Examples:

- $\dim \mathbb{R}^n = n$
- $\dim M(n\times m) = nm$
- $M(n\times n, \text{sym}) :=$ symmetric $(n\times n)$-matrices
  $\Rightarrow \dim M(n\times n, \text{sym}) = \frac{n(n+1)}{2}$.
- $P_n :=$ polynomials of degree $\leq n$.
  $\Rightarrow \dim P_n = n+1$.
- $P =$ polynomials
  $\Rightarrow \dim P = \infty$

  Proof. $\forall n \in \mathbb{N}$ $P_n \subseteq P$ $\Rightarrow \forall n$ $\dim P \geq \dim P_n = n+1$
  $\Rightarrow \dim P = \infty$.
- $C(\mathbb{R})$ = all continuous fct's $\mathbb{R} \to \mathbb{R}$.
  $\Rightarrow \dim C(\mathbb{R}) = \infty$.

  Proof. $P \subseteq C(\mathbb{R}) \Rightarrow \dim C(\mathbb{R}) \geq \dim P = \infty$.
  $\Rightarrow \dim C(\mathbb{R}) = \infty$.

- All possible subspaces of $\mathbb{R}^2$ are: $\{0\}$, lines passing through $(0,0)$, $\mathbb{R}^2$.

  Proof. Let U be a subspace of $\mathbb{R}^2$ and $n = \dim U$.
  - $n = 0 \Rightarrow U = \{0\}$
  - $n = 1 \Rightarrow U = \text{gen}\{u\}$ for some $u \in \mathbb{R}^2 \setminus \{0\}$
    $\Rightarrow U$ is a line containing $(0,0)$, parallel to u.
  - $n = 2 \Rightarrow \dim U = \dim \mathbb{R}^2 \Rightarrow U = \mathbb{R}^2$.
  - $n \geq 3$ is not possible.

- All possible subspaces of $\mathbb{R}^3$ are: $\{0\}$, lines passing through $(0,0,0)$, planes containing $(0,0,0)$, $\mathbb{R}^3$.

  Proof. Let $U \subseteq \mathbb{R}^3$ subspace, $\dim U = n$.
  - $n=0 \Rightarrow U = \{0\}$
  - $n=1 \Rightarrow \exists u \in \mathbb{R}^3 \setminus \{0\}$ s.t. $U = \text{gen}\{u\} \Rightarrow U$ is a line.
  - $n=2 \Rightarrow \exists\, u_1, u_2 \in \mathbb{R}^3$, lin. ind s.t. $U = \text{gen}\{u_1, u_2\} \Rightarrow U$ is a plane
  - $n = 3 \Rightarrow U = \mathbb{R}^3$.
  - $n \geq 4$ is not possible.

4.6. Linear maps.

Definition. Let U, V be vector spaces. A function $A: U \to V$ is called linear if:

(i) $\forall u_1, u_2 \in U$ $\quad A(u_1+u_2) = Au_1 + Au_2$

(ii) $\forall v \in V$, $\forall \lambda \in \mathbb{R}$ $\quad A(\lambda v) = \lambda Av$.

In this case: A is called a linear function, or linear map or linear operator.

A is called injective (or one-to-one) if: $x \neq y \Rightarrow Ax \neq Ay$
(In other words: $Ax = Ay \Rightarrow x = y$).

A is called surjective if: $\forall v \in V$ $\exists x \in U$ s.t. $Ax = v$.

A is called bijective if it is injective and surjective.

The kernel of A (or null space of A) is
$\ker A := N_A := \{x \in U \mid Ax = 0\}$. $\subseteq U$

The range of A (or image of A) is
$\text{Im}(A) := \text{Rg}(A) = \{Ax \mid x \in U\}$
$= \{y \in V \mid \exists x \in U \text{ s.t. } Ax = y\} \subseteq V$.

Observation.

- A injective $\Leftrightarrow$ $\ker A = \{0\}$. Proof. Tutor.
- A surjective $\Leftrightarrow$ $\text{Im} A = V$.
- $A0 = 0$. Proof. $A0 = A(0+0) = A0 + A0 = 2A(0) \Rightarrow A(0) = 0$.

Examples of linear maps:

- Every matrix $A \in M(m\times n)$ can be seen as a linear map $\mathbb{R}^n \to \mathbb{R}^m$.
  (We will show later that every lin. map $\mathbb{R}^n \to \mathbb{R}^m$ "is" indeed a matrix).

We know:

* $m < n \Rightarrow A$ cannot be injective.

  Proof. $Ax = 0$ represents a homogeneous system of m equations for n unknowns. We know that this system has infinitely many solutions because $m < n$. $\Rightarrow \ker(A) \neq \{0\}$

* $m = n$, and $\boxed{\det A \neq 0 \iff A \text{ is bijective} \iff \dim(\operatorname{Im} A) = n \iff \dim(\ker A) = 0}$.

- Let $C^1(\mathbb{R}) = \{$functions $\mathbb{R} \to \mathbb{R}$ which are differentiable$\}$.

  $\to T : C^1(\mathbb{R}) \to C(\mathbb{R})$, $Tf = f'$ is linear.

- Integration is linear.

Proposition. Let $A : U \to V$ be a linear map. Then:

i) $\ker A$ is a subspace of U

ii) $\operatorname{Im}(A)$ is a subspace of V.

(For matrices A this was proved on Taller 8)

Proof. i) Let $u_1, u_2 \in \ker A$ and $\lambda \in \mathbb{R}$.

$$\Rightarrow \begin{cases} A(u_1+u_2) = Au_1 + Au_2 = 0 + 0 = 0 & \Rightarrow u_1 + u_2 \in \ker A \\ A(\lambda u_1) = \lambda A u_1 = \lambda \cdot 0 = 0 & \Rightarrow \lambda u_1 \in \ker A. \end{cases}$$

$\Rightarrow \ker A$ is a subspace of U.

ii) Let $y_1, y_2 \in \operatorname{Im}(A)$ and $\lambda \in \mathbb{R}$.

$\Rightarrow \exists\, x_1, x_2 \in U$ st. $y_1 = Au_1$, $y_2 = Au_2$.

$$\to \begin{cases} y_1 + y_2 = Au_1 + Au_2 = A(u_1+u_2) \in \operatorname{Im}(A) \\ \lambda y_1 = \lambda A u_1 = A(\lambda u_1) \in \operatorname{Im}(A) \end{cases}$$

$\to \operatorname{Im} A$ is a subspace of V. $\square$

Definition. $A : U \to V$ linear.

- $\nu(A) := \dim(\ker A) =:$ nullity of A = nulidad de A
- $\rho(A) = \dim(\operatorname{Im} A) =:$ rank of A = rango de A.

(Obs. Unfortunately, "rango" is also used for "Range of A".)

Observations: Let $A : U \to V$ linear. Then:

(i) $v_1, \dots, v_n \in U$, $\lambda_1, \dots, \lambda_n \in \mathbb{R} \Rightarrow A(\lambda_1 v_1 + \dots + \lambda_n v_n) = \lambda_1 A v_1 + \dots + \lambda_n A v_n$.

(ii) If A is injective, then A can be inverted: $A^{-1} : Rg(A) \to U$.

The inverse function is also linear.

Proposition. Let U, V be vector spaces, $A : U \to V$ linear. Let $x_1, \dots, x_m \in U$, $y_1 := Ax_1, \dots, y_m := Ax_m$. Then:

(i) If $x_1, \dots, x_m$ are lin. dep $\Rightarrow y_1, \dots, y_m$ are lin. dep.

(ii) If $y_1, \dots, y_m$ are lin. indep $\Rightarrow x_1, \dots, x_m$ are lin. indep.

(iii) Suppose that A is invertible. Then:

$x_1, \dots, x_m$ are lin indep $\iff$ $y_1, \dots, y_m$ are lin. indep.

Proof. (i) Suppose that $x_1, \dots, x_m$ are lin. dep.

$\Rightarrow \exists\, \lambda_1, \dots, \lambda_m \in \mathbb{R}$, not all $= 0$, st. $\lambda_1 x_1 + \dots + \lambda_m x_m = 0$

$\Rightarrow \lambda_1 y_1 + \dots + \lambda_m y_m = \lambda_1 A x_1 + \dots + \lambda_m A x_m$

$= A(\lambda_1 x_1 + \dots + \lambda_m x_m) = A0 = 0$.

$\Rightarrow y_1, \dots, y_m$ are lin dep.

(ii) Follows from (i)

(iii) Note. If A is invertible, then $x_1 = A^{-1} y_1, \dots, x_m = A^{-1} y_m$.

The claim follows from (ii), applied to A and A^{-1}.

Observe. "$x_1, \dots, x_m \in U$ lin. ind $\Rightarrow Ax_1, \dots, Ax_m \in V$ lin ind" is in general false.

Theorem. Let U, V vector spaces and let $A: U \to V$,
$E: U \to U$, $F: V \to V$ be linear maps.
Assume that E and F are bijective.

$\Rightarrow$ (i) $\mathcal{Im}(A) = \mathcal{Im}(AE)$, in particular $\mathcal{g}(A) = \mathcal{g}(AE)$

(ii) $\nu(AE) = \nu(A)$. but in general $\ker(AE) \neq \ker(A)$

(iii) $\ker(FA) = \ker(A)$, in particular $\nu(FA) = \nu(A)$

(iv) $\mathcal{g}(FA) = \mathcal{g}(A)$ but in general $\mathcal{Im}(FA) \neq \mathcal{Im}(A)$.

Proof. Observe that by assumption, E and F are invertible and that E^{-1} and F^{-1} are linear maps.

(i) Let $y \in V$. Then: $y \in \mathcal{Im}(A)$

$$\Leftrightarrow \exists x \in U \text{ st. } y = Ax = A\overbrace{EE^{-1}}^{=1}x = AE(\underbrace{E^{-1}x}_{\in U})$$

$\Leftrightarrow y \in \mathcal{Im}(AE)$

(ii) • Let $k = \dim(\ker A)$ and let $u_1, \dots, u_k$ be a basis of $\ker A$.

$\Rightarrow u_1, \dots, u_k$ are lin. ind $\overset{\text{Prop.}}{\Rightarrow} E^{-1}u_1, \dots, E^{-1}u_k$ are lin. ind.

Observe: $AE(E^{-1}u_j) = Au_j = 0 \quad (j = 1, \dots, k)$

$\Rightarrow E^{-1}u_1, \dots, E^{-1}u_k \in \ker(AE)$

$\Rightarrow \dim(\ker(AE)) \geq k = \dim(\ker A)$. (1)

• Let $l = \dim(\ker(AE))$ and let $w_1, \dots, w_l$ be a basis of $\ker(AE)$.

$\Rightarrow w_1, \dots, w_l$ are lin. ind $\overset{\text{Prop.}}{\Rightarrow} Ew_1, \dots, Ew_l$ are lin. indep.

Observe: $A(Ew_j) = (AE)w_j = 0 \quad (j = 1, \dots, l)$

$\Rightarrow Ew_1, \dots, Ew_l \in \ker(A)$

$\Rightarrow \dim \ker(A) \geq l = \dim \ker(AE)$ (2)

(1) and (2) show that $\dim(\ker A) = \dim(\ker(AE))$.

(iii) Let $x \in U$. Then:

$$x \in \ker(FA) \Leftrightarrow FAx = 0 \overset{F \text{ is invertible}}{\Leftrightarrow} Ax = 0 \Leftrightarrow x \in \ker(A).$$

(iv) • Let $k = \dim(\mathcal{Im} A)$ and $v_1, \dots, v_k$ basis of $\mathcal{Im}(A)$.

$\Rightarrow v_1, \dots, v_k$ are lin ind $\overset{\text{Prop.}}{\Rightarrow} Fv_1, \dots, Fv_k$ are lin. ind.

Since $Fv_j \in \mathcal{Im}(FA)$ $(j = 1, \dots, k)$ we have that:

$\dim(\mathcal{Im}(FA)) \geq k = \dim(\mathcal{Im} A)$. (3)

• Let $l = \dim(\mathcal{Im}(FA))$ and $y_1, \dots, y_l$ basis of $\mathcal{Im}(FA)$.

$\Rightarrow y_1, \dots, y_l$ lin ind $\overset{\text{Prop.}}{\Rightarrow} F^{-1}y_1, \dots, F^{-1}y_l$ lin ind.

All $F^{-1}y_j \in \mathcal{Im}(A)$ (because: $y_j \in \mathcal{Im}(FA) \Rightarrow \exists x_j \in U$ st. $y_j = FAx_j$ $\Rightarrow F^{-1}y_j = F^{-1}(FAx_j) = Ax_j \in \mathcal{Im} A$)

$\Rightarrow \dim(\mathcal{Im}(A)) \geq l = \dim(\mathcal{Im}(FA))$ (4)

(3) and (4)

$\Rightarrow \dim(\mathcal{Im}(A)) = \dim(\mathcal{Im}(FA))$.