② Component (i,j) of $B^t A^t = \sum_{\ell=1}^{n} \tilde{b}_{i\ell}\, \tilde{a}_{\ell j} = \sum_{\ell=1}^{n} b_{\ell i}\, a_{j\ell}$

Comparing ① and ② shows that (✱) is true.

Theorem. $\forall A \in M(n\times k)$: $(A^t)^t = A$.

Proof. Clear.

Theorem. Let $A \in M(n\times n)$. Then: A is invertible $\Leftrightarrow$ A^t is invertible.
In this case: $(A^t)^{-1} = (A^{-1})^t$

Proof. "$\Rightarrow$" Assume that A is invertible. Then:

$$\mathbb{1} = AA^{-1} = A^{-1}A$$
$$\Rightarrow \mathbb{1} = \mathbb{1}^t = (AA^{-1})^t = (A^{-1}A)^t$$
$$\Rightarrow \mathbb{1} = (A^{-1})^t A^t = A^t (A^{-1})^t$$
$$\Rightarrow A^t \text{ is invertible and } (A^t)^{-1} = (A^{-1})^t.$$

"$\Leftarrow$" Assume that A^t is invertible. By what we already have shown: $(A^t)^t = A$ is invertible. □

Theorem. Let $A = (a_{ij})_{i,j} \in M(m\times n)$.
$\Rightarrow$ A^t is the unique matrix $\in M(n\times m)$ s.t. $\forall x \in \mathbb{R}^n$, $\forall y \in \mathbb{R}^m$:

$$\langle Ax, y\rangle = \langle x, A^t y\rangle. \quad (✱)$$

Proof. • Let $B \in M(n\times m)$ st $\forall x \in \mathbb{R}^n\ \forall y \in \mathbb{R}^m$: $\langle Ax, y\rangle = \langle x, By\rangle$.
In particular for $x = e_j \in \mathbb{R}^n$, $y = e_h \in \mathbb{R}^m$:

① $\langle Ae_j, e_h\rangle = \left\langle \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}, e_h \right\rangle = a_{hj}$

② $\langle e_j, Be_h\rangle = \left\langle e_j, \begin{pmatrix} b_{1h} \\ \vdots \\ b_{nh} \end{pmatrix} \right\rangle = b_{jh}$

① = ② $\Rightarrow$ $a_{hj} = b_{jh}$ para todo $j = 1,\dots,n$; $h = 1,\dots,m$
$\Rightarrow A = B^t$.

• Easy to check that (✱) is true for A^t. □

3. Determinants.

3.1. Definition of determinants.

Definition. A permutation of a set M is a bijection $M \to M$.
In particular, the set of all permutations of $\{1,\dots,n\}$ is denoted by S_n.
An element $\sigma \in S_n$ is denoted by $\begin{Bmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & & \sigma(n) \end{Bmatrix}$
Its signum is $\mathrm{sgn}(\sigma) = (-1)^{\#\text{inversions}}$, where an "inversion" is a pair i,j such that $i<j$ and $\sigma(i) > \sigma(j)$.

Examples:

• All Elements of S_2 are: $\begin{Bmatrix} 1 & 2 \\ 1 & 2 \end{Bmatrix}$, $\begin{Bmatrix} 1 & 2 \\ 2 & 1 \end{Bmatrix}$

$\mathrm{sgn}(\cdot) = 1$, $\mathrm{sgn}(\cdot) = -1$

• All elements of S_3 are: $\begin{Bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{Bmatrix}$ $\begin{Bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{Bmatrix}$ $\begin{Bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{Bmatrix}$ $\begin{Bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{Bmatrix}$ $\begin{Bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{Bmatrix}$ $\begin{Bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{Bmatrix}$

sgn: 1, -1, -1, 1, -1, 1

Definition. Let $A = (a_{ij})_{i,j=1,\dots,n} \in M(n\times n)$. Then:

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

$=$ determinant of A. $\boxed{\text{Leibniz formula}}$

Other notation: $\boxed{\det A = |A|}$

Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M(2\times 2)$

$\Rightarrow$ $\boxed{\det A = a_{11}a_{22} - a_{12}a_{21}}$ Coincides with the formula on page 36.

<u>Definition</u>. $A \in M(n\times n)$. For $i,j \in \{1,\dots,n\}$ we define

M_{ij} to be the $(n-1)\times(n-1)$-matrix that is obtained from A by deleting the i-th row and the j-th column.

Matrices of the form M_{ij} are called the <u>minors of A</u>.
The numbers $A_{ij} = (-1)^{i+j} \det(M_{ij})$ are called the <u>cofactors of A</u>.

<u>Now</u>: Determinant of a 3x3-matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix}$$

$$= a_{11}\det M_{11} - a_{12}\det M_{12} + a_{13}\det M_{13}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

"expandir por cofactores en la fila 1".

Similarly we can show:

"expandir por cofactores en la fila 2"

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = -a_{21}\det(M_{21}) + a_{22}\det(M_{22}) - a_{23}\det(M_{23})$$

$$= a_{31}\det(M_{31}) - a_{32}\det(M_{32}) + a_{33}\det(M_{33})$$

"expandir por cofactores en la fila 3".

We can also expand by columns:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}\det(M_{11}) - a_{21}\det(M_{21}) + a_{31}\det(M_{31})$$

$$= -a_{12}\det(M_{12}) + a_{22}\det(M_{22}) - a_{32}\det(M_{32})$$

$$= a_{31}\det(M_{31}) - a_{32}\det(M_{32}) + a_{33}\det(M_{33}).$$

<u>Conclusion</u>: $\forall k = 1,2,3$:

$$\boxed{\det A = \underbrace{\sum_{\ell=1}^{3} a_{k\ell}A_{k\ell}}_{\text{expansion by the k-th line}} = \underbrace{\sum_{\ell=1}^{3} a_{\ell k}A_{\ell k}}_{\text{expansion by the k-th column}}.}$$

It can be shown that this formula is true for any n:

Let $A \in M(n\times n)$. Then: For every $k = 1,\dots,n$

$$\boxed{\det A = \sum_{\ell=1}^{n} a_{k\ell}A_{k\ell} = \sum_{\ell=1}^{n} a_{\ell k}A_{\ell k}.}$$

<u>Proof</u>. Long and involves long calculations, but not too hard.
Would be a nice homework.

<u>Examples</u>

Expand by first line

- $\det\begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & 4 \\ 5 & 0 & 3\end{pmatrix} = 0\cdot\det\begin{pmatrix} 3 & 4 \\ 0 & 3\end{pmatrix} - 1\cdot\det\begin{pmatrix} 2 & 4 \\ 5 & 3\end{pmatrix} + 0\det\begin{pmatrix} 2 & 3 \\ 5 & 0\end{pmatrix}$
$= -(6-20) = 14.$

Expand by third column

- $\det\begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & 4 & 0 \\ 4 & 3 & 0 & 5 \\ 1 & 0 & 3 & 4\end{pmatrix} = 0\cdot\det(\cdot\cdot) - 4\det\begin{pmatrix} 1 & 1 & 3 \\ 4 & 3 & 5 \\ 1 & 0 & 4\end{pmatrix} + 0 - 3\det\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 4 & 3 & 5\end{pmatrix}$

$= -4\left(\det\begin{pmatrix} 1 & 3 \\ 3 & 5\end{pmatrix} + 4\det\begin{pmatrix} 1 & 1 \\ 4 & 3\end{pmatrix}\right) - 3\left(3\det\begin{pmatrix} 2 & 1 \\ 4 & 3\end{pmatrix} + 5\det\begin{pmatrix} 1 & 1 \\ 2 & 1\end{pmatrix}\right)$

$= -4[5-9+4(3-4)] - 3[3(6-4)+5(1-2)]$

$= -4[-4-4] - 3[6-5] = 32 - 3 = \cancel{44}\ 29.$

<u>Useful for calculating determinants of a 3x3-matrix</u>:

$A = (a_{ij})_{i,j=1,\dots,3}$. ⤳ $\det A$ is sum of diagonals ↘ minus sum of diagonals ↙

That is:

$$\det A = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{32}.$$

<u>Important</u> This works <u>only</u> for matrices 3x3 !!

Basic properties of the determinant:

(D1) The determinant is linear in the rows of a matrix:

$$\det\begin{pmatrix} \vdots \\ a_i+\lambda b_i \\ \vdots \end{pmatrix} = \det\begin{pmatrix} \vdots \\ a_i \\ \vdots \end{pmatrix} + \lambda \det\begin{pmatrix} \vdots \\ b_i \\ \vdots \end{pmatrix}$$

a_i, b_i: rows of the matrix
$\lambda \in \mathbb{R}$.

Proof. Calculate determinant of $\begin{pmatrix} \vdots \\ a_i+\lambda b_i \\ \vdots \end{pmatrix}$ by expanding by the ith row.

(D2) The determinant is linear in the columns of a matrix:

$$\det(\cdots / a_i+\lambda b_i / \cdots) = \det(\cdots | a_i | \cdots) + \lambda \det(\cdots | b_i | \cdots).$$

a_i, b_i columns of the matrix
$\lambda \in \mathbb{R}$

Proof. Calculate determinant of $(\cdots | a_i+\lambda b_i | \cdots)$ by expanding by the ith column.

(D3) The determinant is alternating; that means:
Let $A \in M(n\times n)$ and let B be the matrix which is obtained from A by exchanging two rows (or columns). Then: $\det A = -\det B$.

Proof. Expand $\det A$ and $\det B$ by the rows (or columns) that have been exchanged. OR: Use Leibniz formula.

(D4) $\det \mathbb{1}_n = 1$.

Proof. Straightforward calculation.

Observation 1. All the properties above can be shown also with the Leibniz formula.

Observation 2. It can be shown that the determinant is the unique map $M(n\times n) \to \mathbb{R}$ which satisfies (D1), (D3) and (D4) [or (D2) (D3) (D4)]. All the following properties of det can be shown using only properties (D1) - (D4).

3.2 More properties of the determinant.

(D5) Let $A \in M(n\times n)$ and suppose that the i-th row of A is a multiple of the j-th row of A. Then: $\det A = 0$.

In particular: If A has a zero row, then $\det A = 0$.

Analogous affirmations hold for the columns of A.

Proof. We prove only the affirmation about columns of A. The proof for rows is analogous.
Suppose there are columns a_i and a_j of A and $\lambda \in \mathbb{R}$ s.t. $a_i = \lambda a_j$.

$$\Rightarrow \det A = \det(\cdots | a_j | \cdots | \underbrace{\lambda a_j}_{=a_i} | \cdots)$$
$$= \lambda \det(\cdots | a_j | \cdots | a_j | \cdots) \quad \text{(D2) linearity in columns}$$
$$= -\lambda \det(\cdots | a_j | \cdots | a_j | \cdots) \quad \text{(D3) exchange columns } i \text{ and } j$$
$$= -\det(\cdots | a_j | \cdots | \lambda a_j | \cdots) \quad \text{(D2) linearity in columns.}$$
$$= -\det A.$$

In summary, we showed: $\det A = -\det A$.
Hence it follows that $\det A = 0$.

(D6) Determinant of upper/lower triangular matrices.

If $A = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$ or $A = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ * & & c_n \end{pmatrix}$, then:

$$\det A = c_1 \cdots c_n.$$

Proof. Upper triangular matrix: Expand by first column or last row.
Lower triangular matrix: Expand by first row or last column.

(D7) Determinant of elementary matrices.

$$\det(S_i(c)) = c, \quad \det Q_{ij}(c) = 1, \quad \det P_{ij} = -1 \qquad (i \neq j).$$

Proof. $S_i(c)$ and $Q_{ij}(c)$ are triangular matrices, so the claim follows from (D6).

P_{ij} is obtained from $\mathbb{1}$ by exchanging rows i and j.

$$\Longrightarrow \det P_{ij} \underset{(D3)}{=} -\det \mathbb{1} \underset{(D4)}{=} -1$$

□

(D8) $A \in M(n \times n)$, E elementary matrix: $\Longrightarrow$ $\boxed{\det(EA) = (\det E)(\det A)}$

Proof. Case 1 $E = S_i(c)$.

$$\Longrightarrow \det(EA) = \det(S_i(c)A) = \det\begin{pmatrix} \vdots \\ c a_i \\ \vdots \end{pmatrix} \underset{(D1)}{=} c \det\begin{pmatrix} \vdots \\ a_i \\ \vdots \end{pmatrix} \overset{(D7)}{=} \det(S_i(c)) \det A = \det E \det A.$$

$S_i(c)$ multiplies i-th row of A by c.

Case 2: $E = Q_{ij}(c)$
Case 3: $E = P_{ij}$
} are proved similar the proof of the case 1.

□

Applying (D8) several times, we obtain:

$A \in M(n \times n)$, $E_1, \dots, E_k$ elementary matrices

$$\Longrightarrow \det(E_1 \cdots E_k A) = (\det E_1) \cdots (\det E_k)(\det A).$$

(D9) $A \in M(n \times n)$. Then: $\boxed{A \text{ is invertible} \iff \det A \neq 0.}$

Proof. Let $E_1, \dots, E_k$ elementary matrices and $\tilde{A}$ a matrix in reduced row echelon form s.t. $A = E_1 \cdots E_k \tilde{A}$

$$\Longrightarrow \det A = \underbrace{(\det E_1) \cdots (\det E_k)}_{\neq 0 \text{ by } (D7)} \det \tilde{A} \quad \text{by (D8)} \qquad (*)$$

Case 1 A is invertible $\Rightarrow \tilde{A} = \mathbb{1} \Rightarrow \det \tilde{A} = 1 \neq 0 \Rightarrow \det A \neq 0$ by $(*)$

Case 2 A is not invertible $\Rightarrow \tilde{A}$ must have a zero row.

$\Rightarrow \det \tilde{A} = 0$ by (D5)

$\Rightarrow \det A = 0$ by $(*)$

□

(D10) $A, B \in M(n \times n)$ $\Longrightarrow$ $\boxed{\det(AB) = (\det A)(\det B)}$

Proof. Let $E_1, \dots, E_k$ elementary matrices and $\tilde{A}$ of reduced row-echelon form such that $A = E_1 \cdots E_k \tilde{A}$.

Case 1. A is invertible $\Rightarrow \tilde{A} = \mathbb{1}$.

$$\Rightarrow \det(AB) = \det(E_1 \cdots E_k B) \overset{(D8)}{=} \det(E_1) \cdots (\det E_k)(\det B)$$

$$\overset{(D8)}{=} \det(E_1 \cdots E_k) \det B = \det A \cdot \det B.$$

Case 2 A is not invertible. Then $\tilde{A}$ must have a zero row.

But then also $\tilde{A}B$ must have a zero row and therefore $\det(\tilde{A}B) = 0$.

Hence: $\det(AB) = \det(E_1 \cdots E_k \tilde{A}B) = \det(E_1) \cdots \det E_k \underbrace{\det(\tilde{A}B)}_{=0} = 0$

And $\det A = \det(E_1 \cdots E_k \tilde{A}) = \det E_1 \cdots \det E_k \underbrace{\det \tilde{A}}_{=0} = 0$

$$\Longrightarrow \det(AB) = 0 = \det A \cdot \det B$$

□

(D11) $A \in M(n \times n)$ invertible. Then: $\boxed{\det A^{-1} = \frac{1}{\det A}}$

Proof. $\det(A) \det(A^{-1}) \underset{(D10)}{=} \det(AA^{-1}) = \det(\mathbb{1}) \underset{(D4)}{=} 1.$

$$\Longrightarrow \det(A^{-1}) = \frac{1}{\det A}.$$

□

(D12) $\boxed{\det A^t = \det A.}$

Proof. Follows easily from the Leibniz formula.

Or: Expand A by 1st row and A^t by 1st column and compare. $\square$

(D13) $\boxed{\det(\lambda A) = \lambda^n \det A}$

Proof. Use (D1) in every row (or (D2) in every column) of A.

OR: $\lambda A = S_1(\lambda) \cdots S_n(\lambda) A$ and use (D8).

Very important: In general $\boxed{\det(A+B) \neq \det A + \det B.}$!

Remark on calculating determinants:

The Leibniz formula is very impractical for numerical calculations because it involves calculations of the order $n!$

If we use Gauß-elimination and then (D8), we only need about n^3 operations

Examples:

- $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{pmatrix} = 0$ because $R_3 = R_1 + R_2$.

- $\det\begin{pmatrix} 14 & 28 & 42 \\ 0 & 1 & 3 \\ 4 & 8 & 16 \end{pmatrix} = 14 \cdot 4 \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} = 56\left[\det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \det\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}\right]$
$= 56\,[\,4-6+6-3\,] = 56.$

- $\det\begin{pmatrix} 0 & 0 & 3 \\ 0 & 4 & 5 \\ 1 & 3 & 8 \end{pmatrix} = -1 \cdot 4 \cdot 3 = -12$ (Expand by first row)
or: Exchange first and third row to obtain a triangular matrix.

- $\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 0$ because $R_1 = R_2$.

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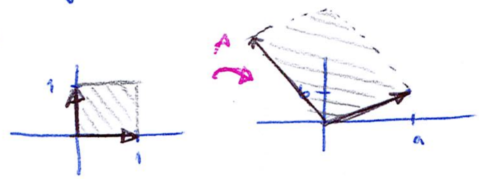
Geometric interpretation of determinant:

• $\mathbb{R}^2$. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

A maps the ~~square~~ spanned by $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ to a parallelogramm spanned by $\begin{pmatrix}a\\c\end{pmatrix}, \begin{pmatrix}b\\d\end{pmatrix}$.



Area of the parallelogramm spanned by $\begin{pmatrix}a\\c\end{pmatrix}, \begin{pmatrix}b\\d\end{pmatrix}$:

$$\text{Area } = \left\| \begin{pmatrix}a\\c\\0\end{pmatrix} \times \begin{pmatrix}b\\d\\0\end{pmatrix} \right\| = \left\| \begin{pmatrix}0\\0\\ad-bc\end{pmatrix} \right\| = |ad-bc| = \det A$$

$\rightsquigarrow$ $\boxed{\det A \text{ describes how are changes under transformation by } A}$

• $\mathbb{R}^3$. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

A maps the cube spanned by $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ to parallelopiped spanned by $\underbrace{\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix}}_{\vec{u}}, \underbrace{\begin{pmatrix}a_{12}\\a_{22}\\a_{32}\end{pmatrix}}_{\vec{v}}, \underbrace{\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix}}_{\vec{w}}$.

Its volume is:

$$\text{Volume} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = \left| \begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix} \cdot \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} \\ a_{32}a_{13} - a_{12}a_{23} \\ a_{12}a_{23} - a_{13}a_{22} \end{pmatrix} \right|$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{21}(a_{32}a_{13} - a_{12}a_{23}) + a_{31}(\cdots)$$

$$= \det A \qquad \text{(expansion by 1st row)}.$$

$\rightsquigarrow$ $\boxed{\det A \text{ describes how volumes change under transformations with } A.}$

Similar results are true in higher dimensions.
Will be important in Vector Calculus.

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3.3. Determinants and inverses of matrices.

Recall. $\det A = \sum_{l=1}^{n} a_{kl} A_{kl}$ where: a_{kl} : components of A
$A_{kl} = (-1)^{k+l} \det M_{kl}$
$=$ cofactors of A.

New: $m \neq k$.

$$\Rightarrow \boxed{\sum_{l=1}^{n} a_{kl} A_{ml} = 0} \qquad (*)$$

Proof. Let B be the matrix that is obtained from A by replacing the m-th row with the k-th row.

$\rightarrow \det B = 0$ because B has two equal rows.

On the other hand:

$$\det B = \underbrace{\sum_{l=1}^{n} b_{ml} B_{ml}}_{\text{expand by } m\text{-th row}} = \sum_{l=1}^{n} \overset{b_{ml}}{\overset{\parallel}{a_{kl}}} \overset{B_{ml}}{\overset{\parallel}{A_{ml}}},$$

So $(*)$ is proved.

From the above, it follows that:

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{= A} \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} = (\det A)\, \mathbb{1}$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{\det A} \underbrace{\begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \\ A_{1n} & \cdots & A_{nn} \end{pmatrix}}_{=: \operatorname{adj}(A) \; = \text{ la adjunta de } A} = \frac{1}{\det A} \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}^{t}.}$$