4.9. Orthonormal basis and projections.

Recall. $\vec{x}, \vec{y} \in \mathbb{R}^n \implies \|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle,$

$$\vec{x} \perp \vec{y} \iff \langle \vec{x}, \vec{y} \rangle = 0$$

Definition.

- $\vec{x}_1, \dots, \vec{x}_k$ is called an orthogonal set if $\vec{x}_j \perp \vec{x}_k$ for $j \neq k$.

  In formulas: $\langle \vec{x}_j, \vec{x}_k \rangle = 0$ for $j \neq k$.

- $\vec{x}_1, \dots, \vec{x}_k$ is called an orthonormal set (ONS) if they from an orthogonal set and $\|\vec{x}_j\| = 1$ for all $j = 1, \dots, k$.

  In formulas: $$\langle \vec{x}_j, \vec{x}_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0, & \text{if } j \neq k. \end{cases}$$

Lemma. Assume that $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ are an ONS. Then they are linearly independent.

Proof. Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ s.t. $0 = \lambda_1 \vec{x}_1 + \dots + \lambda_k \vec{x}_k$.

Take inner product with $\vec{x}_j$ $(j = 1, \dots, k)$:

$$0 = \langle \lambda_1 \vec{x}_1 + \dots + \lambda_k x_k, \vec{x}_j \rangle$$
$$= \lambda_j \langle \vec{x}_1, \vec{x}_j \rangle + \dots + \lambda_k \langle x_k, x_j \rangle = \lambda_j.$$

$\rightarrow$ All $\lambda_j = 0 \implies \vec{x}_1, \dots, \vec{x}_k$ lin. ind. $\square$

Corollary. An ONS consisting of n vectors is a basis of $\mathbb{R}^n$.

Such a basis is called an orthogonal basis (ONB).

Examples.

- The standard unit vectors $\vec{e}_1, \dots, \vec{e}_n$ are an ONB of $\mathbb{R}^n$.
- $\vec{v}_1 = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$ is an ONB of $\mathbb{R}^2$
- $\vec{w}_1 = \begin{pmatrix} \frac{1}{2}\sqrt{3} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix} = \vec{w}_2$ is an ONB of $\mathbb{R}^2$.

Theorem (Repr. of a vector with respect to an ONB)

Let $\vec{x}_1, \dots, \vec{x}_n \in \mathbb{R}^n$ ONB.

$\implies \vec{w} = \langle \vec{x}_1, \vec{w} \rangle \vec{x}_1 + \dots + \langle \vec{x}_n, \vec{w} \rangle \vec{x}_n$.

Proof. Since $\vec{x}_1, \dots, \vec{x}_n$ is a basis of $\mathbb{R}^n$, there exist $\lambda_1, \dots, \lambda_n$ st.

$$\vec{w} = \lambda_1 \vec{x}_1 + \dots + \lambda_n \vec{x}_n.$$

Multiply by $\vec{x}_j$ and use orthogonality of the $\vec{x}_j$: $\langle \vec{x}_j, \vec{w} \rangle = \lambda_j \|x_j\|^2 = \lambda_j$. $\square$

Definition. Let $U \subseteq \mathbb{R}^n$ be a subspace. Then:

$$U^\perp := \{ x \in \mathbb{R}^n \mid x \perp U \} := \{ x \in \mathbb{R}^n \mid \forall u \in U \ \vec{x} \perp \vec{u} \}$$
$$= \text{orthogonal complement of } U.$$

Easy to see: $(U^\perp)^\perp = U$.

Definition. Let V be a vector space, $U_1, U_2 \subseteq V$ subspace.

Then: $U_1 + U_2 := \{ u_1 + u_2 \mid u_1 \in U_1 \text{ and } u_2 \in U_2 \}$.
$= \text{sum of } U_1 \text{ and } U_2$.

If $U_1 \cap U_2 = \{0\}$ then we write $U_1 \oplus U_2$ and call it the direct sum of U_1 and U_2.

It is easy to see that $U_1 + U_2$ is a vector space with $\dim(U_1 + U_2) \leq \dim U_1 + \dim U_2$, with equality if and only if $U_1 \cap U_2 = \{0\}$.

Theorem. Let $U \subseteq \mathbb{R}^n$. Then:

(i) $U^\perp$ is a vector space.
(ii) $U \cap U^\perp = \{0\}$.
(iii) $U \oplus U^\perp = H$
(iv) $\dim U^\perp = n - \dim U$.

Proof.
(i) Let $\vec{x}, \vec{y} \in U^\perp, \lambda \in \mathbb{R}$.
$\Rightarrow \forall u \in U: \ \langle \vec{x}+\lambda\vec{y}, u\rangle = \overbrace{\langle \vec{x}, \vec{u}\rangle}^{=0} + \lambda \overbrace{\langle \vec{y}, \vec{u}\rangle}^{=0} = 0$
$\Rightarrow \vec{x} + \lambda \vec{y} \in U^\perp$.

(ii) Let $\vec{x} \in U \cap U^\perp$. We have to show: $\vec{x} = \vec{0}$.
Since $\vec{x} \in U \cap U^\perp \Rightarrow \vec{x} \perp \vec{x} \Rightarrow \|\vec{x}\|^2 = \langle \vec{x}, \vec{x}\rangle = 0 \Rightarrow \vec{x} = 0$ □

(iv) Let $\vec{u}_1, \dots, \vec{u}_k$ be a basis of U, and let A be the matrix whose rows consist of the $\vec{u}_1, \dots \vec{u}_k$: $A = \begin{pmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_k \end{pmatrix}$.
Then: $U = \text{gen}\{\vec{u}_1, \dots, \vec{u}_k\} = R_A$.
We already know (p. 91): $\ker A = R_A^\perp = U^\perp$
and $\dim(\ker A) = n - \dim(\operatorname{Im}(A))$
$= n - \dim(R_A)$
$= n - k$.
$\Rightarrow \dim(U^\perp) = \dim(\ker A) = n - k = n - \dim U$. □

(iii) Let $u_1, \dots, u_k$ be an ONB of U, and $w_{k+1}, \dots, w_n$ be an ONB of $U^\perp$ (here we use that $\dim U^\perp = n-k$).
Clearly, $\text{gen}\{u_1, \dots, u_k, w_{k+1}, \dots, w_n\} \subseteq U + U^\perp \subseteq \mathbb{R}^n$. (*)
We will show that $u_1, \dots, u_k, w_{k+1}, \dots, w_n$ are lin. indep.
Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ s.t. $\lambda_1 u_1 + \dots + \lambda_k u_k + \lambda_{k+1} w_{k+1} + \dots + \lambda_n w_n = 0$ (**)
Observe: $u_1, \dots, u_k$ ONB $\Rightarrow \langle u_j, u_k\rangle = 0$ for $j \neq k$
$u_j \in U \Rightarrow u_j \perp U^{\perp\perp} \Rightarrow \langle u_j, w_\ell\rangle = 0$ for $j = 1, \dots, k$ and $\ell = k+1, \dots, n$.
$w_{k+1}, \dots, w_n$ ONB $\Rightarrow \langle w_j, w_\ell\rangle = 0$ for $j \neq \ell$.

Take inner product of (**) with $u_j \Rightarrow \lambda_j = 0 \quad (j = 1, \dots, k)$
Take with $w_j \Rightarrow \lambda_j = 0 \quad (j = k+1, \dots, n)$.
$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0 \Rightarrow \vec{u}_1, \dots, \vec{u}_k, \vec{w}_{k+1}, \dots, \vec{w}_n$ are lin. ind.
$\Rightarrow \dim(\text{gen}\{\vec{u}_1, \dots, \vec{u}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}) = n$
$\overset{(*)}{\Rightarrow} \dim(U + U^\perp) \geq n = \dim(\mathbb{R}^n)$.
$\Rightarrow \dim(U + U^\perp) = n$ and $U + U^\perp = \mathbb{R}^n$. □

Theorem. $U \subseteq \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$.
$\Rightarrow \exists! \ \vec{v}_\parallel \in U, \vec{v}_\perp \in U^\perp$ such that $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$

Proof. Existence. Let $\vec{u}_1, \dots, \vec{u}_k$ basis of U, $\vec{w}_{k+1}, \dots, \vec{w}_n$ basis of $U^\perp$.
By (iii) of the previous theorem, we know that $\vec{u}_1, \dots, \vec{u}_k, \vec{w}_{k+1}, \dots, \vec{w}_n$ is a basis of $\mathbb{R}^n$
$\Rightarrow \exists \lambda_1, \dots, \lambda_k \in \mathbb{R}$ s.t. $\vec{v} = \lambda_1 \vec{u}_1 + \dots + \lambda_k \vec{u}_k + \lambda_{k+1}\vec{w}_{k+1} + \dots + \lambda_n \vec{w}_n$
Let $\vec{v}_\parallel = \lambda_1 \vec{u}_1 + \dots + \lambda_k \vec{u}_k$, $\quad \vec{v}_\perp = \lambda_{k+1}\vec{w}_{k+1} + \dots + \lambda_n \vec{w}_n$
$\Rightarrow \vec{v}_\parallel \in U$, $\vec{v}_\perp \in U^\perp$ and $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$

Uniqueness: Let $\vec{v}_\parallel, \vec{a}_\parallel \in U$, $\vec{v}_\perp, \vec{a}_\perp \in U^\perp$ with $\vec{v}_\parallel + \vec{v}_\perp = \vec{v}$ and $\vec{a}_\parallel + \vec{a}_\perp = \vec{v}$.
We have to show: $\vec{v}_\parallel = \vec{a}_\parallel$ and $\vec{v}_\perp = \vec{a}_\perp$.
We know: $\vec{v}_\parallel + \vec{v}_\perp = \vec{a}_\parallel + \vec{a}_\perp$
$\Rightarrow \underbrace{\vec{v}_\parallel - \vec{a}_\parallel}_{\in U} = \underbrace{\vec{a}_\perp - \vec{v}_\perp}_{\in U^\perp}$.
$\Rightarrow \vec{a}_\perp - \vec{v}_\perp \in U \cap U^\perp = \{0\} \Rightarrow \vec{a}_\perp - \vec{v}_\perp = 0 \Rightarrow \vec{a}_\perp = \vec{v}_\perp$
$\vec{a}_\parallel - \vec{v}_\parallel \in U \cap U^\perp = \{0\} \Rightarrow \vec{a}_\parallel - \vec{v}_\parallel = 0 \Rightarrow \vec{a}_\parallel = \vec{v}_\parallel$. □

Definition $\vec{v}_\parallel =: \text{proj}_U \vec{v}$ = projection of $\vec{v}$ onto U
$\vec{v}_\perp =: \text{proj}_{U^\perp} \vec{v}$ = projection of $\vec{v}$ onto $U^\perp$.

Observe that these definitions make sense because of the theorem above.

Clearly: $P: V \longrightarrow V$, $Pv := \vec{v}_{\parallel}$ is a linear operator.

It is called the orthogonal projection onto U.

Properties:

- $P^2 = P$
- $\ker P = U^\perp$, $\operatorname{Im} P = U$.
- Representation of U with respect to the basis $\vec{v}_1, \dots, \vec{v}_k, \vec{w}_{k+1}, \dots, \vec{w}_n$:

$$P = \begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ & 0 \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}.$$

Observations: Let $\vec{v} \in \mathbb{R}^n$.

- $U \subseteq \mathbb{R}^n$ subspace and $\vec{v}_\parallel$ and $\vec{v}_\perp$ as above.

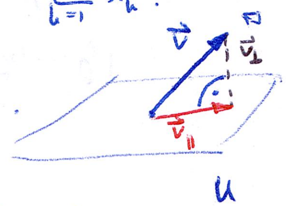
$\Rightarrow \|\vec{v}\|^2 = \|\vec{v}_\parallel\|^2 + \|\vec{v}_\perp\|^2$.

Proof. $\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle = \langle \vec{v}_\parallel + \vec{v}_\perp, \vec{v}_\parallel + \vec{v}_\perp \rangle$
$= \langle \vec{v}_\parallel, \vec{v}_\parallel \rangle + \underbrace{\langle \vec{v}_\parallel, \vec{v}_\perp \rangle}_{=0} + \underbrace{\langle \vec{v}_\perp, \vec{v}_\parallel \rangle}_{=0} + \langle \vec{v}_\perp, \vec{v}_\perp \rangle$
$= \|\vec{v}_\parallel\|^2 + \|\vec{v}_\perp\|^2$. ☐

- $\vec{u}_1, \dots, \vec{u}_n$ ONB in $\mathbb{R}^n$, $\vec{w} = \lambda_1 \vec{u}_1 + \dots + \lambda_n \vec{u}_n$

$\Rightarrow \|\vec{w}\|^2 = |\lambda_1|^2 + \dots + |\lambda_n|^2$.

Proof. $\|\vec{w}\|^2 = \langle \vec{w}, \vec{w} \rangle = \sum_{k,\ell=1}^{n} \lambda_k \lambda_\ell \overbrace{\langle \vec{u}_k, \vec{u}_\ell \rangle}^{=\delta_{k\ell}} = \sum_{k=1}^{n} \lambda_k^2$. ☐



Theorem. $U \subseteq \mathbb{R}^n$ subspace, $\vec{v} \in \mathbb{R}^n$, $\vec{v}_\parallel := \operatorname{proj}_U \vec{v}$.

$\Rightarrow \forall x \in U \quad \|\vec{v} - \operatorname{proj}_U \vec{v}\| \le \|\vec{v} - \vec{x}\|$

In words: $\operatorname{proj}_U \vec{v}$ is the vector in U which is closest to $\vec{v}$.

Proof. Let $\vec{x} \in U$.

$\Rightarrow \|\vec{v} - \vec{x}\|^2 = \|\underbrace{\vec{v} - \operatorname{proj}_U \vec{v}}_{\in U^\perp} + \operatorname{proj}_U \vec{v} - \vec{x}\|^2 = \|\vec{v} - \operatorname{proj}_U \vec{v}\|^2 + \|\operatorname{proj}_U \vec{v} - x\|^2$
$\ge \|\vec{v} - \operatorname{proj}_U \vec{v}\|^2$. ☐

For given $U \subseteq \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$, how do we calculate $\operatorname{proj}_U \vec{v}$?

$\rightsquigarrow$ Find basis $\vec{u}_1, \dots, \vec{u}_k$ of U and $\vec{w}_{k+1}, \dots, \vec{w}_n$ of $U^\perp$ and write $\vec{v}$ as $\quad \vec{v} = \lambda_1 \vec{u}_1 + \dots + \lambda_k \vec{u}_k + \lambda_{k+1} \vec{w}_{k+1} + \dots + \lambda_n \vec{w}_n$.

$\Rightarrow \operatorname{proj}_U \vec{v} = \lambda_1 \vec{u}_1 + \dots + \lambda_k \vec{u}_k$.

Problem: The calculations to obtain $\lambda_1, \dots, \lambda_k$ may be long.

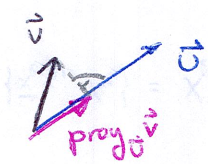
They are easy to calculate if we knew that the $\vec{u}_1, \dots, \vec{u}_k$ are an ONB of U, because in that case we simply have:

$$\lambda_j = \langle \vec{v}, \vec{u}_j \rangle. \qquad (j = 1, \dots, k).$$

Question: How to find an ONB for a given subspace U?

$\rightsquigarrow$ Find any basis $\vec{x}_1, \dots, \vec{x}_k$ of U and then apply the so-called Gram-Schmidt orthogonalization process to obtain an ONB.

Recall: $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\vec{u} \ne 0 \quad \Rightarrow \operatorname{proj}_{\vec{u}} \vec{v} = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2} \vec{u}$.



$\rightsquigarrow$ If $U = \operatorname{gen}\{\vec{u}, \vec{v}\} \subseteq \mathbb{R}^n$ with $\vec{u}, \vec{v}$ lin. ind., then we obtain an ONB as follows:

$$\vec{u}_1 = \frac{\vec{u}}{\|\vec{u}\|}$$

$$\vec{v}_2 = \vec{v} - \operatorname{proj}_{\vec{u}} \vec{v} \ne 0, \qquad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}.$$

Then $\{\vec{u}_1, \vec{u}_2\}$ is an ONB of U.

[Clearly, $\|\vec{u}_1\| = \|\vec{u}_2\| = 1$ by construction and

$$\langle \vec{u}_1, \vec{u}_2 \rangle = \frac{1}{\|\vec{u}\|} \frac{1}{\|\vec{v}_2\|} \langle \vec{u}, \underbrace{\vec{v} - \operatorname{proj}_{\vec{u}} \vec{v}}_{\perp \vec{u}} \rangle = 0.$$

Now for higher dimensional U:

Theorem. (Gram - Schmidt process)

Let $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$ linearly independent.

$\Rightarrow$ There exists an ONS $\vec{u}_1, \dots, \vec{u}_k$ s.t.

$$\forall j = 1, \dots, k \qquad \operatorname{span}\{\vec{x}_1, \dots, \vec{x}_j\} = \operatorname{span}\{\vec{u}_1, \dots, \vec{u}_j\}$$

In particular: $\{\vec{u}_1, \dots, \vec{u}_k\}$ is an ONB for $\operatorname{span}\{\vec{x}_1, \dots, \vec{x}_k\}$.

Proof. For $j = 1, \dots, k$ set $U_j = \operatorname{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_j\}$, $X_j := \operatorname{span}\{\vec{x}_1, \dots, \vec{x}_j\}$.

- $\vec{v}_1 = \vec{x}_1$, $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$. Clearly $\operatorname{span}\{\vec{x}_1\} = \operatorname{span}\{\vec{u}_1\}$.

- Obtain $\vec{v}_2 \in X_2$, orthogonal to U_1: $\vec{v}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{u}_1} \vec{x}_2 \neq 0$

  Normalize: $\vec{u}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$

  $*$ $\vec{u}_2 = \vec{v}_2 - \operatorname{proj}_{\vec{u}_1} \vec{v}_2 \in \operatorname{span}\{\vec{x}_2, \vec{x}_1\} \Rightarrow \operatorname{span}\{\vec{u}_1, \vec{u}_2\} \subseteq \operatorname{span}\{\vec{x}_1, \vec{x}_2\}$

  $*$ $\vec{x}_2 = \vec{v}_2 + \operatorname{proj}_{\vec{u}_1} \vec{v}_2 \in \operatorname{span}\{\vec{u}_1, \vec{u}_2\} \Rightarrow \operatorname{span}\{\vec{x}_1, \vec{x}_2\} \subseteq \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$

  $*$ $\langle \vec{u}_1, \vec{v}_2 \rangle = \langle \vec{u}_1, \vec{x}_2 - \operatorname{proj}_{\vec{u}_1} \vec{x}_2 \rangle = \langle \vec{u}_1, \vec{x}_2 \rangle - \langle \vec{u}_1, \operatorname{proj}_{\vec{u}_1} \vec{x}_2 \rangle$

  $$= \langle \vec{u}_1, \vec{x}_2 \rangle - \frac{\langle u_1, \vec{x}_2 \rangle}{\|\vec{u}_1\|^2} \langle \vec{x}_1, \vec{x}_1 \rangle = 0.$$

- Obtain $\vec{v}_3 \in X_3$, orthogonal to U_2:

  $$\vec{v}_3 = \vec{x}_3 - \operatorname{proj}_{U_2} \vec{x}_3 = \vec{x}_3 - (\operatorname{proj}_{\vec{u}_1} \vec{x}_3 + \operatorname{proj}_{\vec{u}_2} \vec{x}_3)$$

  Normalize: $\vec{u}_3 = \dfrac{\vec{v}_3}{\|\vec{v}_3\|}$.

  $*$ $\vec{v}_3 = \vec{x}_3 - \underbrace{\operatorname{proj}_{U_2} \vec{x}_3}_{\subseteq U_2} \in \operatorname{span}\{\vec{x}_3, U_2\} = \operatorname{span}\{\vec{x}_3, X_2\} = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = X_3$

  $*$ $\vec{x}_3 = \vec{v}_3 + \operatorname{proj}_{U_2} \vec{x}_3 \in \operatorname{span}\{v_3, U_2\} = \operatorname{span}\{\vec{u}_3, \vec{u}_2, \vec{u}_1\} = U_3$.

  $\rightarrow X_3 = U_3$.

  $*$ $\vec{v}_3 = \vec{x}_3 - \operatorname{proj}_{U_2} \vec{x}_3 = \vec{x}_3 - \langle \vec{x}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{x}_3, \vec{u}_2 \rangle \vec{u}_2$

  $$\Rightarrow \langle \vec{v}_3, \vec{u}_1 \rangle = \langle \vec{x}_3, \vec{u}_1 \rangle - \langle \vec{x}_3, \vec{u}_1 \rangle \underbrace{\langle \vec{u}_1, \vec{u}_1 \rangle}_{=1} - \langle \vec{x}_3, \vec{u}_2 \rangle \underbrace{\langle \vec{u}_2, \vec{u}_1 \rangle}_{=0}$$

  $$= \langle \vec{x}_3, \vec{u}_1 \rangle - \langle x_3, \vec{u}_1 \rangle = 0.$$

  Similar: $\langle \vec{v}_3, \vec{u}_2 \rangle = 0$.

$\Rightarrow U_3 = X_3$ and $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an ONB of X_3.

$*$ Now assume we already have $U_\ell = X_\ell$ and $\{\vec{u}_1, \dots, \vec{u}_\ell\}$ ONB of X_ℓ for some $\ell < k$.

Obtain $\vec{v}_{\ell+1} \in X_{\ell+1}$, orthogonal to U_ℓ:

$$\vec{v}_{\ell+1} = \vec{x}_{\ell+1} - \operatorname{proj}_{U_\ell} \vec{x}_{\ell+1} = \vec{x}_{\ell+1} - (\operatorname{proj}_{\vec{u}_1} \vec{x}_{\ell+1} + \dots + \operatorname{proj}_{\vec{u}_\ell} \vec{x}_{\ell+1}) \neq 0$$

Normalize: $\vec{u}_{\ell+1} = \dfrac{\vec{v}_{\ell+1}}{\|\vec{v}_{\ell+1}\|}$

$*$ $\vec{v}_{\ell+1} = \vec{x}_{\ell+1} - \underbrace{\operatorname{proj}_{U_\ell} \vec{v}_{\ell+1}}_{\in U_\ell} \in \operatorname{span}\{\vec{x}_{\ell+1}, U_\ell\} = \operatorname{span}\{\vec{x}_{\ell+1}, X_\ell\} = X_{\ell+1}$

$\Rightarrow U_{\ell+1} \subseteq X_{\ell+1}$

$*$ $\vec{x}_{\ell+1} = \vec{v}_{\ell+1} + \operatorname{proj}_{U_\ell} \vec{v}_{\ell+1} \in \operatorname{span}\{\vec{v}_{\ell+1}, U_\ell\} = \operatorname{span}\{\vec{u}_{\ell+1}, U_\ell\} = U_{\ell+1}$.

$\Rightarrow X_{\ell+1} \subseteq U_{\ell+1}$.

$\rightarrow U_{\ell+1} = X_{\ell+1}$.

$*$ $\vec{v}_{\ell+1} = \vec{x}_{\ell+1} - \operatorname{proj}_{U_\ell} \vec{x}_{\ell+1} = \vec{x}_{\ell+1} - (\langle \vec{x}_{\ell+1}, \vec{u}_1 \rangle \vec{u}_1 + \dots + \langle \vec{x}_{\ell+1}, \vec{u}_\ell \rangle \vec{u}_\ell)$

$\Rightarrow \forall j = 1, \dots, \ell$:

$$\langle \vec{v}_{\ell+1}, \vec{u}_j \rangle = \langle \vec{x}_{\ell+1}, \vec{u}_j \rangle - (\langle \vec{x}_{\ell+1}, \vec{u}_1 \rangle \langle \vec{u}_1, \vec{u}_j \rangle + \dots + \langle \vec{x}_{\ell+1}, \vec{u}_\ell \rangle \langle \vec{u}_\ell, \vec{u}_j \rangle)$$

$$\xrightarrow{u_j \perp u_m \text{ for } j \neq m} = \langle \vec{x}_{\ell+1}, \vec{u}_j \rangle - \langle \vec{x}_{\ell+1}, \vec{u}_j \rangle \underbrace{\langle \vec{u}_j, \vec{u}_j \rangle}_{=1}$$

$$= 0.$$

$\Rightarrow \{u_1, \dots, u_{\ell+1}\}$ is an ONB of $U_{\ell+1}$.

$\square$

Example. Let $\vec{x}_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 3 \\ -4 \\ -1 \\ -2 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}$, $U = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Use Gram-Schmidt to find an ONB of U.

- $\vec{v}_1 = \vec{x}_1$, $\vec{u}_1 = \dfrac{\vec{x}_1}{\|\vec{x}_1\|} = \dfrac{1}{\sqrt{12}}\vec{x}_1 = \dfrac{1}{2\sqrt{3}}\vec{x}_1$.
- $\vec{v}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{u}_1}\vec{x}_2 = \vec{x}_2 - \langle \vec{x}_2, \vec{u}_1\rangle \vec{u}_1 = \vec{x}_2 - \sqrt{12}^{-2}\langle \vec{x}_2, \vec{x}_1\rangle \vec{x}_1$

$$= \begin{pmatrix} 3 \\ -4 \\ -1 \\ -2 \end{pmatrix} - \frac{1}{12}\underbrace{(-3-12-1-2)}_{=-18}x_1 = \begin{pmatrix} 3 \\ -4 \\ -1 \\ -2 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

$$\Rightarrow \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{12}}\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

- $\vec{v}_3 = \vec{x}_3 - \operatorname{proj}_{\vec{u}_1}\vec{x}_3 - \operatorname{proj}_{\vec{u}_2}\vec{x}_3 = \vec{x}_3 - \langle \vec{x}_3, \vec{u}_1\rangle \vec{u}_1 - \langle \vec{x}_3, \vec{u}_2\rangle \vec{u}_2$

$$= \vec{x}_3 - \frac{1}{12}\overbrace{(-2+3+2-1)}^{=2}\vec{x}_1 - \frac{1}{12}\overbrace{(6+1+2+1)}^{=10}\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{6}\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -2 \\ -2 \\ 6 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix}$$

$$\Rightarrow \vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{12}}\begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix}.$$

$$\rightarrow \vec{u}_1 = \frac{1}{\sqrt{12}}\begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix},\quad \vec{u}_2 = \frac{1}{\sqrt{12}}\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix},\quad \vec{u}_3 = \frac{1}{\sqrt{12}}\begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix}.$$

Change of base between two ONB:

Let $v_1, \dots, v_n$ and $w_1, \dots, w_n$ be ONB of $\mathbb{R}^n$.

Let Q ~~matrix~~ transition matrix from B_v to B_w.

$$\Rightarrow Q = \left(\langle \vec{w}_j, \vec{v}_k\rangle\right)_{j,k=1}^{n} \qquad (1)$$

Transition matrix from B_w to B_v:

$$Q^{-1} = \left(\langle \vec{v}_j, \vec{w}_k\rangle\right)_{j,k=1}^{n} \qquad (2)$$

Compare (1) & (2) $\Rightarrow$ $\boxed{Q^{-1} = Q^t}$.

Definition. A matrix $Q \in M(n \times n)$ is called an *orthogonal matrix* if $Q^{-1} = Q^t$.

- The calculations above show: transition matrix from one ONB to another is always an orthogonal matrix.
- Given an orthogonal matrix Q; then its rows (and its columns) form an ONB of $\mathbb{R}^n$.