5. Eigenvectors & Eigenvalues

In this chapter we work mainly with complex vector spaces.

$*$ Inner product on $\mathbb{C}^n$: $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$

$\leadsto \langle x, y \rangle := \sum_{j=1}^n x_j \overline{y_j}$. ← linear in first component, "antilinear" in second comp.

$\|x\| := \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}$.

Properties:

- $\forall x \in \mathbb{C}^n$: $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0 \iff x = 0$
- $\forall x, y \in \mathbb{C}^n$: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$
- Cauchy-Schwarz: $|\langle x, y \rangle| \leq \|x\| \, \|y\|$.

$*$ $A \in M(n\times n, \mathbb{C})$ is called hermitian $\iff A = A^*$

where A^* = transposed and conjugated matrix of A

$\left(A = (a_{ij})_{i,j=1,\dots,n} \Rightarrow A^* = (\overline{a}_{ji})_{i,j=1,\dots,n} \right)$

$*$ $U \in M(n\times n, \mathbb{C})$ is called unitary $\iff U^* = U^{-1}$.

It can be shown: $A, B \in M(n \times n) \Rightarrow \left(B = A^* \iff \forall x, y \in \mathbb{K}^n \ \langle Ax, y \rangle = \langle x, By \rangle \right)$.

Observe: $\forall A \in M(n\times n)$: $\det(A^*) = \overline{\det(A)}$.

In particular
- A hermitian $\Rightarrow \det(A^*) = \det(A) \in \mathbb{R}$.
- U unitary $\Rightarrow |\det U| = 1$.

Some facts on polynomials:

Let p be a polynomial of degree n with complex coefficients.

$\Rightarrow \exists \lambda_1, \dots, \lambda_k \in \mathbb{C}$, $r_1, \dots, r_k \in \mathbb{N}$, $a \in \mathbb{C}$ st.

$$p(x) = a(x-\lambda_1)^{r_1} \cdot \ldots \cdot (x-\lambda_k)^{r_k}$$

If p has real coefficients and λ is a zero of p, then $\overline{\lambda}$ is a zero of p too.

Proof. Assume $p(\lambda) = 0$

$\Rightarrow p(\overline{\lambda}) = \overline{p(\lambda)} = \overline{0} = 0$. ← coeff. of p are real. □

5.1. Eigenvectors & Eigenvalues

Definition. Let V be a finite-dimensional $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

Let $T: V \to V$ linear. Then $\lambda \in \mathbb{K}$ is called an eigenvalue of T if the equation $\boxed{(T - \lambda \mathbb{1}) v = 0}$ $(*)$ has a non-trivial solution v.

Every non-trivial solution v of $(*)$ is called eigenvector of T.

For fixed λ, we set $E_\lambda := E_\lambda(T) := \operatorname{Eig}(\lambda, T) = \{ v \in V \mid (T-\lambda) v = 0 \}$

E_λ is called the eigenspace of T.

The set of all eigenvectors of T is called the spectrum of T, denoted by $\sigma(T)$.

Notation. Instead of $\lambda \mathbb{1}$ we write simply λ.

Lemma. For every eigenvalue λ of T, we have

$E_\lambda = \{\text{all eigenvectors of } T \text{ for } \lambda\} \cup \{0\}$
$= \ker(T - \lambda)$.

In particular, E_λ is a subspace of V.

Proof. Clear. □

Remark: 0 EVal. of A $\iff$ A is not inj and $\ker A = E_0(A)$.

Definition $\dim(E_\lambda(T)) =:$ geometric multiplicity of the eigenvalue λ.

Examples:

- $A: \mathbb{R}^6 \to \mathbb{R}^6$, $\quad A = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 3 & \\ 0 & & & & & 3 \end{pmatrix}$

  $\Rightarrow$ A has three eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.
  geometric multiplicity of λ_1 is 3
  of λ_2 is 1
  of λ_3 is 2.

- $A: \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

  $\Rightarrow$ A has exactly one eigenvalue: $\lambda = 1$, and its geometric multiplicity is 1.

- $A: \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ [Reflection on x-axis]

  $\Rightarrow$ A has two eigenvalues: $\lambda_1 = 1, \lambda_2 = -1$.

- $A: \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [Rotation by $-90°$].

  Clearly, A cannot have eigenvalues.

  BUT: $A: \mathbb{C}^2 \to \mathbb{C}^2$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has the eigenvalues

  $\lambda_\pm = \pm i$ with eigenvectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ y $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

- P orth. proj on $U \Rightarrow \sigma(P) \subseteq \{0,1\}$. $E_0(P) = \ker P = \underbrace{(\operatorname{Rg} P)^\perp}_{= U^\perp}$, $E_1(P) = \underbrace{\operatorname{Rg}(P)}_{= U}$

Theorem. Let $T: V \to V$ linear and let $\lambda_1, \dots, \lambda_k$ be different eigenvalues of T with eigenvectors $v_1, \dots, v_k$.

$\Rightarrow \{v_1, \dots, v_k\}$ are lin. independent.

Proof. By induction:

- Base case: We will show that v_1 and v_2 are lin. indep.

  Assume they are lin. dep $\Rightarrow \exists c \in \mathbb{K}$ s.t. $v_1 = c v_2$ (recall: $v_1, v_2 \neq 0$).

  $\Rightarrow \lambda_1 v_1 = A v_1 = A(c v_2) = c A v_2 = c \lambda_2 v_2 = \lambda_2 v_1$

  $\Rightarrow \lambda_1 = \lambda_2$ ↯.

- Induction step: Let $2 \le j < k$ and assume that $\{v_1, \dots, v_j\}$ are lin. ind. We will show that $\{v_1, \dots, v_{j+1}\}$ are lin. ind.

  Let $c_1, \dots, c_{j+1} \in \mathbb{K}$ s.t.

  $0 = c_1 v_1 + \dots + c_{j+1} v_{j+1}$. $\quad (*)$

  $\Longrightarrow 0 = A(c_1 v_1 + \dots + c_{j+1} v_{j+1})$ $\quad$ Apply A to $(*)$

  $= c_1 \lambda_1 v_1 + \dots + c_{j+1} \lambda_{j+1} v_{j+1}$ $\quad (1)$

  and $0 = \lambda_{j+1} c_1 v_1 + \dots + \lambda_{j+1} c_{j+1} v_{j+1}$ $\quad (2)$ Multiply $(*)$ by λ_{j+1}

  $\xrightarrow{(1)-(2)} 0 = c_1(\lambda_1 - \lambda_{j+1}) v_1 + \dots + c_j (\lambda_j - \lambda_{j+1}) v_j$

  $\xrightarrow[\text{lin. ind}]{v_1, \dots, v_j} c_1(\lambda_1 - \lambda_{j+1}) = 0, \dots, c_j(\lambda_j - \lambda_{j+1}) = 0$

  $\Longrightarrow c_1 = 0, \dots, c_j = 0$ because $\lambda_k \neq \lambda_\ell$ for $k \neq \ell$. $\square$

Remark $A|_{E_\lambda(A)}$ acts as $\lambda \cdot \mathbb{1}|_{E_\lambda(A)}$

5.2. The characteristic polynomial.

How do we find the eigenvalues of a linear map $A: \mathbb{R}^n \to \mathbb{R}^n$?

We may assume that A is given as a matrix.

Then:

$\lambda \in \sigma(A) \iff \dim(E_\lambda(A)) > 0$

$\iff \dim(\ker(A - \lambda)) > 0$

$\iff A - \lambda$ is not injective

$\iff \det(A - \lambda) = 0$.

Observation. For $A \in M(n \times n)$ fixed,

$p(\lambda) := \det(A - \lambda)$

is a polynomial of degree n, called the characteristic polynomial of A.

It is of the form $p(\lambda) = (-1)^n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0$,

where $a_0 = p(0) = \det A$, and $a_{n-1} = \sum_{j=1}^n a_{jj} =: \operatorname{tr}(A)$

$=:$ trace of A.

In summary, we obtain the following Theorem:

Theorem. $A \in M(n\times n, \mathbb{C}) \Rightarrow \left(\begin{array}{l} \lambda \in \mathbb{C} \text{ is an eigenvalue of } A \\ \Longleftrightarrow \lambda \text{ is a zero of the charact.} \\ \quad \text{polynomial of } A. \end{array} \right)$

Definition. $A \in M(n\times n, \mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_k$. $(\lambda_j \neq \lambda_\ell)$

$\Rightarrow \exists r_1, \dots, r_k \in \mathbb{N}$ s.t. $\det(A-\lambda) = (\lambda_1-\lambda)^{r_1} \cdot \ldots \cdot (\lambda_k-\lambda)^{r_k}$.

Then r_j is called the algebraic multiplicity of the eigenvalue λ_j.

It can be shown: For every eigenvalue: $\boxed{\text{geometric mult} \leq \text{algebraic mult}}$

Example. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\Rightarrow \det(A-\lambda) = (1-\lambda)^2 \rightsquigarrow A$ has only eigenvalue, $\lambda = 1$.

- algebraic mult. of $\lambda = 1$: 2.
- geometric mult of $\lambda = 1$: 1 because: $A - 1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
  Clearly: $\dim(\ker(A-1)) = 1$.
- $E_1(A) = \ker(A-1) = \text{gen}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

Receipt for finding eigenvectors & eigenvalues of A:

1. Calculate $p(\lambda) = \det(A-\lambda)$
2. Find all zeros $\lambda_1, \dots, \lambda_k$ of p. These are the eigenvalues of A
3. Find $\ker(A-\lambda_j)$ for $j = 1, \dots, k$.

Examples:

- A diagonal matrix: $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

  $\Rightarrow$ all eigenvalues are $\lambda_1, \dots, \lambda_n$ and for every eigenvalue:
  geometric mult = algebraic mult.

  Moreover: $\forall_j$ $\vec{e}_j$ is eigenvector of A with eigenvalue λ_j

- A upper triangular matrix: $A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

  $\Rightarrow$ The eigenvalues of A are $\lambda_1, \dots, \lambda_n$.

Theorem. $A \in M(n\times n)$. Then:

A has n lin. ind. Eigenvectors $\Longleftrightarrow$ $\forall \lambda$ eigenvalue of A: algebr. mult = geom. mult.

$\Longleftrightarrow$ $\mathbb{R}^n$ has a basis consisting of Evectors of A.

Corollary. A has n different eigenvalues

$\Rightarrow$ $\mathbb{R}^n$ has a basis of Eigenvectors of A.

With respect to this basis, A is of the form $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$,

and $\mathbb{R}^n = E_{\mu_1}(A) \oplus \dots \oplus E_{\mu_k}(A)$ where $\mu_1, \dots, \mu_k$ are the different Evalues of A.

Useful theorems:

Theorem. (Cayley-Hamilton) $A \in M(n\times n)$ and $p(\lambda) = \det(A-\lambda)$.

$\Rightarrow p(A) = 0$.

Definition. $A, B \in M(n\times n)$. A, B are called similar if there exists a $C \in M(n\times n)$, invertible, such that

$$\boxed{A = C^{-1} B C} \quad (*)$$

C is called a similarity transformation.

If A and B are similar, we write $A \sim B$.

Observe: $(*) \iff CA = BC$.

Properties.

(i) $\sim$ is an equivalence relation; that is:

- $R \sim R$
- $R \sim S,\ S \sim T \Rightarrow R \sim T$
- $R \sim S \Rightarrow S \sim R$.

(ii) For $C \in M(n\times n)$, C invertible, fixed, the map

$$T: M(n\times n) \to M(n\times n),\quad A \mapsto C^{-1}AC$$

is a linear map.

Observation. Let $A, B, C \in M(n\times n)$, C invertible.

$\leadsto$ C can be seen as the matrix of a change of bases and "$A = C^{-1}BC$" says that A and B describe the same linear map, but with respect to two different bases.

Theorem. Let $A, B \in M(n\times n)$ s.t. $A \sim B$.

$\Rightarrow$ A and B have the same characteristic polynomial.
In particular, they have the same eigenvalues. (but in general not the same eigenvectors).

Proof. Let $C \in M(n\times n)$ s.t. $A = C^{-1}BC$

$$\Rightarrow p_A(\lambda) = \det(A-\lambda) = \det(C^{-1}BC - \lambda C^{-1}C) = \det(C^{-1}(B-\lambda)C)$$
$$= \underbrace{(\det C^{-1})}_{=(\det C)^{-1}} \det(B-\lambda)(\det C) = \det(B-\lambda) = p_B(\lambda).$$

$\square$

Relation between eigenvectors: Let $v \in \mathbb{R}^n$ s.t. $(A-\lambda)v = 0$

$$\leadsto (A-\lambda)v = 0 \iff (A-\lambda)CC^{-1}v = 0 \iff C^{-1}(A-\lambda)CC^{-1}v = 0$$
$$\iff (C^{-1}AC - \lambda C^{-1}C)C^{-1}v = 0$$
$$\iff (B-\lambda)C^{-1}v = 0$$

$\leadsto$ (v Eigenvector of A with Evalue λ $\iff$ $C^{-1}v$ eigenvector of B with Evalue λ)

Definition. $A \in M(n\times n)$ is called diagonalizable if and only if A is similar to a diagonal matrix.

Observation. A diagonalizable $\Rightarrow$ $\det A = \lambda_1 \cdot \ldots \cdot \lambda_n$ (λ_j: eigenvalues of A)

Theorem. A is diagonalizable $\iff$ $\mathbb{R}^n$ has a basis consisting of Eigenvectors of A.

In particular: A has n different eigenvalues $\Rightarrow$ A is diagonalizable

Proof "$\Rightarrow$" Suppose that A is diagonalizable.

Let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $C \in M(n\times n)$ s.t. $D = C^{-1}AC$

$\Rightarrow CD = AC$. $(*)$

Let $\vec{c}_1, \ldots, \vec{c}_n$ be the columns of C.

$$\Rightarrow A\vec{c}_j = j\text{-th column of } AC$$
$$= j\text{-th column of } CD \quad \text{by } (*)$$
$$= \lambda_j \cdot (j\text{-th column of } C)$$
$$= \lambda_j \vec{c}_j.$$

$\rightarrow$ $\vec{c}_j$ is an eigenvector of A with eigenvalue λ_j.

Since C is invertible, the $\vec{c}_1, \ldots, \vec{c}_n$ form a basis of $\mathbb{R}^n$ and "$\Rightarrow$" is proved.

"$\Leftarrow$" Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis of Evectors of A with Eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively.

$\Rightarrow \forall j: \quad A\vec{v}_j = \lambda \vec{v}_j$.

Let $C = (\vec{v}_1 | \ldots | \vec{v}_n)$

$\Rightarrow C^{-1}AC = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ = A represented in the basis $\vec{v}_1, \ldots, \vec{v}_n$.

$\square$

Eigenvectors & Eigenvalues for linear maps (not necessarily matrices):

Let V be a vector space with $\dim V = n < \infty$ and $T: V \to V$ linear.

Choose a basis $\mathcal{B} := \{\vec{v}_1, \dots \vec{v}_n\}$ of V and let A be the matrix repr. of T with respect to $\mathcal{B}$.

Then: characteristic polynomial of $T := p_T(\lambda) := \det(A-\lambda)$.

Observe: p_T does not depend on the chosen basis.

Because: Let $\mathcal{B}' := \{\vec{w}_1, \dots \vec{w}_n\}$ basis of V and A' the matrix repr. of T wrt. $\mathcal{B}'$.

Let C = transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

$\Rightarrow \quad A = C^{-1}A'C \quad \Rightarrow$ A and A' are similar.

$\Rightarrow$ A and A' have the same char. polyn.

Clearly: λ Eigenvalue of $T \Leftrightarrow T-\lambda$ not inj

$\Leftrightarrow A-\lambda$ not injective

$\Leftrightarrow \det(A-\lambda) = 0$

$\Leftrightarrow p_T(\lambda) = 0$. □

Examples

- $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y+z \\ -3x-2y+3z \\ -2x-2y+3z \end{pmatrix}$

Find Eigenvalues & Eigenvectors of T.

Solution. Matrix repr. of T: $\quad A = \begin{pmatrix} 0 & -1 & 1 \\ -3 & -2 & 3 \\ -2 & -2 & 3 \end{pmatrix}$

Char. polynomial:

$$p(\lambda) = \det(A-\lambda) = \det\begin{pmatrix} -\lambda & -1 & 1 \\ -3 & -2-\lambda & 3 \\ -2 & -2 & 3-\lambda \end{pmatrix}$$

$$= \lambda(2+\lambda)(3-\lambda) + 6 + 6 - [2(2+\lambda) + 6\lambda + 3(3-\lambda)]$$

$$= -\lambda^3 + \lambda^2 + 6\lambda + 12 - 5\lambda - 13$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1 = (\lambda-1)[-\lambda^2+1] = -(\lambda-1)^2(\lambda+1)^2.$$

$\Rightarrow$ Eigenvalues of T: $\lambda_1 = 1$, $\lambda_2 = -1$.

Eigenspaces:

- λ_1: $\quad A - \lambda_1 = \begin{pmatrix} -1 & -1 & 1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \dim E_{\lambda_1} = 2$, $\quad E_{\lambda_1} = \operatorname{gen}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\vec{v}_2} \right\}$.

- λ_2: $\quad A - \lambda_2 = \begin{pmatrix} 1 & -1 & 1 \\ -3 & -1 & 3 \\ -2 & -2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & -4 & 6 \\ 0 & -4 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \dim E_{\lambda_2} = 1$, $\quad E_{\lambda_2} = \operatorname{gen}\left\{ \underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_{\vec{v}_3} \right\}$.

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis of $\mathbb{R}^3 \Rightarrow A$ is diagonalizable.

Let $C = (\vec{v}_1 | \vec{v}_2 | \vec{v}_3)$

$\Rightarrow \quad C^{-1}AC = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

5.3. Symmetric and Hermitian Matrices.

Observe: A real and symmetric $\Rightarrow$ A hermitian

Theorem. Let $A \in M(n\times n, \mathbb{C})$ be a hermitian matrix.

$\Rightarrow$ All eigenvalues of A are real.

Proof. Let λ be an eigenvalue of A. We have to show: $\lambda = \bar\lambda$.
Let $x \in \mathbb{C}^n \setminus \{0\}$ s.t. $(A-\lambda)x = 0$

$$\Rightarrow \lambda \|x\|^2 = \lambda \langle x, x\rangle = \langle \lambda x, x\rangle = \langle Ax, x\rangle = \langle x, A^* x\rangle$$
$$= \langle x, Ax\rangle = \langle x, \lambda x\rangle = \bar\lambda \|x\|^2.$$
$$\Rightarrow \lambda \|x\|^2 = \bar\lambda \|x\|^2 \Rightarrow \lambda = \bar\lambda \quad (\text{because } \|x\| \neq 0).$$
$\square$

Theorem. Let $A \in M(n\times n, \mathbb{C})$ hermitian. Let $\lambda \neq \mu$ be eigenvalues of A and $\vec v, \vec w$ eigenvectors respectively. Then: $\vec v \perp \vec w$.

Proof. Assume $\vec v, \vec w \in \mathbb{C}^n \setminus \{0\}$, $(A-\lambda)\vec v = 0$, $(A-\mu)\vec w = 0$

$$\Rightarrow \lambda \langle \vec v, \vec w\rangle = \langle \lambda \vec v, \vec w\rangle = \langle A\vec v, \vec w\rangle = \langle \vec v, A^* \vec w\rangle$$
$$= \langle \vec v, A\vec w\rangle = \langle \vec v, \mu \vec w\rangle = \bar\mu \langle \vec v, \vec w\rangle \overset{\mu \in \mathbb{R}}{=} \mu \langle \vec v, \vec w\rangle$$
$$\Rightarrow (\lambda-\mu)\langle \vec v, \vec w\rangle = 0 \overset{\lambda \neq \mu}{\Longrightarrow} \langle \vec v, \vec w\rangle = 0 \Rightarrow \vec v \perp \vec w.$$
$\square$

This shows: If A is hermitian and diagonalizable, then $\mathbb{C}^n$ has an ONB of eigenvectors of A (For every $\lambda \in \sigma(A)$, choose an ONB in $E_\lambda(A)$. The union of these bases gives then a basis of $\mathbb{C}^n$)

It can be shown:

Thm. A hermitian $\Rightarrow$ A diagonalizable.

With the above, it follows that every hermitian Matrix A has an ONB of eigenvectors.

Definition. $A \in M(n\times n, \mathbb{C})$ admits an orthogonal diagonalization if there is a diagonal matrix D and an orthogonal matrix Q such that $D = Q^t A Q$

Clearly this is the case if and only if has an ONB of Eigenvectors of A

Theorem. $A \in M(n\times n, \mathbb{C})$. Then:

A hermitian $\Leftrightarrow$ A has an orthogonal diagonalization. $\square$

Other useful theorems:

Let $A \in M(n\times n, \mathbb{K})$. A is called triangularizable if it is similar to an upper triangular matrix.

Theorem. $A \in M(n\times n, \mathbb{K})$. Then the following is equivalent:

(i) A is triangulizable

(ii) The characteristic polynomial of A is product of linear factors:

$$p_A(\lambda) = (\lambda_1 - \lambda)\cdot \ldots \cdot (\lambda_n - \lambda).$$

Corollary. Every $A \in M(n\times n, \mathbb{C})$ is triangulizable.

This is false for $\mathbb{K} = \mathbb{R}$. For example, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not triangulizable over $\mathbb{R}$. [Suppose it were. $\Rightarrow$ A must have eigenvalues; but:
$$p_A(\lambda) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 \text{ has no zeros in } \mathbb{R}]$$