2. Linear Systems & Matrices.

2.1. m equations with n unknowns; Gauß and Gauß-Jordan elimination.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

(*)

R = row = renglón

C = Column = columna.

Where: a_{ij}, b_j are given.

Idea: Use row operations to simplify the system.

Each row carries information. Clearly, something like:

"Multiply one row by 0" is not allowed because it destroys the Information of that row.

Allowed row operations are such that the original information can be restored.

Allowed operations: Inverse operation:

- Exchange two rows. $R_1 \leftrightarrow R_2$ $R_1 \leftrightarrow R_2$
- Multiply one row by a number $\lambda \in \mathbb{R}\setminus\{0\}$. $R \to \lambda R$, $R \to \frac{1}{\lambda} R$
- Sum a multiple of one row to another. $R_1 \to R_1 + \lambda R_2$. $R_1' \to R_1' - \frac{1}{\lambda} R_2$

  or: $R_1 \to \mu R_1 + \lambda R_2$ $(\mu \neq 0)$

(*) The system is called consistent if it has at least one solution for $(x_1, \dots, x_n)$.

It is called inconsistent if it has no solution.

Example: 3 equations with 3 unknowns.

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ 4x_2 + x_3 &= 5 \end{aligned}\right\} \xrightarrow{R_2 \to R_2 - 2R_1} \left\{\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_2 + 3x_3 &= 1 \\ 4x_2 + x_3 &= 5 \end{aligned}\right.$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \left\{\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_2 + 3x_3 &= 1 \\ -11x_3 &= 1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x_3 &= -1/11 \\ x_2 &= 1 - 3x_3 = 14/11 \\ x_1 &= 1 + x_3 - x_2 = -4/11. \end{aligned}\right.$$

In reality, we need only the coefficients. The system above can be written as:

$$\left(\begin{array}{ccc:c} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 3 \\ 0 & 4 & 1 & 5 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc:c} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 1 & 5 \end{array}\right) \xrightarrow{R_3 \to R_3 - 4R_2} \left(\begin{array}{ccc:c} 1 & 1 & -1 & \\ 0 & 1 & 3 & \\ 0 & 0 & -11 & \end{array}\right.$$

Definition. A ~~matrix~~ m×n-matrix is set of numbers ordered like this:

$$A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

It is called coefficient matrix of the system

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \\ a_{m1}x_1 + \dots + a_{mn}x_m &= b_m \end{aligned}$$

The augmented matrix is $\left(\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$

Special forms of matrices:

- Row-echelon form: (Forma escalonada por renglones)

  i) All rows with only zeros are the last rows.

  ii) The first non-zero element in every row is 1.

  iii) In two following rows which are not zero, the 1 of the higher row is left of the 1 of the lower rows.

- Reduced Row-echelon form (Forma escalonada reducida por renglones)

  On addition to the above:

  iv) All entries above the first 1 in a row are zero.

  pivot = first non-zero element in a row of a reduced-echelon-form-matrix.

Examples. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, ~~Forma~~ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ reduced row echelon form

$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ echelon, but no reduced echelon, form

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ NOT in row echelon form.

Clearly. If a matrix is in (reduced) row echelon form, then the corresponding solution is easy to find.

Example.

* Reduced row-echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right) \Longrightarrow \begin{array}{l} x_1 = b_1 \\ x_2 = b_2 \\ x_3 = b_3 . \end{array}$$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right)$: has solution if and only if $b_3 = 0$.
On this case: $x_3 = b_2$, $x_1 = b_1$, x_2: arbitrary.

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right)$: has solution $\Leftrightarrow b_3 = 0$
On this case:
~~b_3~~ $x_3 = b_2$
$x_1 + x_2 = 1 \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 1-t \\ b_2 \end{pmatrix} \quad (t \in \mathbb{R})$.

---

* Row-echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 5 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right) \rightsquigarrow \begin{array}{l} x_3 = b_3, \\ x_2 = b_2 - 5x_3 \\ x_1 = b_1 - x_2 . \end{array}$$

$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 5 & b_2 \\ 0 & 1 & b_3 \end{array}\right) \rightsquigarrow$ has solution only if $b_3 = x_2 = b_2/5 . \Leftrightarrow b_2 = 5b_3$
In this case: $x_2 = b_3$, $x_1 = b_1 - x_2$.

Gauss elimination (Gauss eliminación).
Use allowed row operations to transform a given matrix in row-echelon form.

Scheme which always works:

① Find first non-zero column.
Bring a row which has a non-zero entry in that ~~col~~ column in the first row.

② Divide first row by its first non-zero entry.

③ Use this row to eliminate all the entries below its first non-zero entry: $\rightsquigarrow \begin{pmatrix} 0 \cdots 1 & * \;\; * \;\; * \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 \cdots & 0 \end{matrix} & \boxed{\begin{matrix} * & \\ & A' \end{matrix}} \end{pmatrix}$

④ Repeat the process with the matrix A'

Gauss-Jordan elimination (Gauss-Jordan eliminación).
Use allowed row operations to transform a given matrix in reduced row-echelon form.

Observation The row-echelon form of a given matrix is not unique.

Ex. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are equivalent.

Important From the row-echelon form (or the reduced row-echelon form) it is immediately clear that ~~exactly~~ exactly one of the following true:
* the system has no solution
* the system has exactly one solution
* the system has no solution

Examples

①

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 12 \\ -x_1 + 2x_2 + 3x_3 &= 15 \\ 3x_1 \quad\quad - x_3 &= 1 . \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 12 \\ -1 & 2 & 3 & 15 \\ 3 & 0 & -1 & 1 \end{array}\right) \xrightarrow[\substack{R_3 \to R_3 + 3R_2 \\ R_2 \to -R_2}]{R_1 \to R_1 + 2R_2} \left(\begin{array}{ccc|c} 0 & 7 & 7 & 42 \\ 1 & -2 & -3 & -15 \\ 0 & 6 & 8 & 46 \end{array}\right) \xrightarrow{\substack{R_1 \to \frac{1}{7}R_1 \\ R_3 \to \frac{1}{2}R_3}} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 1 & -2 & -3 & -15 \\ 0 & 3 & 4 & 23 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 1 & -2 & -3 & -15 \\ 0 & 0 & 1 & 5 \end{array}\right) \longrightarrow \underbrace{\left(\begin{array}{ccc|c} 1 & -2 & -3 & -15 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right)}_{\text{Forma escalonada por renglones.}} \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$$\longrightarrow \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right)}_{\text{Forma escalonado reducida}}$$

$$\Longrightarrow \boxed{x_1 = 2, x_2 = 1, x_3 = 5}$$

②

$$\begin{aligned} 3x_1 - 2x_2 + 3x_3 + 3x_4 &= 3 \\ 2x_1 + 6x_2 + 2x_3 - 9x_4 &= 2 \\ x_1 + 2x_2 + x_3 - 3x_4 &= 1 . \end{aligned}$$

$$\left(\begin{array}{cccc|c} 3 & -2 & 3 & +3 & 3 \\ 2 & 6 & 2 & -9 & 2 \\ 1 & 2 & 1 & -3 & 1 \end{array}\right) \xrightarrow{\substack{R_1 \to R_1 - 3R_3 \\ R_2 \to R_1 - 2R_3}} \left(\begin{array}{cccc|c} 0 & -8 & 0 & 12 & 0 \\ 0 & 2 & 0 & -3 & 0 \\ 1 & 2 & 1 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{\text{Reordering}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 1 \\ 0 & -8 & 0 & 12 & 0 \\ 0 & 2 & 0 & -3 & 0 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{4}R_2} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 2 & 0 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \underbrace{\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)}_{\substack{\text{Forma escalonada} \\ \text{por renglones}}} \xrightarrow{R_1 \to R_1 - R_2} \underbrace{\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)}_{\substack{\text{Forma escalonada p} \\ \text{reducida}}}$$

$$\longrightarrow \boxed{x_1 + x_3 = 1, \ 2x_2 - 3x_4 = 0}$$

In vector form: (x_1, x_2, x_3, x_4) is a solution $:\Leftrightarrow$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix}.$$

③

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 7 \\ 3x_1 + 2x_2 - 2x_3 &= 7 \\ -x_1 + 3x_2 - 3x_3 &= 2 . \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 3 & 2 & -2 & 7 \\ -1 & 3 & -3 & 2 \end{array}\right) \xrightarrow{\substack{R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + 3R_3}} \left(\begin{array}{ccc|c} 0 & 7 & -7 & 11 \\ 0 & 11 & -11 & 13 \\ -1 & 3 & -3 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 \to 11R_1 - 7R_2} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 30 \\ 0 & 11 & -11 & 13 \\ -1 & 3 & -3 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 3 & -2 \\ 0 & 11 & -11 & 13 \\ 0 & 0 & 0 & 30 \end{array}\right).$$

Inconsistente!

$\longrightarrow$ The system has no solution.

2.2. Hogeneous systems of equations.

A homogeneous linear system is a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots &+ a_{1n}x_n = 0 \\ \vdots \qquad\qquad &\qquad\quad \vdots \\ a_{m1}x_1 + \cdots \qquad &+ a_{mn}x_n = 0 \end{aligned}$$

Clearly, it always has the so-called trivial solution $x_1 = x_2 = \cdots = x_n = 0$.

Observations.

- For a homogeneous system, exactly one of the following is true:
  - i) It has only one solution (which then must be the trivial solution)
  - ii) It has infinitely many solutions.
- A homogeneous system with more unknowns than equations always has infinitely many solutions.

  Justification: Consider the augmented matrix of the system and bring it in reduced echelon-form. It will look like this:

$$m\left\{\left(\underbrace{\begin{matrix} \cdots 1 & * & * & 0 & & & \\ 0 & 0 & 0 & 1 & * & * & \cdots \\ \vdots & & & & \vdots & & \\ 0 & & & & & & \end{matrix}}_{n}\;\middle|\;\begin{matrix}0\\0\\\vdots\\0\end{matrix}\right)\right. \qquad n > m.$$

Note: There can be at most m pivots.

$\Rightarrow$ There must be at least one row with a pivot and one element which is not above ~~a~~ a pivot ~~more than two entries $\neq 0$~~.

Take last such row. It gives infinitely many solutions which are consistent with the rest of equations.

Easier proof: Suppose the system has only the trivial solution.

$\Rightarrow$ The reduced row-echelon form must look like this: $\left(\begin{array}{ccc|c} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & \vdots \\ & & 1 & 0 \end{array}\right)$

$\Rightarrow m = n$ ↯. □

2.3. Vectors and matrices.

Definition. Let $n \in \mathbb{N}$. $\mathbb{R}^n := \{(x_1, \dots, x_n) \mid \forall j = 1, \dots, n : x_j \in \mathbb{R}\}$

A vector in $\mathbb{R}^n$ is $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Algebraic operations:

- Sum of two vectors: $\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$
- product with $\lambda \in \mathbb{R}$: $\lambda\vec{x} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$.

Definition. Let $m, n \in \mathbb{N}$. A $(m\times n)$-matrix is an ordered rectangular array of $m \cdot n$ numbers

Notation:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{i=1\dots m \\ j=1,\dots,n}}.$$

The set of all $m\times n$-matrices is denoted by $M(m\times n, \mathbb{R})$ or $M(m\times n)$.

Algebraic operations: $A, B \in M(m\times n)$

- Sum of two matrices: $A + B = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & & a_{mn} + b_{mn} \end{pmatrix}$
- Multiplication by $\lambda \in \mathbb{R}$: $\lambda A = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & & \\ \lambda a_{mn} & \cdots & \lambda a_{mn} \end{pmatrix}$.

Observation: A vector $\vec{x} \in \mathbb{R}^n$ can be viewed as a $1\times n$ matrix.

Proposition. With the above definition of sum and product, $\mathbb{R}^n$ and $M(m\times n)$ are vector spaces. ($\rightarrow$ Commutativity, Associativity, Distributivity hold).

~~2.4 Matrix and~~

2.4. Products of matrices & vectors.

Definition. $A = (a_{ij}) \in M(m\times n)$, $\vec{x} \in \mathbb{R}^n$.

$$\rightarrow \boxed{A\vec{x} := \begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{pmatrix} \in \mathbb{R}^m.}$$

Observation $A\vec{x} = x_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1}\end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ \vdots \\ a_{m2}\end{pmatrix} + \dots + x_n\begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn}\end{pmatrix}$ "weighted sum of the columns of A".

(will be useful later).

Properties: $A, B \in M(m\times n)$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

- $(A+B)\vec{x} = A\vec{x} + B\vec{x}$
- $A(\vec{x}+\vec{y}) = A\vec{x} + B\vec{y}$
- $(\lambda A)\vec{x} = \lambda(A\vec{x})$, $A(\lambda\vec{x}) = \lambda(A\vec{x})$.

Application to systems of linear equations:

Consider the linear system:

$$\left.\begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{matrix}\right\} (*)$$

This can be written as $\boxed{A\vec{x} = \vec{b}}$

where: $A = (a_{ij})$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m\end{pmatrix}$

It follows from the properties above:

If $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{c}$ $\Rightarrow A(\vec{x}+\vec{y}) = \vec{b} + \vec{c}$.

In particular:

Theorem: i) ~~$A\vec{x}$~~ $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$ $\Rightarrow A(\vec{x}-\vec{y}) = 0$

ii) $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{0}$ $\Rightarrow A(\vec{x}+\vec{y}) = \vec{b}$

In other words: i) If $\vec{x}$ and $\vec{y}$ are a solution of $(*)$ then $\vec{x}-\vec{y}$ is a solution of the corresponding homogeneous system.

ii) If $\vec{x}$ is a solution of $(*)$ and $\vec{y}$ is a solution of the corresponding homogeneous system, then $\vec{x}+\vec{y}$ solves the system $(*)$

So far: A matrix A together with $\vec{b} \in \mathbb{R}^n$ represents a linear system of equations and we try to find a solution $\vec{x}$ of $A\vec{x} = \vec{b}$

But now we can also view A as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$:

$$A: \mathbb{R}^n \to \mathbb{R}^m, \quad \vec{x} \longmapsto A\vec{x}.$$

$\rightsquigarrow$ Uniqueness of solutions has to do with injectivity of A

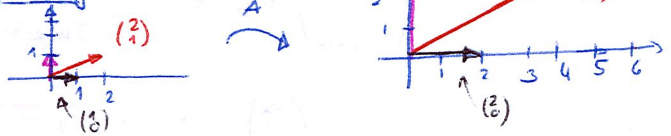
All $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution, is exactly the range of A.

Examples: Let $m = n = 2$.

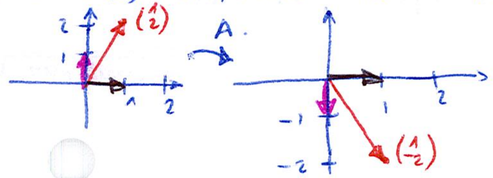
- $A = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \rightarrow \forall \vec{x} \in \mathbb{R}^2$: $A\vec{x} = \vec{x}$.
- $A = \begin{pmatrix} 3 & 0 \\ 0 & 3\end{pmatrix} \Rightarrow \forall \vec{x} = \begin{pmatrix} x_1 \\ x_2\end{pmatrix} \in \mathbb{R}^2$: $A\vec{x} = 3\vec{x}$ $\rightsquigarrow$ scales vector.
- $A = \begin{pmatrix} 2 & 0 \\ 0 & 3\end{pmatrix} \Rightarrow \forall \vec{x} = \begin{pmatrix} x_1 \\ x_2\end{pmatrix} \in \mathbb{R}^2$: $A\vec{x} = \begin{pmatrix} 2x_1 \\ 3x_2\end{pmatrix}$

Note: $A\left(x_1\begin{pmatrix} 1 \\ 0\end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1\end{pmatrix}\right)$ $= 2\begin{pmatrix} 1 \\ 0\end{pmatrix} + 3\begin{pmatrix} 0 \\ 1\end{pmatrix}$

Graphically:

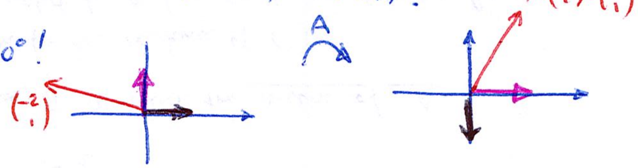


- $A = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} \Rightarrow A\begin{pmatrix} 1 \\ 0\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1\end{pmatrix} = \begin{pmatrix} 0 \\ -1\end{pmatrix}$, $A\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2\end{pmatrix}$ $\rightsquigarrow$ Reflection on x-axis



- $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightsquigarrow A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix},\ A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix},\ A\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_2\\-x_1\end{pmatrix}.$

$\rightsquigarrow$ A is rotation by 90°!



Composition (=Multiplication) of matrices.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A"] \arrow[rr, bend right, "BA"'] & \mathbb{R}^m \arrow[r, "B"] & \mathbb{R}^k \end{tikzcd}$$

Theorem. Let $A \in M(m \times n)$, $B \in M(k, m)$

$\longrightarrow$ ~~A~~ $BA \in M(n, k)$ and

$BA = (c_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,k}}$ with $c_{ij} = \sum_{\ell=1}^{m} b_{i\ell}\, a_{\ell j}$

Proof. $\forall \vec{x} \in \mathbb{R}^n$:

$$BA\vec{x} = B(A\vec{x}) = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{k1} & & b_{km} \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} \sum_{j=1}^{n} a_{1j} x_j + b_{12} \sum_{j=1}^{n} a_{2j} x_j + \dots + b_{1m} \sum_{j=1}^{n} a_{mj} x_j \\ \vdots \\ b_{k1} \sum_{j=1}^{n} a_{1j} x_j + b_{k2} \sum_{j=1}^{n} a_{2j} x_j + \dots + b_{km} \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{(b_{11} a_{11} + b_{12} a_{21} \dots + b_{1m} a_{m1})}^{= c_{11}} x_1 + \overbrace{(b_{11} a_{12} + b_{12} a_{22} + \dots + b_{1m} a_{m2})}^{= c_{12}} x_2 + \dots + \overbrace{(b_{11} a_{1n} + \dots + b_{1m} a_{mn})}^{c_{1n}} x_n \\ \vdots \\ \underbrace{(b_{k1} a_{11} + b_{k2} a_{21} + \dots + b_{km} a_{m1})}_{= c_{k1}} x_1 + \dots \quad + \underbrace{(b_{k1} a_{1n} + \dots + b_{km} a_{mn})}_{c_{kn}} x_n \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \\ c_{k1} & \cdots & c_{kn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

□

Examples:

- $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}1 & 0\\2 & 5\end{pmatrix} = \begin{pmatrix}1+4 & 0+10\\3+8 & 0+20\end{pmatrix} = \begin{pmatrix}5 & 10\\11 & 20\end{pmatrix}.$
- $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\begin{pmatrix}1 & 2 & 5 & 4\\1 & 3 & 3 & 2\\1 & 1 & 1 & 3\end{pmatrix} = \begin{pmatrix}1+2+3 & 2+6+3 & 5+6+3 & 4+4+9\\4+5+6 & 8+15+6 & 20+15+6 & 16+10+18\end{pmatrix} = \begin{pmatrix}6 & 11 & 14 & 17\\15 & 29 & 41 & 44\end{pmatrix}$
- $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}.$

$\boxed{\text{Properties.}}$ A, B, C, D matrices of suitable sizes.

- $A(B+C) = AB + AC;$
  $(B+C)D = BD + CD.$ (Distributivity)
- $A(BD) = (AB)D.$ (Associativity)

Proof. Easy (but long and cumbersome).

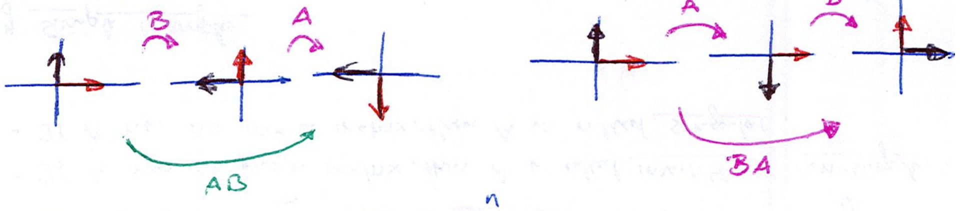
Recall. $B + C = C + B$, but in general $\boxed{BC \neq CB}$!

(If B and C are not square matrices, then one of the products is not even defined!)

Example $A = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ = Reflection on x-axis

$B = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ = Rotation by 90°.

$\Rightarrow AB = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$ $\qquad BA = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = -AB$



Important matrix: $\mathcal{I}_n := \mathbb{1}_n := n\Big\{\overbrace{\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}}^{n}$ = identity matrix.

If n is clear, one writes simply $\mathcal{I}$ or $\mathbb{1}$.

Observe: $\forall \vec{x} \in \mathbb{R}^n$: $\mathcal{I}_n \vec{x} = \vec{x}$.

2.5. Inverse of a quadratic matrix.

Motivating Example. A shop sells two types of packages of sweets:

type A contains 1 piece of chocolate and 3 bubble gums.
type B " 2 pieces " and 1 bubble gum.

(a) How many pieces of chocolate and how many bubble gums are contained in:

(i) 7 boxes of type A, 8 boxes of type B
(ii) 10 —"— , 9, —"—
(iii) 1 " , 13 —"—

Solution Easy:

$$\begin{array}{c} \text{chocolate} \ \ \cancel{\text{bubble gum}} \\ \text{chocolate} \rightarrow \\ \text{bubble gum} \rightarrow \end{array} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \#\text{ boxes of type A} \\ \#\text{ boxes of type B} \end{pmatrix} = \begin{pmatrix} \#\text{ chocolate} \\ \#\text{ bubble gum} \end{pmatrix}.$$

(b) How many boxes do we have to buy of each type if we want:

(i) 5 CH, 15 BG
(ii) 15 CH, 30 BG
(iii) 10 CH, 10 BG
(iv) 9 CH, 13 BG.

Solution. Not so easy because for every item (i)..,(iv) we have to solve a system lin. equations:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} \#\text{ chocolate} \\ \#\text{ bubble gum} \end{pmatrix} \text{ given!}$$

We have to solve for $\vec{x}$ s.t. $A\vec{x} = \vec{b}$.

It would be very useful to have a matrix C s.t. $CA = 1\!\!1$; Because in that case:

$$A\vec{x} = \vec{b} \Rightarrow \underbrace{CA}_{=1\!\!1}\vec{x} = C\vec{b} \Rightarrow \vec{x} = C\vec{b}.$$

If such a matrix exists, it is called the inverse of A.

Observe: Necessary condition for the existence of C:
The linear system associated to A has exactly one solution for any given right hand side $\vec{b}$.

Definition. Let $A \in M(n\times n)$. Suppose that there exists a matrix $B \in M(n\times n)$ s.t. $AB = BA = 1\!\!1$.
Then B is called the inverse matrix of A and it is denoted by A^{-1}.

- If A has an inverse matrix, then A is called invertible or non-singular.
- If A has no inverse matrix, then A is called singular.

Very Simple examples:

- $A = 1\!\!1$. Clearly A is invertible with inverse $1\!\!1$ because $1\!\!1 \cdot 1\!\!1 = 1\!\!1$.
- $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ A is invertible and $A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$
- $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ A is not invertible because: $\forall B \in M(2\times 2)$: $BA = AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq 1\!\!1$.
- $A = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \Rightarrow$ A is not invertible because $\forall B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M(2\times 2)$ $AB = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \neq 1\!\!1$.

Properties of invertible matrices:

Theorem. Let $A \in M(n\times n)$ be an invertible matrix. Then:
(i) A^{-1} is invertible and $(A^{-1})^{-1} = A$.
(ii) The inverse of A is unique.

Proof. (i) By definition of A^{-1}: $AA^{-1} = A^{-1}A = 1\!\!1$
$\Rightarrow A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.
(ii) Let $B, C \in M(n\times n)$ s.t. $AB = BA = 1\!\!1$ and $AC = CA = 1\!\!1$.
$\Rightarrow C = C(AB) = (CA)B = B$.

Theorem Let $A, B \in M(n\times n)$ be invertible matrices. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof.
- $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = 1\!\!1$
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = 1\!\!1$.

$\longrightarrow$ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

$\square$.

Relation between invertibility of a matrix and existence & uniqueness of the associated linear system of equations.

Recall. Let $A \in M(m\times n)$, $\vec{b} \in \mathbb{R}^m$. Solutions of $\boxed{A\vec{x} = \vec{b}}$? $(*)$

(1) $(*)$ has no solution $\Longleftrightarrow$ the reduced row-echelon form of $(A \vdots b)$ has a line of the form $(0 \cdots 0 \vdots \tilde{b})$ with $\tilde{b} \neq 0$.

(2) $(*)$ has at least one sol. $\Leftrightarrow$ the reduced row-echelon form of $(A \vdots b)$ has <u>no</u> line of the form $(0 \cdots 0 \vdots \tilde{b})$ with $\tilde{b} \neq 0$.

In this case:

(2.1) # pivots = # columns $\Rightarrow$ $(*)$ has exactly one solution

(2.2) # pivots < # columns $\Rightarrow$ $(*)$ has infinitely many solutions.

Observe. For a homogeneous system, (1) cannot occure!

Theorem. Let $A \in M(n\times n)$. Then the following is equivalent:

i) A is invertible

ii) $\forall \vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has exactly one solution.

iii) The homogeneous system $A\vec{x} = \vec{0}$ has exactly one solution.

iv) A is "row-equivalent to $\mathbb{1}$" ($\hat{=}$ A can be transformed to $\mathbb{1}$ by row operations)

v) Every row-echelon form of A has n pivots.

Proof. ii) $\Rightarrow$ iii) $\Rightarrow$ v) $\Rightarrow$ iv) $\Rightarrow$ ii) is clear from the "Recall" above.

i) $\Rightarrow$ ii) Let $\vec{b} \in \mathbb{R}^n$. Consider the equation $A\vec{x} = \vec{b}$.

- Existence of solution: Sol $\vec{x} = A^{-1}\vec{b}$ $\Rightarrow$ $A\vec{x} = \vec{b}$.
- Uniqueness: Suppose $\exists \vec{x}, \vec{y}$ s.t. $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$ $\Rightarrow$ $A(\vec{x} - \vec{y}) = \vec{0}$
  $\Rightarrow \vec{x} - \vec{y} = A^{-1}\vec{0} = \vec{0} \Rightarrow \vec{x} = \vec{y}$.

ii) $\Rightarrow$ i) For $k = 1, \dots, n$ let $\vec{e}_k = (\overbrace{0, \dots, 0}^{k-1}, 1, 0, \dots, 0)$ and $\vec{c}_k \in \mathbb{R}^n$ s.t. $A\vec{c}_k = \vec{e}_k$.

Let $C = (\vec{c}_1 \vdots \dots \vdots \vec{c}_n)$ be the matrix whose columns are formed by the vectors $\vec{c}_k$.

Then clearly $AC = \mathbb{1}$.

It remains to show: $CA = \mathbb{1}$. We know: $AC = \mathbb{1} \Rightarrow$

$\Rightarrow ACA = A \Rightarrow A(CA - \mathbb{1}) = 0 \Rightarrow \forall \vec{x} \in \mathbb{R}^n : A(CA - \mathbb{1})\vec{x} = 0$

$\Rightarrow (CA - \mathbb{1})\vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^n$ (because sol. of $A\vec{y} = \vec{0}$ is uniquely $\vec{y} = 0$)

$\Rightarrow CA\vec{x} = \vec{x}$ $(\vec{x} \in \mathbb{R}^n)$ $\Rightarrow CA = \mathbb{1}$.

□

Theorem. Let $A \in M(n\times n)$. Then:

i) If $\exists B \in M(n\times n)$ s.t. $BA = \mathbb{1}$ $\Rightarrow$ A is invertible and $B = A^{-1}$.

ii) If $\exists C \in M(n\times n)$ s.t. $AC = \mathbb{1}$ $\Rightarrow$ A is invertible and $C = A^{-1}$.

Proof. i) Let $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{0} \Rightarrow B A\vec{x} = B\vec{0} \Rightarrow \mathbb{1}\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$.

$\Rightarrow$ the homogeneous system $A\vec{x} = \vec{0}$ has only ones solution.

$\Rightarrow$ A is invertible and $B = B\underbrace{(AA^{-1})}_{=\mathbb{1}} = \underbrace{(BA)}_{=\mathbb{1}}A^{-1} = A^{-1}$.

ii) By (i): C is invertible and $A = C^{-1}$. Then: A is invertible (by Thm. on page 32) and $A^{-1} = (C^{-1})^{-1} = C$.

□

How do we calculate the inverse matrix of a given invertible matrix?

- The proof of Thm. on page 35 says how!
  Let $A \in M(n\times n)$ be an invertible matrix, and let $\vec{e}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← k-th line
- If we have solutions $\vec{c}_k$ of $A\vec{c}_k = \vec{e}_k$ then $C = (\vec{c}_1 \vdots \vec{c}_2 \vdots \dots \vdots \vec{c}_n)$ is the inverse matrix of A.
- How do we obtain the $\vec{c}_k$?
  ⇝ look at $(A \vdots \vec{e}_k)$ and bring it in reduced row-echelon form
  ⇝ we obtain $\left(\begin{array}{ccc|c} 1 & & & \gamma_{1k} \\ & \ddots & & \vdots \\ & & 1 & \gamma_{nk} \end{array}\right)$ (the row-echelon form of A must be $\mathbb{1}$, because A is assumed to be invertible
  $\Rightarrow \vec{c}_k = \begin{pmatrix} \gamma_{1k} \\ \vdots \\ \gamma_{nk} \end{pmatrix}$
- Instead of repeating the matrix reduction n times, we can do it only one time, if we write as right hand side the collection of all vectors $\vec{e}_1, \dots, \vec{e}_n$:

$$\left(\begin{array}{c|cccc} & 1 & 0 & & 0 \\ A & 0 & 1 & \ddots & \vdots \\ & \vdots & \vdots & & 0 \\ & 0 & 0 & & 1 \end{array}\right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|ccc} 1 & & & \gamma_{11} & \cdots & \gamma_{1n} \\ & \ddots & & \vdots & & \\ & & 1 & \gamma_{n1} & \cdots & \gamma_{nn} \end{array}\right)$$

this is A^{-1}!

Examples: Given A, find A^{-1}.

- $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Sol: $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$

$\xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{R_3 \to -\frac{1}{2}R_3} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$

$\Rightarrow A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

Check: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. ✓

$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -2+3 & -4+4 \\ 3/2-3/2 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. ✓

- $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$

Solution. $\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - 5R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$

$\xrightarrow{R_2 \to 4R_2 + 3R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 8 & 0 & -15 & 4 & 3 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to 4R_1 + R_3} \left(\begin{array}{ccc|ccc} 4 & 4 & 0 & -1 & 0 & 1 \\ 0 & 8 & 0 & -15 & 4 & 3 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$

$\xrightarrow{R_1 \to 2R_1 - R_2} \left(\begin{array}{ccc|ccc} 8 & 0 & 0 & 13 & -4 & -1 \\ 0 & 8 & 0 & -15 & 4 & 3 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right)$

Check: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}\begin{pmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}\begin{pmatrix} 13 & -4 & -1 \\ -15 & 4 & 3 \\ 10 & 0 & -2 \end{pmatrix}$

$= \frac{1}{8}\begin{pmatrix} 13-15+10 & 0 & -1+3-2 \\ -30+30 & 8 & 6-6 \\ 5 \cdot 13 - 5 \cdot 15 + 10 & 0 & -5+15-2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \mathbb{1}$. ✓

- $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$.

Solution. $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to 2R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$.

A is not row equivalent to $\mathbb{1}_2$ $\Rightarrow$ A is not invertible. !

Special case: $A \in M(2\times 2)$.

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2\times 2)$.

Case 1: $a \neq 0$ $\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow[R_2 \to aR_2 - cR_1]{} \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right)$ (*)

$\rightsquigarrow$ A is invertible $\Leftrightarrow$ $ad-bc \neq 0$ and:

(*) $\xrightarrow{R_1 \to R_1 - \frac{b}{ad-bc}R_2} \left(\begin{array}{cc|cc} a & 0 & 1+\frac{bc}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & ad-bc & -c & a \end{array}\right)$

$\xrightarrow[R_2 \to \frac{1}{ad-bc}R_2]{R_1 \to \frac{1}{a}R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$

$\Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Case 2. $c \neq 0$ $\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ a & b & 1 & 0 \end{array}\right)$

$\xrightarrow{R_2 \to aR_1 - cR_2} \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & ad-bc & -c & a \end{array}\right)$ (**)

$\rightsquigarrow$ A is invertible if and only if $ad-bc \neq 0$. and:

(**) $\xrightarrow{R_1 \to R_1 - \frac{d}{ad-bc}R_2} \left(\begin{array}{cc|cc} c & 0 & \frac{cd}{ad-bc} & 1-\frac{ad}{ad+bc} \\ 0 & ad-bc & -c & a \end{array}\right)$

$\xrightarrow[R_2 \to \frac{1}{ad-bc}R_2]{R_1 \to \frac{1}{c}R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$

$\Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (as in case 1!)

Case 3 $a = c = 0$. $\rightsquigarrow$ $A = \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$. Clearly in this case, A is not invertible. and $ad-bc = 0$.

Conclusion. Let $A \in M(2\times 2)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det A := ad-bc =:$ determinant of A.

Then: i) A is invertible $\Leftrightarrow$ $\det A \neq 0$.

ii) If A is invertible then $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Check. $\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix} = \mathbb{1}$

$\frac{1}{\det A}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{pmatrix} = \mathbb{1}$.