4.8. Representation of linear maps as matrices.

Recall.

- U, V vector spaces, $T: U \to V$ linear ; $u_1, \dots, u_n$ basis of U.
  $\rightsquigarrow$ T is completely determined by its values on $u_1, \dots, u_n$.
  That means: If we know $Tu_1, \dots, Tu_n$, then we know Tu for any $u \in U$.
- $A \in M(m \times n)$ $\rightsquigarrow$ A is determined by $A\vec{e}_j$ $\quad \left(\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\right)$
  because: $A\vec{e}_j = j$-th column of A.
  Recall: $\vec{e}_1, \dots, \vec{e}_n$ is a basis of $\mathbb{R}^n$.

Theorem. Let U, V be finite dimensional vector spaces and choose bases $B_U := \{u_1, \dots, u_n\}$ of U and $B_V := \{v_1, \dots, v_m\}$ of V.

Then

(i) Every $A \in M(m \times n)$ determines a unique linear map $T_A: U \to V$ by $T_A u_j = \sum_{k=1}^{m} a_{kj} v_k \quad (j = 1, \dots, n)$.

(ii) Every $T: U \to V$ has a unique matrix representation $A_T \in M(m \times n)$ such that $\quad A_T (u)_{B_U} = (Tu)_{B_V}$. $\quad$ (*)

Proof. (i) Clear.
(ii) Existence:
Define $A_T := (a_{jk})_{\substack{k=1,\dots,n \\ j=1,\dots,m}} \in M(m \times n)$ by $Tu_k = a_{1k} v_1 + \dots + a_{mk} v_m$

Then clearly: $A_T (u_k)_{B_U} = A_T \vec{e}_k = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix} = (Tu_k)_{B_V} \quad (k = 1, \dots, n)$

By linearity then: $A_T (u)_{B_U} = (Tu)_{B_V}$.

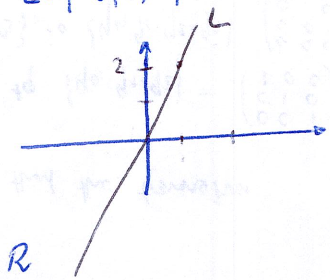
Uniqueness: This follows from the "Recall" on p. 105:
Let $A_T, \tilde{A}_T \in M(m \times n)$ s.t. (*) holds.
Then, for every $k = 1, \dots, n$:

$$k\text{-th column of } A_T = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix} = A_T (u_k)_{B_U} = (Tu)_{B_V} = \tilde{A}_T (u_k)_{B_U} = \begin{pmatrix} \tilde{a}_{1k} \\ \vdots \\ \tilde{a}_{mk} \end{pmatrix} = k\text{-th column of } \tilde{A}_T.$$

$\Rightarrow A_T = \tilde{A}_T$. $\quad \square$

Observation. It is easy to see that $T_{A_T} = T$ and $A_{T_A} = A$.

Examples: Reflection in $\mathbb{R}^2$ on the line $L: \{ t\binom{1}{2} \mid t \in \mathbb{R} \}$.



Clearly, "reflection" is linear.
Denote it by R.

Find matrix representation of R.

Since R is linear, it suffices to know R on a basis of $\mathbb{R}^2$.
Take, e.g., $\vec{v}_1 = \binom{1}{2}$, $\vec{v}_2 = \binom{2}{-1}$.
Clearly, $\vec{v}_1, \vec{v}_2$ are a basis of $\mathbb{R}^2$ and $\vec{v}_1 \parallel L$, $\vec{v}_2 \perp L$

$\Rightarrow R\vec{v}_1 = \vec{v}_1, \quad R\vec{v}_2 = -\vec{v}_2$. $\quad$ (1)

Solution 1. Let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
(1) gives: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{1}{2} = \binom{1}{2}$, $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{2}{-1} = \binom{-2}{1}$.

$\Rightarrow$ System of equations:

$$\begin{aligned} a + 2b &= 1 \\ c + 2d &= 2 \\ 2a - b &= -2 \\ 2c - d &= 1 \end{aligned}$$

$$\Rightarrow \begin{cases} a + 2b = 1 & c + 2d = 2 \\ 2a - b = -2 & 2c - d = 1. \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} c \\ d \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\Rightarrow R = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

Solution 2. Let R_V = matrix repr. of R wrt. basis $\{\vec{v}_1, \vec{v}_2\}$.

$$\Rightarrow R_V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now: Change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to standard basis:

$$R = (\vec{v}_1 | \vec{v}_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\vec{v}_1, \vec{v}_2)^{-1}$$

$$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$

$\square$

In general U, V vector spaces with bases B_U, B_U' of U and B_V, B_V' of V. Let $T: U \to V$ linear and A_T be the matrix repr. of T with respect to B_U and B_V.

Then: A_T' = matrix repr. of T wrt. B_U' and B_V' is

$$\boxed{A_T' = C_V A_T C_U^{-1}.}$$

where C_U = transition matrix from B_U to B_U'

C_V = " B_V to B_V'.

Example: $T: P_2 \to P_3$, $(Tp)(x) := \int_0^x p(s)\,ds$.

- Representation of T wrt. $p_0 = 1$, $p_1 = X$, $p_2 = X^2$, $p_3 = X^3$.

Observe: $Tp_0 = X = p_1$, $Tp_1 = \frac{1}{2}X^2 = \frac{1}{2}p_2$

$Tp_2 = \frac{1}{3}X^3 = \frac{1}{3}p_3$.

$$\Rightarrow A_T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

- Repr. of T wrt. $q_0 = X^3$, $q_1 = X^2$, $q_2 = X$, $q_3 = 1$

Solution 1: As above; a direct calculation shows:

$$A_T' = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution 2: Use A_T and observe that the transition matrices are:

C_2 = trans. matrix from $\{p_0, p_1, p_2\}$ to $\{q_0, q_1, q_2\}$ = $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

C_3 = " $\{p_0, p_1, p_2, p_3\}$ to $\{q_0, q_1, q_2, q_3\}$ = $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Note: $C_3^{-1} = C_3$.

$$\Rightarrow A_T' = C_4 A_T C_3^{-1} = C_4 A_T C_3 = \ldots = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\square$