2.6. Elementary matrices.

Let $n \in \mathbb{N}$ and define the following matrices in $M(n \times n)$:

- $$S_i(c) = \begin{array}{c} \\ \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \leftarrow i\text{ th row} \\ \uparrow \\ i\text{-th column.} \end{array}, \quad c \neq 0.$$

- $$Q_{ij}(c) = \begin{array}{c} \begin{pmatrix} 1 & & & & \\ & \ddots & & c & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \leftarrow i\text{ th row} \\ \uparrow \\ j\text{-th column.} \end{array} \qquad (\text{Clearly: } Q_{ii}(c) = S_i(c))$$

- $$P_{ij} = \begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i\text{ th row} \\ \\ \leftarrow j\text{ th row} \\ \\ \\ \end{array} \\ i\text{-th column.} \quad j\text{-th column} \end{array}$$

Matrices of this form are called elementary matrices.

How do they act on other matrices?

Let $A \in M(n,k)$.

$$\Rightarrow S_i A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k} \\ c\,a_{i1} & \cdots & c\,a_{ik} \\ a_{i+1,1} & \cdots & a_{i+1,1} \\ \vdots & & \\ a_{n1} & \cdots & a_{nk} \end{pmatrix} \rightsquigarrow \boxed{\begin{array}{l} S_i(c) \text{ multiplies the} \\ i\text{-th rows of } A \text{ by } c \\ \text{and leaves the other} \\ \text{rows.} \end{array}}$$

$$Q_{ij} A = \left(\mathbb{1} + \begin{pmatrix} \cdots & c & \cdots \\ & \vdots & \end{pmatrix} \right) A = A + \begin{pmatrix} \cdots & c & \cdots \\ & \vdots & \end{pmatrix} A$$

$$= A + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ c\,a_{j1} & \cdots & c\,a_{jk} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \leftarrow i \text{ th row}$$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{i-1,1} & & \\ a_{i1} + c\,a_{j1} & \cdots & a_{ik} + c\,a_{jk} \\ a_{i+1,1} & \cdots & a_{i+1,k} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} \end{pmatrix} \rightsquigarrow \boxed{\begin{array}{l} Q_{ij}(c) \text{ sums } c \text{ times} \\ \text{the } j\text{-th row to the} \\ i\text{-th row of } A \end{array}}$$

$$P_{ij} A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jk} \\ a_{i1} & \cdots & a_{ik} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{n1} \end{pmatrix} \rightsquigarrow \boxed{\begin{array}{l} P_{ij} \text{ exchanges the } i\text{-th and} \\ j\text{-th row of } A. \end{array}}$$

Theorem. Every elementary matrix is invertible. More precisely:

- $\forall c \neq 0: \ (S_i(c))^{-1} = S_i(1/c)$
- $\forall i \neq j: \ (Q_{ij}(c))^{-1} = Q_{ij}(-c)$
- $(P_{ij})^{-1} = P_{ij}$.

Proof. Easy calculations.

Observe: Row operations on A correspond to multiplication from the left by elementary matrices!

Recall.

- For every matrix $A \in M(n,k)$ exist row operations which take A into row-echelon form.
- A matrix $A \in M(n\times n)$ is invertible if and only if there exist row operations which take A into $\mathbb{1}$.

Since row operations correspond to multiplication from the left by the corresponding elementary matrix, we obtain:

Theorem. Let $A \in M(n\times n)$. Then A is invertible if and only if A is product of elementary matrices.

Proof. • Suppose A is invertible. Then A is row-equivalent to $\mathbb{1}$.

$\Rightarrow$ There are elementary matrices $E_1, \dots, E_m$ s.t.

$$E_1 \cdot \ldots \cdot E_m A = \mathbb{1}$$

$$\Rightarrow A = E_m^{-1} E_{m-1}^{-1} \cdot \ldots \cdot E_1^{-1}.$$

Since every E_j^{-1} is again an elementary matrix, it follows that A is product of invertible matrices.

• Suppose A is product of the elementary matrices $F_1, \dots, F_k$. Since every F_j is invertible, A must be invertible and

$$A^{-1} = (F_1 \cdot \ldots \cdot F_k)^{-1} = F_k^{-1} \cdot \ldots \cdot F_1^{-1}.$$

$\square$

Definition. A matrix $B \in M(n\times n)$ is called upper triangular (triangular superior) if it is of the form $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$.

It is called lower triangular (triangular inferior) if it is of the form

$$\begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}.$$

An argument similar to the one in the proof the theorem shows:

Let $A \in M(m\times n)$. Then there exist elementary matrices $E_1, \dots, E_m$ and an upper triangular matrix B s.t.

$$A = E_1 \cdots E_m B.$$

2.7 Transpose of a matrix.

Definition. Let $A \in M(m\times n)$. Then its transpose is

$$A^t = (\tilde a_{ji})_{\substack{j=1,\dots,n\\ i=1,\dots,m}} \in M(n\times m), \quad \text{donde } \tilde a_{ji} = a_{ij}.$$

That is: the rows of A are the columns of A^t, the columns of A are the rows of A^t.

Examples: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$; $\mathbb{1}^t = \mathbb{1}$

Definition. Let $A \in M(n\times n)$. Then A is called symmetric if $A^t = A$.

Theorem. Let $A \in M(m\times n)$, $B \in M(n\times k)$ $\Rightarrow (AB)^t = B^t A^t$.

Proof. Let $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, $B = (b_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,k}}$

$AB \in M(m\times k)$
$(AB)^t \in M(k\times m)$

and $A^t = (\tilde a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$, $B^t = (\tilde b_{ij})_{\substack{i=1,\dots,k\\ j=1,\dots,n}}$.

$\rightsquigarrow \tilde a_{ij} = a_{ji}$, $\tilde b_{ij} = b_{ji}$.

We have show: component (i,j) of $(AB)^t$ = component (i,j) of $B^t A^t$ for $i = 1, \dots, k$, and $j = 1, \dots, m$. $(*)$

(i) Component (i,j) of $(AB)^t$ = component (j,i) of AB

$$= \sum_{\ell=1}^{n} a_{j\ell} b_{\ell i}$$

② Component (ij) of $B^t A^t = \sum_{\ell=1}^n \tilde{b}_{i\ell}\, \tilde{a}_{\ell j} = \sum_{\ell=1}^n b_{\ell i}\, a_{j\ell}$

Comparing ① and ② shows that (✱) is true.

Theorem. $\forall A \in M(n\times k)$: $(A^t)^t = A$.

Proof. Clear.

Theorem. Let $A \in M(n\times n)$. Then: A is invertible $\iff A^t$ is invertible.
In this case $(A^t)^{-1} = (A^{-1})^t$

Proof. "$\Rightarrow$" Assume that A is invertible. Then:

$$\mathbb{1} = AA^{-1} = A^{-1}A$$
$$\Rightarrow \mathbb{1} = \mathbb{1}^t = (AA^{-1})^t = (A^{-1}A)^t$$
$$\Rightarrow \mathbb{1} = (A^{-1})^t A^t = A^t (A^{-1})^t$$
$$\Rightarrow A^t \text{ is invertible and } (A^t)^{-1} = (A^{-1})^t.$$

"$\Leftarrow$" Assume that A^t is invertible. By what we already have shown: $(A^t)^t = A$ is invertible. $\square$

3. Determinants.

3.1. Definition of determinants.

Definition. A permutation of a set M is a bijection $M \to M$.

In particular, the set of all permutations of $\{1,\dots,n\}$ is denoted by S_n.

An element $\sigma \in S_n$ is denoted by $\begin{Bmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & & \sigma(n) \end{Bmatrix}$

Its signum is $\operatorname{sgn}(\sigma) = (-1)^{\#\text{inversions}}$, where an "inversion" is a pair i,j such that $i<j$ and $\sigma(i) > \sigma(j)$.

Examples:

- All Elements of S_2 are: $\begin{Bmatrix} 1 & 2 \\ 1 & 2 \end{Bmatrix}$, $\begin{Bmatrix} 1 & 2 \\ 2 & 1 \end{Bmatrix}$

  $\operatorname{sgn}(\cdot) = 1$, $\operatorname{sgn}(\cdot) = -1$

- All elements of S_3 are: $\begin{Bmatrix} 1&2&3 \\ 1&2&3 \end{Bmatrix}$ $\begin{Bmatrix} 1&2&3 \\ 1&3&2 \end{Bmatrix}$ $\begin{Bmatrix} 1&2&3 \\ 2&1&3 \end{Bmatrix}$ $\begin{Bmatrix} 1&2&3 \\ 2&3&1 \end{Bmatrix}$ $\begin{Bmatrix} 1&2&3 \\ 3&2&1 \end{Bmatrix}$ $\begin{Bmatrix} 1&2&3 \\ 3&1&2 \end{Bmatrix}$

  sgn: 1, -1, -1, 1, -1, 1

Definition. Let $A = (a_{ij})_{i,j=1,\dots,n} \in M(n\times n)$. Then:

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$
$$= \text{determinant of } A.$$

$\boxed{\text{Leibniz formula}}$

Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M(2\times 2)$.

$\Rightarrow \boxed{\det A = a_{11}a_{22} - a_{12}a_{21}}$

Coincides with the formula on page 36.

Definition. $A \in M(n\times n)$. For $i,j \in \{1,\dots,n\}$ we define

M_{ij} to be the $(n-1)\times(n-1)$-matrix that is obtained from A by deleting the i-th row and the j-th column.

Matrices of the form M_{ij} are called the minors of A.

The numbers $A_{ij} = (-1)^{i+j}\det(M_{ij})$ are called the cofactors of A.

Now: Determinant of a 3×3-matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}.$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix}$$

$$= a_{11}\det M_{11} - a_{12}\det M_{12} + a_{13}\det M_{13}.$$

"expandir por cofactores en la fila 1".

Similarly we can show:

"expandir por cofactores en la fila 2"

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = -a_{21}\det(M_{21}) + a_{22}\det(M_{22}) - a_{23}\det(M_{23})$$

$$= a_{31}\det(M_{31}) - a_{32}\det(M_{32}) + a_{33}\det(M_{33})$$

"expandir por cofactores en la fila 3".

We can also expand by columns:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}\det(M_{11}) - a_{21}\det(M_{21}) + a_{31}\det(M_{31})$$

$$= -a_{12}\det(M_{12}) + a_{22}\det(M_{22}) - a_{32}\det(M_{32})$$

$$= a_{31}\det(M_{31}) - a_{32}\det(M_{32}) + a_{33}\det(M_{33}).$$

Conclusion: $\forall k = 1,2,3$:

$$\boxed{\det A = \underbrace{\sum_{\ell=1}^{3} a_{k\ell}A_{k\ell}}_{\text{expansion by the }k\text{th line}} = \underbrace{\sum_{\ell=1}^{3} a_{\ell k}A_{\ell k}}_{\text{expansion by the }k\text{th column}}.}$$

It can be shown that this formula is true for any n:

Let $A \in M(n\times n)$. Then: For every $k = 1,\dots,n$

$$\boxed{\det A = \sum_{\ell=1}^{n} a_{k\ell}A_{k\ell} = \sum_{\ell=1}^{n} a_{\ell k}A_{\ell k}.}$$

Proof. Long and involves long calculations, but not too hard. Would be a nice homework.

Examples:

- Expand by first line

$$\det\begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & 4 \\ 5 & 0 & 3\end{pmatrix} = 0\det\begin{pmatrix} 3 & 4 \\ 0 & 3\end{pmatrix} - 1\cdot\det\begin{pmatrix} 2 & 4 \\ 5 & 3\end{pmatrix} + 0\det\begin{pmatrix} 2 & 3 \\ 5 & 0\end{pmatrix}$$
$$= -(6-20) = 14.$$

- Expand by third column

$$\det\begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & 4 & 0 \\ 4 & 3 & 0 & 5 \\ 1 & 0 & 3 & 4\end{pmatrix} = 0\cdot\det(\dots) - 4\det\begin{pmatrix} 1 & 1 & 3 \\ 4 & 3 & 5 \\ 1 & 0 & 4\end{pmatrix} + 0 - 3\det\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 4 & 3 & 5\end{pmatrix}$$

$$= -4\left(\det\begin{pmatrix} 1 & 3 \\ 3 & 5\end{pmatrix} + 4\det\begin{pmatrix} 1 & 1 \\ 4 & 3\end{pmatrix}\right) - 3\left(3\det\begin{pmatrix} 2 & 1 \\ 4 & 3\end{pmatrix} + 5\det\begin{pmatrix} 1 & 1 \\ 2 & 1\end{pmatrix}\right)$$

$$= -4[5-9+4(3-4)] - 3[3(6-4)+5(1-2)]$$

$$= -4[-4-4] - 3[6-5] = 32 - 3 = 29.$$