4. Vector spaces

4.1 Definition and basic properties

Recall. Let $\mathbb{K}$ be a field ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

A $\mathbb{K}$-Vector space is a triple $(V, +, \cdot)$ where V is a set and $+ : V \times V \to V$, $\cdot : \mathbb{K} \times V \to V$ s.t.

for all $x, y, z \in V$, $\lambda, \mu \in \mathbb{K}$:

- Associativity: $(x+y)+z = x+(y+z)$
- Commutativity: $x+y = y+x$
- Identity element: $\exists\, 0 \in V$ s.t. $\forall x \in V$ $x + 0 = x$
- Inverse element: $\forall x \in V$ $\exists x' \in V$ s.t. $x + x' = 0$

  Notation: $x' = -x$.

- Multiplication by $1 \in \mathbb{K}$: $1 \cdot x = x$. $(x \in V)$
- Associativity of multiplication: $\mu(\lambda x) = (\mu\lambda)x$
- Distributivity: $(\mu + \lambda)x = \mu x + \lambda x$, $\mu(x+y) = \mu x + \mu y$

For us always: $\boxed{\mathbb{K} = \mathbb{R}}$

Observations:

i) The identity element is unique.

Proof. Suppose $\exists\, 0, 0' \in V$ s.t. $\forall x \in V$: $0 + x = 0' + x = x$.

$\Rightarrow 0' = 0' + 0 = 0 + 0' = 0$.

ii) $\forall x \in V$, the inverse is unique.

Proof. Suppose $\exists x \in V$ and $x', x'' \in V$ s.t. $x + x' = 0$ and $x + x'' = 0$

$\Rightarrow x' = x' + 0 = x' + (x + x'') = (x' + x) + x'' = (x + x') + x'' = 0 + x'' = x''$.

Theorem. Let V be a vector space and $0 \in V$ the zero of V.

$\Rightarrow$ i) $\forall \lambda \in \mathbb{R}$: $\lambda 0 = 0$

ii) $\forall x \in V$ $0 \cdot x = 0 \in \mathbb{R}$

iii) $\forall x \in V$ $(-1) \cdot x = -x$.

iv) $\lambda x = 0 \Rightarrow \lambda = 0$ or $x = 0$.

Proof. i) $\lambda \cdot 0 = \lambda(0 + 0) = \lambda 0 + \lambda 0 \quad / + [-\lambda 0]$

$\Rightarrow \underbrace{\lambda \cdot 0 + [-\lambda 0]}_{=0} = \lambda 0 + \underbrace{\lambda 0 + [-\lambda 0]}_{=0}$

$\Rightarrow 0 = \lambda 0$.

ii) $0x = (0+0)x = 0x + 0x \quad / + [-0x]$

$\Rightarrow 0x + [-0x] = 0x + 0x + [-0x]$

$\Rightarrow 0 = 0x$.

iii) $0 \overset{(ii)}{=} 0x = (1-1)x = x + (-1)\cdot x \Rightarrow -1 \cdot x = -x$.

iv) Suppose $\lambda x = 0$ and $\lambda \neq 0$.

$\Rightarrow x = \frac{1}{\lambda}\lambda x = \frac{1}{\lambda} 0 = 0$ $\square$

Definition. Let V be a vector space and $W \subseteq V$ subset.

W is called a subspace of V if it is a vector space with sum and product inherited from V. A subspace W is called a proper subspace if $W \neq \{0\}$ and $W \neq V$.

Observations. Let V be a vector space.

- $W \subseteq V$ subspace $\Rightarrow 0 \in W$.
- $W \subseteq V$ subset. Then: W is a subspace $\Leftrightarrow \begin{cases} x, y \in W \Rightarrow x + y \in W \\ x \in W, \lambda \in \mathbb{R} \Rightarrow \lambda x \in W \\ W \neq \emptyset. \end{cases}$

  Proof. "$\Rightarrow$" is clear.

  "$\Leftarrow$" Let $x \in W \Rightarrow \begin{cases} 0 = 0x \in W \\ -x = (-1)x \in W \end{cases}$

  Commutativity, associativity and distributivity are clear. $\square$

- Every subspace of a vector space is a vector space.

Examples:

Let V be a vector space.

$\Rightarrow$ • $\{0\}$ is a subspace of V. It is called the trivial vector space.

• V is a subspace of V.

• $x \in V$ arbitrary. Then: $\{\lambda x \mid \lambda \in \mathbb{R}\}$ is a subspace of V.

• More general: $x_1, \dots, x_n \in V$ arbitrary

Then: $\{\lambda_1 x_1 + \dots + \lambda_n x_n \mid \lambda_1, \dots, \lambda_n \in \mathbb{R}\}$ is a subspace of V.

More examples:

* $\mathbb{R}^n$ is a vector space with the usual operations.

* $\mathbb{R}^2$ is a vector space.

Subspaces of $\mathbb{R}^2$: For example: $\left\{\begin{pmatrix}\lambda\\0\end{pmatrix} \mid \lambda \in \mathbb{R}\right\}$, $\left\{\lambda\begin{pmatrix}x_0\\y_0\end{pmatrix} \mid \lambda \in \mathbb{R}\right\}$

They describe lines in $\mathbb{R}^2$!

It can be shown that every proper subspace of $\mathbb{R}^2$ is a line.

Example: $W := \left\{\begin{pmatrix}a\\b\end{pmatrix} + \lambda\begin{pmatrix}x_0\\y_0\end{pmatrix} \mid \lambda \in \mathbb{R}\right\} \subseteq \mathbb{R}^2$ is NOT a subspace if $\begin{pmatrix}a\\b\end{pmatrix} \not\parallel \begin{pmatrix}x_0\\y_0\end{pmatrix}$. (In this case, W is an affine space)

* $\mathbb{R}^3$ is a vector space.

Examples of subspaces of $\mathbb{R}^3$:

• $\left\{\lambda\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} \mid \lambda \in \mathbb{R}\right\}$ describes a line if $\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} \neq 0$

• $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \mid ax+by+cz = 0\right\}$ is a VS and describes a plane if $\begin{pmatrix}a\\b\\c\end{pmatrix} \neq 0$.

Proof. Let $W := \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \mid ax+by+cz = 0\right\}$.

Let $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \in W, \lambda \in \mathbb{R}$.

$\Rightarrow a(x_1+\lambda y_1) + b(y_1+\lambda y_2) + c(z_1+\lambda z_2) = ax_1+by_1+cz_1 + \lambda(ax_2+by_2+cz_2) = 0$

$\Rightarrow \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} + \lambda\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \in W$.

$\Rightarrow W$ is subsp by the observation on p. 58. $\square$

• Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and let $a, b, c \in \mathbb{R}$.

$\Rightarrow W := \{ax+by+cz = \alpha\}$ is NOT a vector space.

Proof. Let $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \in \mathbb{R}^3$.

$\Rightarrow a(x_1+x_2) + b(y_1+y_2) + c(z_1+z_2) = 2\alpha \overset{\alpha \neq 0}{\neq} \alpha$

$\Rightarrow \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \notin W \Rightarrow W$ cannot be a vector space.

• $W := \left\{\begin{pmatrix}\lambda\\\lambda^2\\\lambda^3\end{pmatrix} \mid \lambda \in \mathbb{R}\right\}$ is not a vector space.

Proof. Clearly, $\begin{pmatrix}1\\1\\1\end{pmatrix} \in W$; but $2 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix} \notin W$

$\Rightarrow W$ is not a vector space.

* The space of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space.

Examples of subspaces:

- All bdd functions
- All continuous functions
- All continuous function with $f(3) = 0$
- All polynomials
- All even functions
- All odd functions.

Subsets which are NOT subspaces:

- All fct's with $f(3) = 12$.
- All fct's with $f(3) > 0$
- All polynomials of degree ≥ 5.

* The space of all $(m\times n)$-matrices over $\mathbb{R}$ (or $\mathbb{C}$ or $\mathbb{K}$) is a vector space with the usual sum and product with $\lambda \in \mathbb{K}$.

Examples of subspaces.

- All matrices with $a_{11} = 0$
- All matrices with $a_{12} = \mu\, a_{11}$
- All matrices st. 1st column = last column.
- All symmetric matrices (if $m=n$)
- All antisymmetric matrices (if $m=n$)
- All diagonal matrices (if $m=n$)
- All upper triangular matrices
- All lower triangular matrices.

Subsets which are NOT subspaces:

* All invertible matrices
- All non-invertible matrices
- All non-symmetric matrices
- All matrices with $\det A = 1$.

* All polynomials of degree $\le n$

* All polynomials of degree $\le n$ and only even powers.

* All polynomials of degree $\ge n$ is NOT a vector space.

* All polynomials of degree $= n$ is NOT a vector space.

* $\mathbb{R}$ is a $\mathbb{Q}$-vector space

* $\mathbb{Q}$ is NOT an $\mathbb{R}$-vector space.

(No hay elemento neutral: $\nexists \begin{pmatrix} a \\ b\end{pmatrix}$ st. $\begin{pmatrix} 0 \\ 1\end{pmatrix} + \begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} 0 \\ 1\end{pmatrix}$)

* $\mathbb{R}^2$ with the sum $\begin{pmatrix} x \\ y\end{pmatrix} \oplus \begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} x+a \\ 0\end{pmatrix}$ is not a vector space

* $\mathbb{R}^2$ with the sum $\begin{pmatrix} x \\ y\end{pmatrix} \oplus \begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} x+b \\ y+a\end{pmatrix}$ is NOT a vector space.

Proof. $\oplus$ is not commutativ. because: $\begin{pmatrix} 1 \\ 0\end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix} \neq \begin{pmatrix} 0 \\ 1\end{pmatrix} = \begin{pmatrix} 0 \\ 1\end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0\end{pmatrix}$

* $V = \left\{ \begin{pmatrix} x \\ y\end{pmatrix} \,\middle|\, x \ge 0 \right\}$ with the usual sum is NOT a vector space.

Proof. Clearly, the identity element is $\begin{pmatrix} 0 \\ 0\end{pmatrix}$.
Clearly, $\begin{pmatrix} 1 \\ 0\end{pmatrix}$ has no inverse in V. □

* $V = \mathbb{R}^+ \times \mathbb{R}^+ = \left\{ \begin{pmatrix} x \\ y\end{pmatrix} \,\middle|\, x>0,\ y>0 \right\}$.

Define $\begin{pmatrix} x \\ y\end{pmatrix} \oplus \begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} xa \\ yb\end{pmatrix} \in V$, $\lambda \cdot \begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} x^\lambda \\ y^\lambda\end{pmatrix} \in V$

Then: $(V, \oplus, \cdot)$ is a VS.

Proof. Commutativity and associativity are clear.

- Neutral element: $0 = \begin{pmatrix} 1 \\ 1\end{pmatrix}$
- Inverse element: $\begin{pmatrix} x \\ y\end{pmatrix} \oplus \begin{pmatrix} 1/x \\ 1/y\end{pmatrix} = \begin{pmatrix} 1 \\ 1\end{pmatrix} = 0$
- $1\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} x^1 \\ y^1\end{pmatrix} = \begin{pmatrix} x \\ y\end{pmatrix}$
- $\lambda, \mu \in \mathbb{R} \leadsto (\lambda\mu)\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} x^{\lambda\mu} \\ y^{\lambda\mu}\end{pmatrix} = \mu\begin{pmatrix} x^\lambda \\ y^\lambda\end{pmatrix} = \mu\lambda\begin{pmatrix} x \\ y\end{pmatrix}$.
- Distributivity: $(\lambda+\mu)\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} x^\lambda x^\mu \\ y^\lambda y^\mu\end{pmatrix} = \begin{pmatrix} x^\lambda \\ y^\lambda\end{pmatrix} \oplus \begin{pmatrix} x^\mu \\ y^\mu\end{pmatrix} = \lambda\begin{pmatrix} x \\ y\end{pmatrix} \oplus \mu\begin{pmatrix} x \\ y\end{pmatrix}$

$$\lambda\left(\begin{pmatrix} x \\ y\end{pmatrix} \oplus \begin{pmatrix} a \\ b\end{pmatrix}\right) = \lambda\begin{pmatrix} xa \\ yb\end{pmatrix} = \begin{pmatrix} x^\lambda a^\lambda \\ y^\lambda b^\lambda\end{pmatrix} = \begin{pmatrix} x^\lambda \\ y^\lambda\end{pmatrix} \oplus \begin{pmatrix} a^\lambda \\ b^\lambda\end{pmatrix}$$
$$= \lambda\begin{pmatrix} x \\ y\end{pmatrix} \oplus \lambda\begin{pmatrix} a \\ b\end{pmatrix}.$$

□

* More general: Let $f: \mathbb{R} \to (a,b)$ injective. Then (a,b) becomes a vector space with the multiplication and product

$$x \oplus y := f\left(f^{-1}(x) + f^{-1}(y)\right) \qquad x, y \in (a,b)$$
$$\lambda x = f\left(\lambda f^{-1}(x)\right) \qquad \lambda \in \mathbb{R}.$$

4.2. Linear Combinations.

Definition. Let V be a vector space and $v_1, \dots, v_n \in V$. and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

Then the vector $v = \alpha_1 v_1 + \dots + \alpha_n v_n \in V$ is called a linear combination of $v_1, \dots, v_n$.

Example * $V = \mathbb{R}^3$; $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

Then: $w = \begin{pmatrix} 9 \\ 12 \\ 15 \end{pmatrix}$, $z = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ are linear comb. of v_1 and v_2 because

$w = v_1 + 2v_2$, $z = -v_1 + v_2$.

* $V = M(2\times 2)$; $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\rightarrow$ $R = \begin{pmatrix} 5 & 7 \\ -7 & 5 \end{pmatrix} = 5A + 7B$ is a linear comb. of A and B

$S = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$ is NOT linear combination of A and B.

Theorem. Let V be a vector space and $v_1, \dots, v_n \in V$.

Then $\mathrm{gen}(v_1, \dots, v_n) := \{ \lambda_1 v_1 + \dots + \lambda_n v_n \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \}$

is a subspace of V. It is called the subspace generated by $v_1, \dots, v_n$ and $v_1, \dots, v_n$ are called its generators.

Proof. Let $v = \lambda_1 v_1 + \dots + \lambda_n v_n$, $w = \mu_1 v_1 + \dots + \mu_n v_n \in V$; $\alpha \in \mathbb{R}$.

$$\rightarrow \quad v + \alpha w = \lambda_1 v_1 + \dots + \lambda_n v_n + \alpha\mu_1 v_1 + \dots + \alpha\mu_n v_n$$
$$= (\lambda_1 + \alpha\mu_1) v_1 + \dots + (\lambda_n + \alpha\mu_n) v_n \in V.$$

By the observation on page 58, V is a vector space. $\square$

Observe: If a vector space V is generated by vectors $v_1, \dots, v_n$, then every $v \in V$ is linear combination of the $v_1, \dots, v_n$.
The linear combination is not necessarily unique.

Observe: * The generators of a given subspace are not unique.
* If $V = \mathrm{gen}\{v_1, \dots, v_n\}$ and $w \in V$, then also $V = \mathrm{gen}\{v_1, \dots, v_n, w\}$.

Examples

* $V = M(2\times 2)$. Then $\mathrm{gen}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$

$= \mathrm{gen}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\}$.

* $V = P_n := \{\text{polynomials of degree} \le n\}$.

How many elements do we need ~~[illegible]~~ in order to generate P_n?

Answer: $P_n = \mathrm{gen}\{1, x, x^2, \dots, x^n\}$.

* $V = \mathbb{R}^3$; $v, w, z \in V \setminus \{0\}$.

$\rightsquigarrow$ $\mathrm{gen}\{v\}$ is the line passing through $(0,0,0)$, parallel to v

$\mathrm{gen}\{v, w\} = \begin{cases} \text{line passing through } (0,0,0), \text{ parallel to } v \text{ if } v \parallel w \\ \text{plane passing through } (0,0,0), \text{ parallel to } v \text{ and } w \text{ if } v \nparallel w. \end{cases}$