4.6. Range and kernel of a matrix.

Let $A \in M(m \times n) \rightsquigarrow A$ is a linear map $\mathbb{R}^n \to \mathbb{R}^m$.

$\Rightarrow \ker A,\ \nu(A),\ \operatorname{Im}(A),\ \rho(A)$ are defined.

Observation.

- A injective $\iff \ker(A) = \{0\} \iff A\vec{x} = \vec{0}$ has only the trivial solution.
- A surjective $\iff \operatorname{Im}(A) = \mathbb{R}^m \iff \forall \vec{b} \in \mathbb{R}^m \quad A\vec{x} = \vec{b}$ has at least one solution.

Definition. Let $A \in M(m \times n)$ and let $\vec{c}_1, \dots, \vec{c}_n$ be the columns of A and $\vec{r}_1, \dots, \vec{r}_m$ the rows of A.

Then: $R_A := \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_m\} =:$ row space of A = espacio de renglones de A.

$C_A := \operatorname{gen}\{\vec{c}_1, \dots, \vec{c}_n\} =:$ column space of A = espacio de columnas de A.

Proposition. $C_A = \operatorname{Im}(A)$, $R_A = \operatorname{Im}(A^t)$.

Proof. Let $y \in \mathbb{R}^m$. Then:

$$\vec{y} \in \operatorname{Im}(A) \iff \exists \vec{x} \in \mathbb{R}^n \text{ s.t. } \vec{y} = A\vec{x} = (\vec{c}_1 | \dots | \vec{c}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{c}_1 + \dots + x_n\vec{c}_n$$

$$\iff \vec{y} \in \operatorname{gen}\{\vec{c}_1, \dots, \vec{c}_n\} = C_A.$$

This also shows $R_A = \operatorname{Im}(A^t)$ because clearly $R_A = C_{A^t}$. ☐

Proposition. Let $A \in M(m \times n)$, $E \in M(n \times n)$, $F \in M(m \times m)$. Assume that E, F are invertible. Then:

(i) $C_A = C_{AE}$

(ii) $R_A = R_{FA}$.

Proof. Since E, F are invertible, they are products of elementary matrices, so it suffices to show the claim for elementary matrices (because if we know that $C_A = C_{AE_j}$ for an elementary matrix E_j, then:

$C_{AE} = C_{AE_1 \cdots E_k} = C_{AE_1 \cdots E_{k-1}} = \dots = C_{AE_1} = C_A$. if $E = E_1 \cdots E_k$.)

$R_{FA} = R_{F_1 \cdots F_k A} = R_{F_2 \cdots F_k A} = \dots = C_{F_k A} = C_A$ if $F = F_1 \cdots F_k$)

(ii) • If $F = P_{ij}$ then: $R_A = \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_m\} = R_{FA}$ (F changes $\vec{r}_i$ and $\vec{r}_j$)

• If $F = S_j(c)$, $c \neq 0$, then: $R_{FA} = \operatorname{gen}\{\vec{r}_1, \dots, c\vec{r}_j, \dots, \vec{r}_m\}$
$= \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_j, \dots, \vec{r}_m\} = R_A$.

• If $F = Q_{ij}(c)$, then: $R_{FA} = \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_i + c\vec{r}_j, \dots, \vec{r}_j, \dots, \vec{r}_m\}$

$\vec{r}_i = (r_i + cr_j) - c\vec{r}_j \in R_{FA}$ $\longrightarrow = \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_i, \vec{r}_i + c\vec{r}_j, \dots, \vec{r}_j, \dots, \vec{r}_m\}$

$= \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_i, \cancel{\vec{r}_i + c\vec{r}_j}, \dots, \vec{r}_j, \dots, \vec{r}_m\}$

$\vec{r}_i + c\vec{r}_j$ can be deleted because it is lin. comb. of $\vec{r}_i, \vec{r}_j \in \operatorname{gen}\{\dots\}$ $\longrightarrow = \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_i, \dots, \vec{r}_j, \dots, \vec{r}_m\}$

$= R_A$.

(i) Observe: E invertible $\Rightarrow E^t$ invertible.

$$\Rightarrow C_{AE} = R_{(AE)^t} = R_{E^t A^t} \overset{(ii)}{=} R_{A^t} = C_A.$$ ☐

Note. Multiplication of A from the right by an elementary matrix operates on the columns of A.

Important consequences of the proposition and the theorem on p. 83:

Theorem. $A, B \in M(m\times n)$

$\Rightarrow$ i) A, B are row-equivalent $\Rightarrow$ $\underset{\dim(\operatorname{Im}(A))}{\underbrace{\rho(A)}} = \rho(B)$, $\underset{\dim(\ker A)}{\underbrace{\nu(A)}} = \nu(B)$, $R_A = R_B$

ii) A, B are column-equivalent $\Rightarrow$ $\rho(A) = \rho(B)$, $\nu(A) = \nu(B)$, $C_A = C_B$.

Proof. (i) A, B row-equivalent $\Rightarrow$ $\exists\, F_1, \dots, F_k$ elementary matrices such that $A = F_1 \cdot \ldots \cdot F_k B$.

Since all F_j are invertible, it follows:

- $\rho(A) = \rho(F_1 \cdot \ldots \cdot F_k B) = \rho(B)$ (Thm on p 83, (iv))
- $\nu(A) = \nu(F_1 \cdot \ldots \cdot F_k B) = \nu(B)$ (Thm on p. 83, (iii))
- $R_A = R_{F_1 \cdot \ldots \cdot F_k B} = R_B$ (Prop on p. 86, (ii)).

(ii) A, B column equivalent $\Rightarrow$ $\exists\, E_1, \dots, E_\ell$ elementary matrices such that $A = B E_1 \cdot \ldots \cdot E_\ell$.

The claims follow as in (i), using (i) and (ii) of the Thm on p. 83, and (i) of the Prop. on page 86. $\square$

Theorem. Let $A \in M(m\times n)$. Then there exist elementary matrices $E_1, \dots, E_k$, $F_1, \dots, F_\ell$ s.t.

$$F_1 \cdot \ldots \cdot F_\ell\, A\, E_1 \cdot \ldots \cdot E_k = \tilde{\tilde{A}} = \begin{array}{c} \\ r\{ \\ m-r\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{n-r} \end{array} \\ \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \end{array} \qquad (*)$$

Proof. Let $\tilde{A}$ be the reduced row-echelon form of A and $F_1, \dots, F_\ell$ elementary matrices s.t. $F_1 \cdot \ldots \cdot F_\ell A = \tilde{A}$.

$\tilde{A}$ is of the form $$\tilde{A} = \begin{pmatrix} 1 & * & * & \cdots & 0 & * & * & & 0 \\ & & & & 1 & * & x & \cdots & 0 \\ & & & & & & & & 1 \\ & 0 & & & & & & & \end{pmatrix}$$

Now clearly we can choose column operations s.t. the first rows becomes $(1\ 0 \cdots 0)$

$$\tilde{A} E_1 \cdot \ldots \cdot E_{k_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & & 0 \\ & & & 1 & * & * \cdots & 0 \\ & & & & & & 1 \\ & & & & & & & \ddots \end{pmatrix}$$

We repeat this for all non-zero rows.

$$\rightsquigarrow \tilde{A} E_1 \cdot \ldots \cdot E_{k_r} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \\ & & & 1 & \cdots & 0 \\ & & & & & 1 \\ & & & & & & \ddots \end{pmatrix}$$

Finally we reorder the columns:

$$\tilde{A} E_1 \cdot \ldots \cdot E_k = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right).$$

$\square$

Lemma. Let $B \in M(m\times n)$ be of the form

$$B = \begin{array}{c} r\{ \\ m-r\{ \end{array}\left(\begin{array}{c|c} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{r} & \overbrace{0}^{n-r} \\ \hline 0 & 0 \end{array}\right)$$

$\Rightarrow$ $\nu(B) = m-r = \#$ zero rows in B (1)

$\varrho(B) = r = \#$ pivots in B. (2)

$\dim(R_B) = \dim C_B$ (3)

Proof. (1) Clearly, $\ker B = \left\{\begin{pmatrix}0\\ \vdots \\ 0 \\ x_{r+1} \\ \vdots \\ x_n\end{pmatrix} \;\middle|\; x_{r+1}, \dots, x_n \in \mathbb{R}\right\}$

$\Rightarrow \nu(B) = \dim(\ker(B)) = n-r$

(2) Clearly, $\operatorname{Im}(B) = C_B = \left\{\begin{pmatrix}x_1\\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0\end{pmatrix} \;\middle|\; x_1, \dots, x_r \in \mathbb{R}\right\}$

$\Rightarrow \varrho(B) = r = \#$ pivots in B.

(3) Clearly, $\dim R_B = r = \dim C_B$.

Immediate Consequences:

Theorem. $A \in M(m\times n) \Rightarrow \varrho(A) = \#$ pivots of the reduced row-echelon form of A.

Proof. Let $\tilde{A}$ and $E_1, \dots, E_k, F_1, \dots, F_\ell$ as in (*) (p 88)

$\Rightarrow \varrho(A) \underset{\substack{\uparrow \\ \text{Thm on p. 87}}}{=} \varrho(\tilde{A}) = \#$ pivots of $\tilde{A}$

$= \#$ pivots of $\tilde{A}$

where $\tilde{A}$ is, as in (*), the reduced row-echelon form of A. $\square$

Theorem. $A \in M(m\times n)$. $\Rightarrow$ $\dim R_A = \dim C_A = \dim(\operatorname{Im}(A))$.

Proof. Let $\tilde{A}, E_1, \dots, E_k, F_1, \dots, F_\ell$ as in (*)

- $\underline{\dim C_A = \dim(\operatorname{Im}(A))}$ is clear because $C_A = \operatorname{Im}(A)$ by Prop. on p. 85
- $\underline{\dim R_A = \dim C_A}$: Let $\tilde{A}, \tilde{\tilde{A}}, E_1, \dots, E_k, F_1, \dots, F_\ell$ as in (*)

$\Rightarrow \dim R_A \underset{\substack{\uparrow \\ R_A = R_{\tilde{A}} \text{ by Thm on p.87}}}{=} \dim R_{\tilde{A}} = \#$ pivots of $\tilde{A}$

$= \#$ pivots of $\tilde{\tilde{A}} = \dim C_{\tilde{\tilde{A}}} = \dim(\operatorname{Im}(\tilde{\tilde{A}}))$

$= \dim(\operatorname{Im}(A))$ by Thm. on page 83

$= \dim C_A$. by Prop. on page 85

Or: Use that $(\tilde{A})^t = \widetilde{(A^t)}$

$\Rightarrow \dim R_A = \#$ pivots of $\tilde{A} = r = \#$ pivots of $(\tilde{A})^t$

$= \#$ pivots $\widetilde{(A^t)} = \dim R_{A^t} = \dim C_A$. ⌋

Theorem. $A \in M(m\times n)$. $\Rightarrow$ $\boxed{\dim(\ker A) + \dim(\operatorname{Im} A) = n}$

Proof. By (*) and theorem on p. 83 we know:

- $\dim(\ker(A)) = \dim(\ker(\tilde{A})) = m-r$
- $\dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(\tilde{A})) = r$

$\Rightarrow \dim(\ker(A)) + \dim(\operatorname{Im}(A)) = m-r+r = m.$ $\square$

This shows again: $A \in M(n\times n)$

$\rightsquigarrow$ A invertible $\Leftrightarrow \dim(\ker A) = 0 \Leftrightarrow \dim(\operatorname{Im}(A)) = n$

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

A injective $\qquad$ A surjective

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$\ker A = \{0\}$ $\qquad$ $\operatorname{Im}(A) = \mathbb{R}^n$.

Sometimes useful:

<u>Proposition</u> $A \in M(m \times n)$. Then:

$$u \in \ker A \overset{(1)}{\Longleftrightarrow} u \perp R_A \overset{(2)}{\Longleftrightarrow} u \perp \mathrm{Im}(A^t).$$

<u>Proof</u>. Observe: $R_A = C_{A^t} = \mathrm{Im}(A^t)$, hence (2) is clear.

Proof of (1): Let $\vec{x} \in \mathbb{R}^n$. Then:

$$\begin{aligned}\vec{x} \in \ker(A) &\Longleftrightarrow A\vec{x} = 0 \Longleftrightarrow \forall j = 1, \dots, m: \ \vec{r}_j \vec{x} = 0 \\ &\Longleftrightarrow \forall j = 1, \dots, m \quad \vec{x} \perp \vec{r}_j \\ &\Longleftrightarrow \vec{x} \perp \mathrm{gen}\{\vec{r}_1, \dots, \vec{r}_n\} \\ &\Longleftrightarrow \vec{x} \perp R_A.\end{aligned}$$

$\square$

<u>Examples:</u>

- $$A = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightsquigarrow$ • $\mathrm{Im}(A) = C_A = \mathrm{gen}\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}4\\2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\}$

$= \mathrm{gen}\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\} = \left\{ \begin{pmatrix}x\\y\\z\\0\end{pmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}.$

• $\rho(A) = \dim(\mathrm{Im}(A)) = 3$.

• $\ker(A) = \left\{ \begin{pmatrix}-2x_2\\x_2\\-2x_4\\x_4\\0\end{pmatrix} \middle| \ x_2, x_4 \in \mathbb{R} \right\}$

$= \left\{ \mu \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + \lambda \begin{pmatrix}0\\0\\-2\\1\\0\end{pmatrix} \middle| \ \mu, \lambda \in \mathbb{R} \right\}$

$= \mathrm{gen}\left\{ \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-2\\1\\0\end{pmatrix} \right\}.$ (*)

• $\nu(A) = \dim(\ker(A)) = 2$ (follows from (*) or: $\nu(A) = n - \rho(A) = 5 - 3 = 2$).

- $$B = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}$$

$\rightsquigarrow$ • $\mathrm{Im}\, B = \mathrm{gen}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} = \mathbb{R}^3$

• $\rho(B) = \dim(\mathrm{Im}(B)) = 3$

• $\ker(B) = \left\{ \begin{pmatrix}-2x_2 - 3x_3\\x_2\\x_3\\0\\-x_5\\x_5\end{pmatrix} \right\} = \mathrm{gen}\left\{ \begin{pmatrix}-2\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\-1\\1\end{pmatrix} \right\}$

• $\nu(B) = \dim(\ker B) = 3$.

• $A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$. Find $\nu(A)$, $\rho(A)$, bases of $\ker(A)$, $\operatorname{Im}(A)$, R_A.

Solution.

• $\ker(A)$ and R_A: Row-operations do not change $\ker(A)$ and R_A

⤳ use row-reduction to transform A in reduced form:

$$(\star)\quad A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow[Q_{31}(-3)]{Q_{21}(-2)} \begin{pmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{pmatrix} \xrightarrow[S_2(-\frac{1}{3})Q_{32}(-2)]{} \begin{pmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[Q_{12}(-4)]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} =: \tilde{A}.$$

$\Rightarrow \ker(A) = \ker \tilde{A} = \left\{ \begin{pmatrix} -c \\ -c \\ c \end{pmatrix} \middle| c \in \mathbb{R} \right\} = \operatorname{gen}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

$\vec{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is a basis of $\ker A$.

$\nu(A) = \dim(\ker A) = 1$.

$R_A = \operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix} \right\} = \operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\vec{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{w}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is basis of R_C.

• $\operatorname{Im}(A)$:

$\rho(A) = \dim(\operatorname{Im} A) = 3 - \nu(A) = 3 - 1 = 2$.

3 possibilities to find a basis of $\operatorname{Im}(A)$:

① Column operations do not change $\operatorname{Im}(A)$

⤳ use column operations to reduce to transform A in reduced form:

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow[Q_{13}(-5)]{Q_{12}(-4)} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -3 \\ 3 & -6 & -6 \end{pmatrix} \xrightarrow[S_2(-\frac{1}{3})Q_{32}(-1)]{} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix} \xrightarrow{Q_{21}(-1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix} =: A'$$

$\operatorname{Im}(A) = \operatorname{Im}(A') = \operatorname{gen}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{\vec{v}_3} \right\}$

Clearly, $\vec{v}_2$ and $\vec{v}_3$ are lin. independent, so the form a basis of $\operatorname{Im}(A)$.

(Another basis would be, e.g., $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.)

② $\operatorname{Im}(A) = C_A = R_{A^t}$. Apply row operations to A^t to obtain a basis of R_{A^t}. This is then a basis of $\operatorname{Im}(A)$.

[The calculations are exactly the same as in ①]

③ We know from $(\star)$:

$\tilde{A} = Q_{12}(-4) S_2(-\frac{1}{3}) Q_{32}(-2) Q_{21}(-2) Q_{31}(-3) A$

$\Rightarrow A = \underbrace{Q_{31}(3) Q_{21}(2) Q_{32}(2) S_2(-3) Q_{12}(4)}_{=:F} \tilde{A}$

Here: $F = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 1 \end{pmatrix}$.

Clearly: $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis of $\operatorname{Im}(A')$

$\Rightarrow \operatorname{Im}(A) = \operatorname{Im}(FA') = F(\operatorname{Im}(A')) = F\left(\operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}\right)$

$= \operatorname{gen}\left\{ F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$.

[Check that $\operatorname{gen}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{\vec{v}_3} \right\} = \operatorname{gen}\left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\vec{w}_2}, \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{\vec{w}_3} \right\}$.

• $\left.\begin{array}{l} \vec{w}_2 = \vec{v}_2 + 2\vec{v}_3 \in \operatorname{gen}\{\vec{v}_2, \vec{v}_3\} \\ \vec{w}_3 = 4\vec{v}_2 + 5\vec{v}_3 \in \operatorname{gen}\{\vec{v}_2, \vec{v}_3\} \end{array}\right\} \Rightarrow \operatorname{gen}\{\vec{w}_2, \vec{w}_3\} \subseteq \operatorname{gen}\{\vec{v}_2, \vec{v}_3\}$. (*)

• Since $\dim(\operatorname{gen}\{\vec{w}_2, \vec{w}_3\}) = 2 = \dim(\operatorname{gen}\{\vec{v}_2, \vec{v}_3\})$ and (*)

$\Rightarrow \operatorname{gen}\{\vec{w}_2, \vec{w}_3\} = \operatorname{gen}\{\vec{v}_2, \vec{v}_3\}$.

(OR: $\left.\begin{array}{l} \vec{v}_2 = \frac{4}{3}\vec{w}_3 - \frac{5}{3}\vec{w}_2 \in \operatorname{gen}\{\vec{w}_2, \vec{w}_3\} \\ \vec{v}_3 = \frac{4}{3}\vec{w}_2 - \frac{1}{3}\vec{w}_3 \in \operatorname{gen}\{\vec{v}_2, \vec{v}_3\} \end{array}\right\} \Rightarrow \operatorname{gen}\{\vec{v}_2, \vec{v}_3\} \subseteq \operatorname{gen}\{\vec{w}_2, \vec{w}_3\}$

with (*) we have then equality.)

$$A = \begin{pmatrix} 1 & 2 & 0 & 5 \\ 2 & 4 & -2 & 4 \\ 1 & 2 & 1 & 8 \\ 2 & 4 & -1 & 7 \end{pmatrix}.$$ Find $\dim(\ker A)$, $\dim(\operatorname{Im} A)$, bases of $\ker A$, $\operatorname{Im} A$, R_A.

- $\ker A$: row-operations lead to:

$$A = \begin{pmatrix} 1 & 2 & 0 & 5 \\ 2 & 4 & -2 & 4 \\ 1 & 2 & 1 & 8 \\ 2 & 4 & -1 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -3 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: \tilde{A}$$

$$\Longrightarrow \ker A = \operatorname{gen}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\},$$

$$\dim(\ker A) = 2$$

$$R_A = \operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

- $\operatorname{Im}(A)$: $\dim(\operatorname{Im}(A)) = 4 - \dim(\ker(A)) = 4 - 2 = 2.$

Column operations to obtain basis for $\operatorname{Im}(A)$:

$$A = \begin{pmatrix} 1 & 2 & 0 & 5 \\ 2 & 4 & -2 & 4 \\ 1 & 2 & 1 & 8 \\ 2 & 4 & -1 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & -6 \\ 1 & 0 & 1 & 3 \\ 2 & 0 & -1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} =: A'$$

$$\Longrightarrow \operatorname{Im}(A) = C_A = C_{A'} = \operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Alternatively: From we know that

$$\tilde{A} = Q_{31}(-1)\, Q_{32}(-1)\, Q_{42}(1)\, Q_{41}(-2)\, Q_{24}(1)\, S_2(-1)\, A$$
$$=: FA$$

$$\Longrightarrow A = F^{-1}\tilde{A} \Longrightarrow \operatorname{Im} A = F^{-1} \operatorname{Im} \tilde{A} = F^{-1}\left(\operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}\right)$$

a straight forward calculation yields: $F^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$

$$\Longrightarrow \operatorname{Im} A = \operatorname{gen}\left\{ F^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, F^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$
$$= \operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

$\square$