)
$$\left.\begin{matrix} x+2y=11 \\ 3x+4y=27 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x=11-2y \\ \Rightarrow 3(11-2y)+4y=27 \end{matrix}\right\} \Rightarrow \begin{cases} x=11-2y \\ -2y=-6 \end{cases}$$

$\Rightarrow \boxed{y=3,\ x=5}$ Easy to check: This is really a solution!

⇝ The system has a unique solution. ⇝ Existence & uniqueness!

(B)
$$\left.\begin{matrix} x+2y=1 \\ 2x+4y=5 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x=1-2y \\ 2(1-2y)+4y=5 \end{matrix}\right\} \Rightarrow \begin{cases} x=1-2y \\ 2=5 \end{cases} ↯$$

⇝ The system has no solution ⇝ No existence!

(C)
$$\left.\begin{matrix} ① & x+2y=1 \\ ② & 3x+6y=3 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x=1-2y \\ 3(1-2y)=3 \end{matrix}\right\} \Rightarrow \begin{cases} x=1-2y \\ 3=3 \end{cases}$$

Evidently, every pair (x,y) that satisfies ①, also satisfies ② and vice versa.

⇝ The system has infinitely many solutions.

⇝ Existence, but not uniqueness.

Geometric interpretation

The solutions of the linear equation $Ax+By=U$ are:

* a line with slope $-A/B$ if $B\neq 0$ or parallel to y-axis if $B=0$ and $A\neq 0$
* $\mathbb{R}^2$ if $A=B=0$, $U=0$
* $\{\}$ (empty set) if $A=B=0$, $U\neq 0$.

⇝ Solution of the system

$$\left.\begin{matrix} ① & Ax+By=U \\ ② & Cx+Dy=V \end{matrix}\right\} \quad (*)$$

is the intersection of two lines, a line with $\{\}$, etc.

Exactly one solution: if ① and ② are non-parallel lines

⇝ ① and ② must describe lines with different slopes.

Case 1. $B\neq 0$ and $D\neq 0$: ⇝ $-A/B \neq -C/D$

$\Leftrightarrow -AD\neq -BC \Leftrightarrow \boxed{AD-BC\neq 0}$

Case 2 $B=0,\ D\neq 0$: ⇝ $A\neq 0$ (otherwise ① is not a line)

$\Rightarrow AD\neq 0 \Rightarrow \boxed{AD-BC\neq 0}$

Case 3. $B\neq 0,\ D=0$ ⇝ as in Case 2: $BC\neq 0$

$\Rightarrow \boxed{AD-BC\neq 0}$

There are no other cases ($B=D=0$ would imply that the lines are parallel).

No solution:

EITHER: (1) and (2) are parallel, but not equal

$\leadsto$ they have the same slope

If $B \neq 0 \Rightarrow D \neq 0$ and $-A/B = -C/D \Rightarrow -AD = -BC$

$\Rightarrow \boxed{AD - BC = 0}$

If $B = 0 \Rightarrow D = 0$ and $\boxed{AD - BC = 0}$

OR (1) or (2) describe the empty set

$\leadsto$ Either $A = B = 0$ or $C = D = 0$ $\leadsto$ $\boxed{AD - BC = 0}$

Infinitely many solutions:

EITHER (1) and (2) describe the same line

$\leadsto$ they have the same slope. As above: $\boxed{AD - BC = 0}$

OR (1) is a line and (2) is $\mathbb{R}^2$ (or vice versa)

$\leadsto$ $C = D = 0$ (or $A = B = 0$) $\leadsto$ $\boxed{AD - BC = 0}$

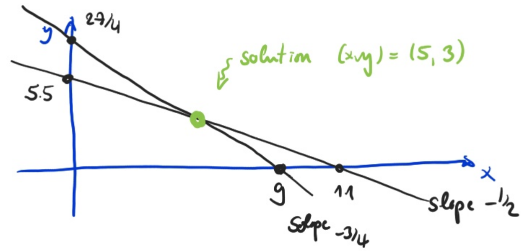
SUMMARY:

- $AD - BC \neq 0$ $\leadsto$ exactly one solution
- $AD - BC = 0$ $\leadsto$ no solution or infinitely many

$AD - BC$ is called the <u>determinant</u> of (*)

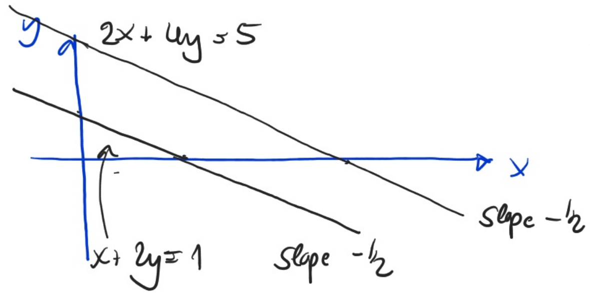
In the examples:

(A)



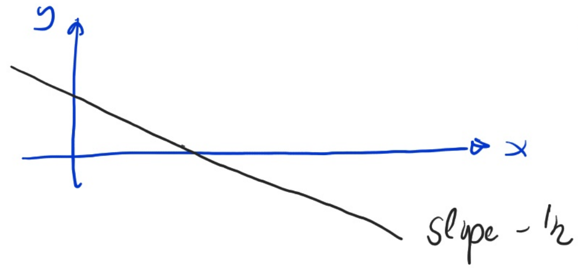
Determinant:
$1 \cdot 4 - 2 \cdot 3 = -2 \neq 0$.

(B)



Determinant:
$1 \cdot 4 - 2 \cdot 2 = 4 - 4 =$

(C)



Determinant:
$1 \cdot 6 - 2 \cdot 3 = 6 - 6 = 0$.