Appendix Proof of "$\dim(\ker A) + \dim(\operatorname{Im} A) = n$" without elementary matrices.

Theorem. Let U, V be vector spaces. $A: U \to V$ linear.
Assume $\dim U =: n < \infty$.
Then: $\dim(\ker A) + \dim(\operatorname{Im}(A)) = n$.

Proof. Let $k = \dim(\ker(A))$ and $u_1, \dots, u_k$ basis of $\ker(A)$.
Complete it to a basis $u_1, \dots, u_k, w_{k+1}, \dots, w_n$ of U and set $W := \operatorname{gen}\{w_{k+1}, \dots, w_n\}$.
Let $\tilde{A} := A|_W$. Then $\tilde{A}$ is injective, hence $\tilde{A}w_{k+1}, \dots, \tilde{A}w_n$ are lin. independent. They also span $\operatorname{Im}(\tilde{A})$.
$\to$ $\operatorname{Im}(\tilde{A}) = \operatorname{gen}\{\tilde{A}w_{k+1}, \dots, \tilde{A}w_n\}$ and $\dim(\operatorname{Im}\tilde{A}) = n-k$.

Now we will show: $\operatorname{Im}(A) = \operatorname{Im}(\tilde{A})$.
"$\supseteq$" is clear (because $\tilde{A}$ is a restriction of A)
"$\subseteq$": Let $y \in \operatorname{Im}(A) \Rightarrow \exists x \in U$ s.t. $y = Ax$.
Write x as lin. comb. of the basis $u_1, \dots, u_k, w_{k+1}, \dots, w_n$:
$$x = \lambda_1 u_1 + \dots + \lambda_k u_k + \mu_{k+1} w_{k+1} + \dots + \mu_n w_n.$$
$$\Rightarrow y = Ax = A(\lambda_1 u_1 + \dots + \lambda_k u_k + \mu_{k+1} w_{k+1} + \dots + \mu_n w_n)$$
$$= \lambda_1 \underbrace{Au_1}_{=0} + \dots + \lambda_k \underbrace{Au_k}_{=0} + \mu_{k+1} Aw_{k+1} + \dots + \mu_n Aw_n$$
$$= \mu_{k+1} Aw_{k+1} + \dots + \mu_n Aw_n$$
$$\in \operatorname{gen}\{Aw_{k+1}, \dots, Aw_n\} = \operatorname{Im}(\tilde{A}).$$
$$\Rightarrow \dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(\tilde{A})) = n-k = n - \dim(\ker(A)).$$
$\blacksquare$

As a corollary, we obtain the thm on p. 90:

Thm. $A \in M(m \times n) \Rightarrow \dim C_A = \dim R_A$.

Proof. We know: $C_A = \operatorname{Im} A$, $R_A = C_{A^t} = \operatorname{Im}(A^t)$
and $\ker A = (R_A)^\perp = (\operatorname{Im}(A^t))^\perp$

We will use: $U \subseteq V$ subspace $\Rightarrow \dim V = \dim U + \dim U^\perp$
(will be shown later)

$$\Rightarrow \dim(\ker A) = \dim(R_A)^\perp = n - \dim R_A$$
$$= \underbrace{\dim(\ker(A)) + \dim(\operatorname{Im}(A))}_{=n} - \dim R_A$$

$$\Rightarrow \dim R_A = \dim(\operatorname{Im} A) = \dim C_A.$$
$\square$