4.3 Linear Independence

Definition. Let V be a vector space and $v_1, \ldots, v_n \in V$.

Consider the equation $\boxed{\alpha_1 v_1 + \ldots + \alpha_n v_n = 0}$ $(*)$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

* The vectors $v_1, \ldots, v_n$ are called linearly independent if the only solution of $(*)$ is $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$
* The vectors $v_1, \ldots, v_n$ are called linearly dependent if they are not linearly independent (that is, there are $\alpha_1, \ldots, \alpha_n$, not all of them 0, st. $(*)$ holds).

Ejemplos.

* $v_1 = \begin{pmatrix}1\\2\end{pmatrix}, v_2 = \begin{pmatrix}-3\\-6\end{pmatrix} \in \mathbb{R}^2$.

  v_1, v_2 are lin. dependent because $3v_1 + v_2 = 0$.

* $v_1 = \begin{pmatrix}1\\2\end{pmatrix}, v_2 = \begin{pmatrix}3\\0\end{pmatrix}$ are lin. independent.

  Proof. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ st. $\alpha_1 v_1 + \alpha_2 v_2 = 0$

  $$\Rightarrow \underbrace{\begin{cases}\alpha_1 + 3\alpha_2 = 0\\ 2\alpha_1 + 0\alpha_2 = 0\end{cases}}_{\text{lin. system of lin. eq. for } \alpha_1, \alpha_2} \Rightarrow \begin{cases}\alpha_1 = -3\alpha_2\\ \alpha_1 = 0\end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0.$$ $\square$

* $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}2\\3\\4\end{pmatrix} \in \mathbb{R}^3$ are lin. independent.

  Proof: As above.

* $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, v_2 = \begin{pmatrix}2\\4\\6\end{pmatrix} \in \mathbb{R}^3$ are lin. dependent because $2v_1 - v_2 = 0$.

* $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}-1\\2\\3\end{pmatrix}, v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ are lin. independent.

  $v_1, v_2, v_4 = \begin{pmatrix}0\\6\\8\end{pmatrix}$ are lin. dependent.

  Proof ① $0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{pmatrix}\alpha_1 - \alpha_2\\ \alpha_1 + 2\alpha_2\\ \alpha_1 + 3\alpha_2 + \alpha_3\end{pmatrix}$

  $$\Rightarrow \begin{cases}\alpha_1 - \alpha_2 = 0 & ①\\ \alpha_1 + 2\alpha_2 = 0 & ②\\ \alpha_1 + 3\alpha_2 + \alpha_3 = 0 & ③\end{cases} \qquad ① - ② \Rightarrow \alpha_2 = 0, \quad \xRightarrow{\text{in } ①} \alpha_1 = 0, \quad \xRightarrow{\text{in } ③} \alpha_3 = 0$$

  $\Rightarrow v_1, v_2, v_3$ are lin. independent.

  ② $0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{pmatrix}\alpha_1 - \alpha_2\\ \alpha_1 + 2\alpha_2 + 6\alpha_3\\ \alpha_1 + 3\alpha_2 + 8\alpha_3\end{pmatrix}$

  $$\Rightarrow \begin{aligned}\alpha_1 - \alpha_2 &= 0\\ \alpha_1 + 2\alpha_2 + 6\alpha_3 &= 0\\ \alpha_1 + 3\alpha_2 + 8\alpha_3 &= 0\end{aligned}$$

  $$\rightsquigarrow \begin{array}{c}\begin{pmatrix}1 & -1 & 0\\ 1 & 2 & 6\\ 1 & 3 & 8\end{pmatrix}\\ \begin{matrix}\uparrow & \uparrow & \uparrow\\ v_1 & v_2 & v_4\end{matrix}\end{array} \longrightarrow \begin{pmatrix}1 & -1 & 0\\ 0 & 3 & 6\\ 0 & 4 & 8\end{pmatrix} \longrightarrow \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 2\\ 0 & 1 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

  $$\longrightarrow \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

  $\Rightarrow \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}-2\\-2\\1\end{pmatrix} \neq 0 \Rightarrow v_1, v_2, v_4$ are lin. dependent

  (e.g. $-2v_1 - 2v_2 + v_3 = 0$).

* $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in M(2\times 2)$ are lin. independent,
* $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \in M(2\times 2)$ are lin. dependent.

<u>Observations</u>.

(1) If $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ and $\alpha_j \neq 0$

$\Longrightarrow v_j = \frac{\alpha_1}{\alpha_j} v_1 + \frac{\alpha_2}{v_j} v_2 + \dots + \frac{\alpha_{j-1}}{\alpha_j} v_{j-1} + \frac{\alpha_{j+1}}{\alpha_j} v_j + \dots + \frac{\alpha_n}{\alpha_j} v_n$

$\rightsquigarrow v_j$ is linear combination of $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n$.

(2) Let $v_1, v_2 \in V$ be vectors. Then:

v_1, v_2 are linearly dependent $\Longleftrightarrow$ one vector is multiple of the other.

<u>Proof</u>. "$\Longleftarrow$" Assume $v_1 = \alpha v_2 \Rightarrow v_1 - \alpha v_2 = 0$.

"$\Longrightarrow$" Assume v_1, v_2 lin. dependent. Then there exist $\alpha_1, \alpha_2 \in \mathbb{R}$, s.t. $\alpha_1 v_1 + \alpha_2 v_2 = 0$ and at least one $\alpha_j \neq 0$. Suppose that $\alpha_1 \neq 0$, then: $v_1 = -\frac{\alpha_2}{\alpha_1} v_2$.

If $\alpha_2 \neq 0$, then $v_2 = -\alpha_1/\alpha_2 \, v_1$. ◻

(3) $v_1, \dots, v_n$ linearly dependent & $w \in V \Rightarrow v_1, \dots, v_n, w$ are lin. dependent

(4) $v_1, \dots, v_n$ linearly independent $\Rightarrow$ Every subsystem is linearly independent.

<u>Observe</u>: Let $v_1, \dots, v_n \in \mathbb{R}^m$. Then:

$v_1, \dots, v_n$ are lin. dependent

$\Longleftrightarrow \exists \alpha_1, \dots, \alpha_n \in \mathbb{R}$ s.t. $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ and not all $\alpha_j = 0$

$\Longleftrightarrow \underbrace{(v_1 | v_2 | \dots | v_n)}_{\text{matrix whose columns are the vectors } v_1, \dots, v_n} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$ and $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \neq 0$

$\Leftrightarrow$ the homogeneous system corresponding to the matrix $(v_1 | v_2 | \dots | v_n)$ has at least one (and therefore infinitely many) non-trivial solutions.

(5) $v_1, \dots, v_n \in V$ and $w \in V$ linear combination of $v_1, \dots, v_n$.

$\Rightarrow v_1, \dots, v_n, w$ are linearly dependent.

Important consequence:

<u>Theorem</u>. Let $V = \mathbb{R}^m$ and $v_1, \dots, v_n \in V$.

* If $n > m$, then $v_1, \dots, v_n$ are lin. dependent.
* If the $v_1, \dots, v_n$ are lin. independent, then $n \leq m$.

<u>Proof</u>. Follows directly from the observation above and the fact that a homogeneous system $m \times n$ always has infinitely solutions if $n > m$. ◻

<u>Theorem</u>. Let $V = \mathbb{R}^m$ and let $v_1, \dots, v_m \in V$.

Then: $v_1, \dots, v_m$ are lin. independent

$\Longleftrightarrow (v_1 | v_2 | \dots | v_m) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \vec{0}$ has only the trivial solution

$\Longleftrightarrow \det(v_1 | v_2 | \dots | v_m) = 0$. ◻

<u>Proof</u>. Follows from the theorem above.

<u>Theorem</u>. Let $V = \mathbb{R}^m$, $v_1, \dots, v_m \in V$ linearly independent.

Then: $\mathbb{R}^m = \text{gen}\{v_1, \dots, v_m\}$.

<u>Proof</u>. Let $A = (v_1 | v_2 | \dots | v_m)$ = matrix whose columns are the given vectors $v_1, \dots, v_m$. Let $w \in V$. We have to show:

There exist $\beta_1, \dots, \beta_m \in \mathbb{R}$ s.t. $w = \beta_1 v_1 + \dots + \beta_m v_m = A \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$

By assumption A is invertible

$\Rightarrow$ Choose $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = A^{-1} w$. ◻

More examples of systemse of linearly independent / dependent vectors:

* P_n = {polynomials of degree $\le n$}.

Consider P_3 and vectors $v_1 = x^3+1$, $v_2 = x^2+1$, $v_3 = x+1 \in P_3$

$\Rightarrow$ v_1, v_2, v_3 are linearly independent.

Proof. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ st. $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\Rightarrow \alpha_1 (x^3+1) + \alpha_2 (x^2+1) + \alpha_3 (x+1) = 0$

$\Rightarrow \alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x + \alpha_1 + \alpha_2 + \alpha_3 = 0$

$\xRightarrow{\text{compare coefficients}} \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$. □

Consider additionally $v_4 = x^2 + x$.

$\Rightarrow$ v_1, v_2, v_3, v_4 are linearly independent.

Proof. Let $\alpha_1, \dots, \alpha_4 \in \mathbb{R}$ st. $\sum_{j=1}^{4} \alpha_j v_j = 0$

$\Rightarrow \alpha_1(x^3+1) + \alpha_2(x^2+1) + \alpha_3(x+1) + \alpha_4(x^2+x) = 0$

$\Rightarrow \alpha_1 x^3 + (\alpha_2+\alpha_4)x^2 + (\alpha_3+\alpha_4)x + \alpha_1+\alpha_2+\alpha_3 = 0$

$$\xRightarrow{\text{Compare coefficients}} \begin{cases} \alpha_1 = 0 \\ \alpha_2 + \alpha_4 = 0 \\ \alpha_3 + \alpha_4 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$$

$$\rightsquigarrow \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1 \\ 0&0&1&1 \\ 1&1&1&0 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_1} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&1&1&0 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&0&1&-1 \end{pmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_3} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&0&0&-2 \end{pmatrix}$$ $\leftarrow$ has 4 pivots $\Rightarrow$ has only the trivial solution.

$\rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. □

• $V = P_2$, $v_1 = x^2 + 2x - 1$, $v_2 = 5x+2$, $v_3 = 2x^2 - 11x - 8$.

$\Rightarrow$ v_1, v_2, v_3 are lin. dependent.

Proof. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ st. $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\Rightarrow \alpha_1(x^2+2x-1) + \alpha_2(5x+2) + \alpha_3(2x^2-11x-8) = 0$

$\Rightarrow (\alpha_1 + 2\alpha_3)x^2 + (2\alpha_1 + 5\alpha_2 - 11\alpha_3)x - \alpha_1 + 2\alpha_2 - 8\alpha_3 = 0$

$$\xRightarrow{\text{Compare coefficients}} \begin{matrix} \alpha_1 + 2\alpha_3 = 0 \\ 2\alpha_1 + 5\alpha_2 - 11\alpha_3 = 0 \\ -\alpha_1 + \alpha_2 - 8\alpha_3 = 0 \end{matrix}$$

$$\rightsquigarrow \begin{pmatrix} 1&0&2 \\ 2&5&-11 \\ -1&2&-8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&2 \\ 0&5&-15 \\ 0&2&-6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&2 \\ 0&1&-3 \\ 0&1&-3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&2 \\ 0&1&-3 \\ 0&0&0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} t \quad (t \in \mathbb{R}).$$

~~$\Rightarrow -2\alpha_1 + 3\alpha_2 + \alpha_3 = 0$~~

$\Rightarrow -2v_1 + 3v_2 + v_3 = 0 \Rightarrow v_1, v_2, v_3$ are lin. dependent. □

• $V = M(2\times2)$. $A = \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix}$, $B = \begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$, $C = \begin{pmatrix} 0&5 \\ 5&0 \end{pmatrix}$

$\Rightarrow$ A, B, C are lin. dependent because $A - B - \frac{2}{5}C = 0$.

• $V = M(2\times3)$, $A = \begin{pmatrix} 1&2&3 \\ 4&5&6 \end{pmatrix}$, $B = \begin{pmatrix} 2&2&2 \\ 1&1&1 \end{pmatrix}$, $C = \begin{pmatrix} 1&2&2 \\ 2&1&1 \end{pmatrix}$

$\Rightarrow$ A, B, C are lin. independent.

Proof. Let $\alpha, \beta, \gamma \in \mathbb{R}$ st. $\alpha A + \beta B + \gamma C = 0$

$$\Rightarrow \begin{pmatrix} \alpha + 2\beta + \gamma & 2\alpha + 2\beta + 2\gamma & 3\alpha + 2\beta + 2\gamma \\ 4\alpha + \beta + 2\gamma & 5\alpha + \beta + \gamma & 6\alpha + \beta + \gamma \end{pmatrix} = 0$$

$$\rightarrow \begin{cases} \alpha + 2\beta + \gamma = 0 \\ 4\alpha + \beta + 2\gamma = 0 \\ 2\alpha + 2\beta + 2\gamma = 0 \\ 5\alpha + \beta + \gamma = 0 \\ 3\alpha + 2\beta + 2\gamma = 0 \end{cases} \quad 6\alpha + \beta + \gamma = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 2 \\ 4 & 1 & 1 \\ 5 & 2 & 2 \\ 6 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -4 & -1 \\ 0 & -7 & -3 \\ 0 & -8 & -4 \\ 0 & -11 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \alpha = \beta = \gamma = 0.$ $\square$

4.4. Basis and dimension.

Definition: Let V be a vector space. A basis of V is a set of vectors $v_1, \dots, v_n \in V$ s.t.

(i) $v_1, \dots, v_n$ are linearly independent

(ii) $v_1, \dots, v_n$ generate V

Observe: Let $v_1, \dots, v_n$ be a basis of V. and $w \in V$;

- $v_1, \dots, v_n, w$ is not a basis of V (because it is no longer linearly independent). [If $v_1, \dots, v_n, w$ were lin. indep, then w cannot be a lin. comb. of $v_1, \dots, v_n \Rightarrow w \notin \text{gen}\{v_1, \dots, v_n\} = V$]
- If we delete one v_j from $v_1, \dots, v_n$, then the remaining vectors are not a basis of V (because they no longer generate V) [Eg $j=1$. Suppose $v_2, \dots, v_n$ generate V; then v_1 is linear combination of $v_2, \dots, v_n$ and therefore $v_1, v_2, \dots, v_n$ are linearly dependent.]

In a certain sense, a basis of V is a "minimal generating system" of V.

Examples

- $\boxed{V = \mathbb{R}^n}$. Then, by Theorem on page 68, every basis of V ~~has~~ consists of exactly n vectors.
  Clearly, $\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is a basis of $\mathbb{R}^n$. It is called the standard basis of $\mathbb{R}^n$.
- $\boxed{V = \mathbb{R}^2}$ Examples for basis: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 13 \\ 8 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
  Examples which are not a basis: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$;
  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;
  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

- $\boxed{V = \mathbb{R}^3.}$

* $v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, v_2 = \begin{pmatrix}4\\5\\6\end{pmatrix}, v_3 = \begin{pmatrix}7\\8\\9\end{pmatrix}$ is NOT a basis of $\mathbb{R}^3$.

<u>Proof 1</u>. Assume that there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t. $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$$\Rightarrow \vec{0} = \begin{pmatrix}\alpha_1 + 4\alpha_2 + 7\alpha_3\\ 2\alpha_1 + 5\alpha_2 + 8\alpha_3 \\ 3\alpha_1 + 6\alpha_2 + 9\alpha_3\end{pmatrix} = \begin{pmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}.$$

$$\leadsto \begin{pmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 4 & 7\\0 & -3 & -6\\0 & -6 & -12\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 4 & 7\\0 & 1 & 2\\0 & 1 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 4 & 7\\0 & 1 & 2\\0 & 0 & 0\end{pmatrix}$$

$$\leadsto \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\alpha_3 \quad\Rightarrow\quad v_1 - 2v_2 + v_3 = \vec{0}$$

$\Rightarrow v_1, v_2, v_3$ are <u>not</u> linearly independent $\Rightarrow$ they are <u>not</u> a basis.

<u>Proof 2</u>. Calculate the determinant of the matrix whose columns are given by the vectors v_1, v_2, v_3:

$$\det\begin{pmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{pmatrix} = 45 + 120 + 84 - 105 - 48 - 72 = 0.$$

$\rightarrow v_1, v_2, v_3$ are not lin. indep.

$\Rightarrow$ they do not form a basis. $\square$

* $v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$ is a basis of $\mathbb{R}^3$.

<u>Proof 1</u>. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t. $\vec{0} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$$\Rightarrow \vec{0} = \begin{pmatrix}\alpha_1 + \alpha_2\\ \alpha_1 + \alpha_3 \\ \alpha_2 + \alpha_3\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}$$

$$\leadsto \begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 0\\0 & -1 & 1\\0 & 1 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 0\\0 & 1 & -1\\0 & 0 & 2\end{pmatrix}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \quad \rightarrow v_1, v_2, v_3$ are lin. indep.

Since they are <u>three</u> vectors, they are a basis of $\mathbb{R}^3$.

<u>Proof 2</u>. Calculate $\det(v_1 | v_2 | v_3) = \det\begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix} = 2 \neq 0$

$\rightarrow v_1, v_2, v_3$ form a basis of $\mathbb{R}^3$.

- $\boxed{V = P_n}$ <u>Canonical basis</u> (or <u>standard basis</u>) is: $1, X, X^2, \dots, X^n$.

Clearly, these vectors are linearly independent and span P_n.

* $p_1 = X, \; p_2 = 2X^2 + 5X - 1, \; p_3 = 3X^2 + X + 2$ is a basis of P_2

<u>Proof</u>. Let $q = aX^2 + bX + c \in P_2$.

Suppose there ex. $\alpha_1, \alpha_2, \alpha_3$ s.t. $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = q$.

$\Rightarrow \alpha_1 X + \alpha_2(2X^2 + 5X - 1) + \alpha_3(3X^2 + X + 2) = aX^2 + bX + c$

$\Rightarrow (2\alpha_2 + 3\alpha_3)X^2 + (\alpha_1 + 5\alpha_2 + \alpha_3)X + (-\alpha_2 + 2\alpha_3) = aX^2 + bX + c$

Compare coefficients $\Rightarrow$

$$\begin{aligned} 2\alpha_2 + 3\alpha_3 &= a\\ \alpha_1 + 5\alpha_2 + \alpha_3 &= b\\ -\alpha_2 + 2\alpha_3 &= c\end{aligned} \quad\Longleftrightarrow\quad \begin{pmatrix}0 & 2 & 3\\1 & 5 & 1\\0 & -1 & 2\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

<u>Observe</u>: The columns represent the vectors p_1, p_2, p_3!

Row reduction:

$$\left(\begin{array}{ccc|c}0 & 2 & 3 & a\\1 & 5 & 1 & b\\0 & -1 & 2 & c\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & 5 & 1 & a\\0 & 1 & -2 & -c\\0 & 2 & 3 & b\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & 5 & 1 & a\\0 & 1 & -2 & -c\\0 & 0 & 7 & b+2c\end{array}\right)$$

This shows:

- Every $q \in P_3$ can be represented as a unique linear combination of p_1, p_2, p_3 (choose $\alpha_3 = \frac{1}{7}(b+2c)$, $\alpha_2 = -c + 2\alpha_3$, $\alpha_1 = a - \alpha_3 - 5\alpha_2$)

  $\leadsto p_1, p_2, p_3$ generate P_2 ①

- In particular, 0 can be represented in a <u>unique</u> way as lin. combination of p_1, p_2, p_3: $0 = 0p_1 + 0p_2 + 0 \cdot p_3$.

  $\Rightarrow p_1, p_2, p_3$ are linear independent. ②

  ① & ② show that p_1, p_2, p_3 is a basis of P_2. $\square$

* $p_1 = 1 + X, \; p_2 = X + X^2, \; p_3 = X^2 + X^3, \; p_4 = 1 + X + X^2 + X^3$ is <u>NOT</u> a basis of P_3.

<u>Proof</u>. Tallas.

- $\boxed{V = M(n \times m)}$ Canonical basis: A_{ij} ~~[illegible]~~ where

  A_{ij} is the $n \times m$-matrix with 1 in the position ij and 0 everywhere else.

  Example: Canonical basis of $M(2\times 3)$: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is basis of $M(2\times 2)$.

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2\times 2)$ and assume that there are $\alpha_1, \dots, \alpha_4$ st. $\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Compare components $\Rightarrow$
$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= a \\ \alpha_2 + \alpha_3 + \alpha_4 &= b \\ \alpha_3 + \alpha_4 &= c \\ \alpha_4 &= d \end{aligned} \qquad (*)$$

$\Rightarrow$ unique solution $\alpha_4 = d,\ \alpha_3 = c-d,\ \alpha_2 = b-c-d,$ $\alpha_1 = a-b-c-d$.

$\Rightarrow$ The four matrices are a generating system (because the system $(*)$ has a solution for every right hand side a, b, c, d); and they are linearly independent because of the uniqueness of the solutions (choose $a=b=c=d$; then all α_j must be 0).

Important theorems:

Theorem. Let V be a vector space and let $v_1, \dots, v_n$ and $w_1, \dots, w_m$ be bases of V. Then: $n = m$.

That means: Every basis of V has the same number of elements. This number is called the dimension of V.

Notación: $\dim V = n$.

Proof. Suppose $m > n$. We will show that then $w_1, \dots, w_m$ is not linearly independent.

We know: $v_1, \dots, v_n$ is a basis of V, hence it is a generating system.

$\Rightarrow$ There exist a_{ij} such that:
$$\left.\begin{aligned} w_1 &= a_{11} v_1 + a_{12} v_2 + \dots + a_{1n} v_n \\ w_2 &= a_{21} v_1 + a_{22} v_2 + \dots + a_{2n} v_n \\ &\vdots \\ w_m &= a_{m1} v_1 + a_{m2} v_2 + \dots + a_{mn} v_n. \end{aligned}\right\} \quad (1)$$

Now we check if the $w_1, \dots, w_m$ are lin. independent. Consider
$$c_1 w_1 + c_2 w_2 + \dots + c_m w_m = 0 \qquad (2)$$

Insert (1) into (2):
$$\Rightarrow 0 = c_1 (a_{11} v_1 + \dots + a_{1n} v_n) + \dots + c_m (a_{m1} v_1 + \dots + a_{mn} v_n)$$
$$= (c_1 a_{11} + c_2 a_{21} + \dots + c_m a_{m1}) v_1 + \dots + (c_1 a_{1n} + \dots + c_m a_{mn}) v_n$$

Since the $v_1, \dots, v_n$ are linearly independent; the terms in parentesis must be 0:
$$\begin{aligned} c_1 a_{11} + c_2 a_{21} + \dots + c_m a_{m1} &= 0 \\ &\vdots \\ c_1 a_{1n} + c_2 a_{2n} + \dots + c_m a_{mn} &= 0 \end{aligned} \qquad (3)$$

(3) is a system of n equations for the m unknowns $c_1, \dots, c_m$.
Since the system is homogeneous and $m > n$, it must have infinitely many solutions.

$\Longrightarrow$ equation (2) has infinitely many solutions
$\Rightarrow$ $w_1, \dots, w_m$ are linearly dependent
$\Rightarrow$ $w_1, \dots, w_m$ is not a basis of V.

So we have shown: $m > n \Rightarrow w_1, \dots, w_m$ is not a basis of V.
Analogously we can show: $n > m \Rightarrow v_1, \dots, v_n$ is not a basis of V.
So it follows that $m = n$.

$\square$

Theorem. V vector space, $v_1, \dots, v_n$ basis of V.
$\Rightarrow \forall\, w \in V$ exist unique $c_1, \dots, c_n \in \mathbb{R}$ s.t. $w = c_1 v_1 + \dots + c_n v_n$.

Proof. Existence is clear because $v_1, \dots, v_n$ is a basis of, hence in particular a generating system.

Uniqueness: Assume there are $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{R}$ s.t.

$w = c_1 v_1 + \dots + c_n v_n$ and
$w = d_1 v_1 + \dots + d_n v_n$.

$\Rightarrow 0 = w - w = (c_1 - d_1) v_1 + \dots + (c_n - d_n) v_n$.
$\Rightarrow c_1 - d_1 = 0,\ c_2 - d_2 = 0, \dots, c_n - d_n = 0$ because the $v_1, \dots, v_n$ are lin. ind.
$\Rightarrow c_1 = d_1,\ c_2 = d_2, \dots, c_n = d_n$.

$\square$

Definition. A vector space V is called finitely generated if there are finitely many elements $v_1, \dots, v_m \in V$ s.t. $V = \text{gen}\{v_1, \dots, v_m\}$.

Clearly: V finitely generated $\Leftrightarrow \dim V < \infty$.

Theorem. Let V be a finitely generated vector space and let $v_1, \dots, v_k \in V$ be a system of lin. ind. vectors.
$\Rightarrow$ exist vectors $w_{k+1}, \dots, w_n$ s.t. $v_1, \dots, v_k, w_{k+1}, \dots, w_n$ are a basis of V.

That means: Every system of lin. ind. vectors can be completed to a basis of V. In particular: Every finitely generated vector space has a basis.

Theorem. Let V be a vector space and $v_1, \dots, v_m \in V$ s.t. $V = \text{gen}\{v_1, \dots, v_m\}$.
$\Longrightarrow$ By removing appropriate vectors from $v_1, \dots, v_m$ we obtain a basis of V.

That means: Every generating system of V contains a basis of V.

Corollary. Let V be a vector space.

(1) $v_1, \dots, v_k \in V$ lin. ind. $\Rightarrow k \le \dim V$.
(2) $v_1, \dots, v_m \in V$ generate V $\Rightarrow m \ge \dim V$.

Observation. V vector space, $U \subseteq V$ subspace.
If $w_1, \dots, w_k \in U$, then $\text{gen}\{w_1, \dots, w_k\} \subseteq U$ and $\dim U \ge \dim(\text{gen}\{w_1, \dots, w_k\})$

Theorem. Let V be a vector space and $U \subseteq V$ subspace.
Then: (i) $\dim U \le \dim V$
(ii) If $\dim V < \infty$, then: $\dim U = \dim V \Leftrightarrow U = V$.

Proof. (i) Let $k = \dim U$ and $u_1, \dots, u_k$ basis of U. $\Rightarrow u_1, \dots, u_k$ are lin. ind. in V $\Rightarrow \dim V \ge k$.

(ii) "$\Leftarrow$" is clear.
"$\Rightarrow$" Let $k = \dim U$ and choose basis $u_1, \dots, u_k \in U$. Then: $u_1, \dots, u_k$ are lin. ind. in V. Since $\dim V = k$, we must have that $u_1, \dots, u_k$ is basis of, hence $V = \text{gen}\{u_1, \dots, u_k\} = U$.

$\square$

Examples:

- $\dim \mathbb{R}^n = n$
- $\dim M(n\times m) = nm$
- $M(n\times n, \text{sym}) :=$ symmetric $(n\times n)$-matrices
  $\Rightarrow \dim M(n\times n, \text{sym}) = \frac{n(n+1)}{2}$.
- $P_n :=$ polynomials of degree $\leq n$.
  $\Rightarrow \dim P_n = n+1$.
- $P =$ polynomials
  $\Rightarrow \dim P = \infty$

  Proof. $\forall n\in\mathbb{N}$ $P_n \subseteq P \Rightarrow \forall n$ $\dim P \geq \dim P_n = n+1$
  $\Rightarrow \dim P = \infty$.
- $C(\mathbb{R}) =$ all continuous fct's $\mathbb{R}\to\mathbb{R}$.
  $\Rightarrow \dim C(\mathbb{R}) = \infty$.

  Proof. $P\subseteq C(\mathbb{R}) \Rightarrow \dim C(\mathbb{R}) \geq \dim P = \infty$.
  $\Rightarrow \dim C(\mathbb{R}) = \infty$.
- All possible subspaces of $\mathbb{R}^2$ are: $\{0\}$, lines passing through $(0,0)$, $\mathbb{R}^2$.

  Proof. Let U be a subspace of $\mathbb{R}^2$ and $n = \dim U$.
  - $n = 0 \Rightarrow U = \{0\}$
  - $n = 1 \Rightarrow U = \text{gen}\{u\}$ for some $u\in\mathbb{R}^2\setminus\{0\}$
    $\Rightarrow U$ is a line containing $(0,0)$, parallel to u.
  - $n = 2 \Rightarrow \dim U = \dim\mathbb{R}^2 \Rightarrow U = \mathbb{R}^2$.
  - $n\geq 3$ is not possible.
- All possible subspaces of $\mathbb{R}^3$ are: $\{0\}$, lines passing through $(0,0,0)$, planes containing $(0,0,0)$, $\mathbb{R}^3$.

  Proof. Let $U\subseteq\mathbb{R}^3$ subspace, $\dim U = n$.
  - $n=0 \Rightarrow U = \{0\}$
  - $n=1 \Rightarrow \exists u\in\mathbb{R}^3\setminus\{0\}$ s.t. $U = \text{gen}\{u\} \Rightarrow U$ is a line.
  - $n=2 \Rightarrow \exists u_1,u_2\in\mathbb{R}^3$, lin. ind. s.t. $U = \text{gen}\{u_1,u_2\} \Rightarrow U$ is a plane
  - $n = 3 \Rightarrow U = \mathbb{R}^3$.
  - $n\geq 4$ is not possible.

4.6. Linear maps.

Definition. Let U,V be vector spaces. A function $A: U\to V$ is called linear if:

(i) $\forall u_1,u_2\in U$ $A(u_1+u_2) = Au_1 + Au_2$

(ii) $\forall u\in V$, $\forall\lambda\in\mathbb{R}$ $A(\lambda u) = \lambda Au$.

In this case: A is called a linear function, or linear map or linear operator.

A is called injective (or one-to-one) if: $x\neq y \Rightarrow Ax\neq Ay$
(In other words: $Ax = Ay \Rightarrow x = y$).

A is called surjective if: $\forall v\in V$ $\exists x\in U$ s.t. $Ax = v$.

A is called bijective if it is injective and surjective.

The kernel of A (or null space of A) is (nucleo di A)

$\ker A := N_A := \{x\in U \mid Ax = 0\}$. $\subseteq U$

The range of A (or image of A) is

$\text{Im}(A) := Rg(A) = \{Ax \mid x\in U\}$
$= \{y\in V \mid \exists x\in U \text{ s.t. } Ax = y\} \subseteq V$

$\square$

Observation.

- A injective $\Leftrightarrow \ker A = \{0\}$. Proof. Later.
- A surjective $\Leftrightarrow \text{Im} A = V$.
- $A0 = 0$. Proof. $A0 = A(0+0) = A0 + A0 = 2A(0) \Rightarrow A(0) = 0$. $\square$

Examples of linear maps:

- Every matrix $A\in M(m\times n)$ can be seen as ~~as~~ a linear map $\mathbb{R}^n\to\mathbb{R}^m$.

  (We will show later that every lin. map $\mathbb{R}^n\to\mathbb{R}^m$ "is" indeed a matrix).