4.7. Change of basis.

Example. Let $\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$.

Clearly, $\vec{v}_1, \vec{v}_2$ are lin. indep; hence they form ~~the~~ a basis of $\mathbb{R}^2$. Let $B := \{\vec{v}_1, \vec{v}_2\}$.

$\Rightarrow$ Every $\vec{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ can be written in a unique way in the form $\vec{x} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2$

$\rightsquigarrow$ If we know λ_1, λ_2, we know $\vec{x}$; and: given $\vec{x}$ we can find λ_1, λ_2 s.t. $\vec{x} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2$.

$\rightsquigarrow$ we can write $\boxed{\vec{x} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}_B}$ where the index "B" indicates that the components of the vector $\begin{pmatrix}a\\b\end{pmatrix}_B$ are with respect to the basis B.

If there is no index, then we always take the components with respect to the standard basis in $\mathbb{R}^n$.

With $\vec{v}_1, \vec{v}_2$ as above, we have for example:

① • $\vec{x} = \begin{pmatrix}3\\-2\end{pmatrix}_B = 3\vec{v}_1 + (-2)\vec{v}_2 = \begin{pmatrix}3\\3\end{pmatrix} + \begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$.

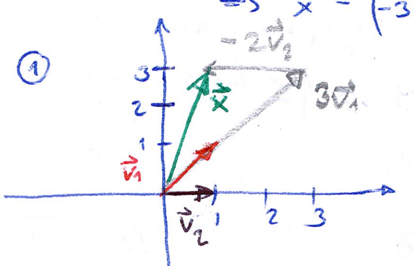
② • Write $\vec{x} = \begin{pmatrix}-1\\2\end{pmatrix}$ in terms of $\vec{v}_1, \vec{v}_2$:

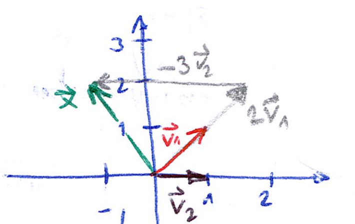
$$\vec{x} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 = (\vec{v}_1 | \vec{v}_2)\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1&1\\1&0\end{pmatrix}^{-1}\vec{x} = \begin{pmatrix}0&1\\1&-1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$$

$$\Rightarrow \vec{x} = \begin{pmatrix}2\\-3\end{pmatrix}_B.$$

①





$\rightarrow$ In the example with the mentos y chicles, the natural basis in the mentos-chicle-space is ...

$\rightsquigarrow$ • The matrix $A := (\vec{v}_1 | \vec{v}_2)$ describes the transition from the basis $\vec{v}_1, \vec{v}_2$ to the standard basis.

• A^{-1} describes the transition from the basis $\vec{e}_1, \vec{e}_2$ (standard basis of $\mathbb{R}^2$) to the basis $\vec{v}_1, \vec{v}_2$.

In general: U vector space and $B = \{u_1, \dots, u_n\}$ basis in U.

$\rightsquigarrow$ $\forall x \in U$ exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ s.t. $x = \lambda_1 u_1 + \dots + \lambda_n u_n =: \begin{pmatrix}\lambda_1\\\vdots\\\lambda_n\end{pmatrix}$

> **This means:** Given a finite dimensional vector space U and a basis (ordered) $u_1, \dots, u_n \in U$, we can always identify U with $\mathbb{R}^n$.

Examples $U = P_2$.

① $B = \{1, x, x^2\}$ clearly is a basis of P_2.

$\rightsquigarrow$ We can identify a polynomial $p = ax^2 + bx + c \in P_2$ with the vector $\begin{pmatrix}c\\b\\a\end{pmatrix} \in \mathbb{R}^3$.

② $B = \{x^2, x, 1\}$ clearly is a basis of P_2

$\rightsquigarrow$ We can identify a polynomial $p = ax^2 + bx + c \in P_2$ with the vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$

③ $B = \{\underbrace{x^2 + x}_{p_1}, \underbrace{x+1}_{p_2}, \underbrace{1}_{p_3}\}$ clearly is a basis of P_2.

Given $p = ax^2 + bx + c$. A short calculation shows:

$$p = a p_1 + (b-a) p_2 + (c-b-a) p_1$$

$\Rightarrow p$ can be identified with $\begin{pmatrix}a\\b-a\\c-b-a\end{pmatrix}$

Notation: $p = \begin{pmatrix}a\\b-a\\c-b-a\end{pmatrix}_B$

<u>Now</u> Change of basis in $\mathbb{R}^n$:

Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ basis of $\mathbb{R}^n$.

Then: Every $\vec{x} \in \mathbb{R}^n$ has a unique representation as
$\vec{x} = \lambda_1 \vec{v}_1 + \ldots + \lambda_n \vec{v}_n$, (*) $\rightsquigarrow$ $\vec{x} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}_B$
and each tupel $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ describes exactly one vector $\vec{x}$ by (*).

In components:

$$(*) \iff \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{(\vec{v}_1 | \ldots | \vec{v}_n)}_{=: A} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

<u>Observe</u>. A is invertible because the $\vec{v}_1, \ldots, \vec{v}_n$ are lin. independent

$$\Rightarrow \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = A^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

$\rightsquigarrow$ A is the <u>transition matrix</u> from the basis B to the standard basis
A^{-1} is the <u>transition matrix</u> from the standard basis to the basis B

<u>Now</u>: Given $B_1 = \{\vec{v}_1, \ldots, \vec{v}_n\}$, $B_2 = \{\vec{w}_1, \ldots, \vec{w}_n\}$ basis in $\mathbb{R}^n$.
How do we get a transition matrix from the basis B_1 to the basis B_2?

<u>Solution</u>. Go from basis B_1 to the standard basis and then to the basis B_2.

<u>That is</u>: Let $A_1 = (\vec{v}_1 | \ldots | \vec{v}_n)$, $A_2 = (\vec{w}_1 | \ldots | \vec{w}_n)$.
Given $\vec{x} \in \mathbb{R}^n$, assume that $\vec{x} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}_{B_1}$.

$$\Rightarrow \quad A_1 \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \text{repr. of } \vec{x} \text{ in the standard basis}$$

$$\Rightarrow \quad A_2^{-1} A_1 \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}_{B_2} = \text{repr. of } \vec{x} \text{ in the basis } B_2.$$

$\Rightarrow$ $\boxed{A_2^{-1} A_1 \text{ is the transition matrix from basis } B_1 \text{ to basis } B_2.}$

<u>Alternative Solution</u> (without using the standard basis).

Since the $\vec{w}_1, \ldots, \vec{w}_n$ are a basis, we know that there exist a_{ij} s.t.

$$\begin{cases} \vec{v}_1 = a_{11} \vec{w}_1 + \ldots + a_{n1} \vec{w}_n \\ \vdots \\ \vec{v}_n = a_{n1} \vec{w}_1 + \ldots + a_{nn} \vec{w}_n \end{cases}$$

$$\rightsquigarrow \vec{v}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}_{B_2}, \ldots, \vec{v}_n = \begin{pmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix}_{B_2}.$$

Now let $\vec{x} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}_{B_1}$

$$\Rightarrow \vec{x} = \alpha_1 \vec{v}_1 + \ldots + \alpha_n \vec{v}_n = \alpha_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}_{B_2} + \alpha_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}_{B_2} + \ldots + \alpha_n \begin{pmatrix} a_{nn} \\ \vdots \\ a_{nn} \end{pmatrix}_{B_2}$$

$$= \left(\underbrace{\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}}_{=: A} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \right)_{B_2}$$

$\rightsquigarrow$ A is the transition matrix from basis B_1 to basis B_2. □

<u>Theorem</u> - Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$. Let $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^n$, given in terms of B. Let $A = ((\vec{x}_1)_v | \ldots | (\vec{x}_n)_v)$.

Then: A is invertible $\underset{\text{clear}}{\iff}$ $\det A \neq 0 \iff \vec{x}_1, \ldots, \vec{x}_n$ are lin. independent.

<u>Proof</u>. Let $M = (\vec{v}_1 | \ldots | \vec{v}_n) \rightsquigarrow (\vec{x}_1)_v = M^{-1}\vec{x}_1, \ldots, (\vec{x}_n)_v = M^{-1}\vec{x}_n$

$$\Rightarrow A = ((\vec{x}_1)_v | \ldots | (\vec{x}_n)_v) = (M^{-1}\vec{x}_1 | \ldots | M^{-1}\vec{x}_n)$$
$$= M^{-1} (\vec{x}_1 | \ldots | \vec{x}_n)$$

$$\Rightarrow \det A = \det(M^{-1}(\vec{x}_1 | \ldots | \vec{x}_n)) = \underbrace{\det M^{-1}}_{\neq 0} \det(\vec{x}_1 | \ldots | \vec{x}_n).$$

$$\rightsquigarrow \left(\det A \neq 0 \iff \det(\vec{x}_1 | \ldots | \vec{x}_n) \neq 0 \iff \vec{x}_1, \ldots, \vec{x}_n \text{ lin. indep.} \right).$$

□

<u>Remark</u> For $\{\vec{v}_1, \ldots, \vec{v}_n\} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ this is nothing else than "$\det A \neq 0 \iff$ its columns are lin. ind.".