$\boxed{\text{LINEAR ALGEBRA}}$

Introduction

Linear Algebra started as method to solve linear systems.

Now it appears in a lot of places:

physics, engineering, multivariate statistics,

google page rank; smoothing algorithm for pictures,

approximation of non-linear problems.....

Examples for linear systems:

- (Ex. 30 of Ch. 1.2 in Grossmann).

  A zoo has birds and 4-legged animals

  In total, there are 60 heads and 200 legs.

  How many birds and 4-legged animals are there?

Solution x = number of birds

y = number of 4-legged animals.

We know:
$$\left.\begin{aligned} x+y&=60\\ 2x+4y&=200\end{aligned}\right\} \Rightarrow \begin{aligned} &x = 60-y\\ \Rightarrow\ &2(60-y)+4y = 200\\ \Rightarrow\ &2y = 200-120 = 80\\ \Rightarrow\ &\boxed{y = 40.}\\ \Rightarrow\ &\boxed{x = 20.}\end{aligned}$$

- Find a polynomial P of degree ≤ 3 s.t. $P(0)=1$, $P(1)=7$, $P'(0)=3$, $P'(2)=23$

Solution. We know: $P(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$.

We have to find $\alpha, \beta, \gamma, \delta$.

Note: $P'(x) = 3\alpha x^2 + 2\beta x + \gamma$.

$$\Rightarrow \left.\begin{cases} (1) & \delta = 1\\ (2) & \alpha+\beta+\gamma+\delta = 7\\ (3) & \gamma = 3\\ (4) & 12\alpha+4\beta+\gamma = 23\end{cases}\right\} \underset{\text{in } (2),(4)}{\overset{(1),(3)}{\Longrightarrow}} \begin{cases} (2') & \alpha+\beta = 3\\ (4') & 12\alpha+4\beta = 20.\end{cases}$$

$(2') \Rightarrow \alpha = \cancel{\beta}\ 3-\beta$ in $(4')$: $12(3-\beta)+4\beta = 20 \Rightarrow 8\beta = 56 \Rightarrow \beta = 7.$

$\Rightarrow \alpha = -4,$

$\Rightarrow \boxed{P(x) = -4x^3 + 7x^2 + 3x + 1}$

Important Questions: Existence and uniqueness of solutions ??

Important facts: ($A=B$ and $C=D \Rightarrow A+C = B+D$) and ($A=B \Rightarrow AC = BC$).
"$\Leftarrow$" if $C \neq 0$

Special Case: 2 equations with 2 unknowns.

Examples. ~~x+2y~~

(A) •
$$\begin{aligned} x+2y&=11\\ 3x+4y&=27\end{aligned} \Rightarrow \left.\begin{cases} x = 11-2y\\ \Rightarrow 33-6y+4y = 27\end{cases}\right\} \Rightarrow \begin{cases} x = 11-2y\\ -2y = -6\end{cases}$$

$\Rightarrow \boxed{y=3,\ x=5.}$

Easy to check that this is a solution. (or: convince yourself that all "$\Rightarrow$" in reality are "$\Leftrightarrow$").

$\longrightarrow$ The system has exactly one solution.

(B) •
$$\left.\begin{aligned} (1)\ & x+2y = 1\\ (2)\ & 2x+4y = 5\end{aligned}\right\} \Rightarrow \begin{cases} x = 1-2y\\ \Rightarrow 2-4y+4y = 5 \Rightarrow 2 = 5\ \lightning\end{cases}$$

or: $2\cdot(1)$: $2x+4y = 2$
(2): $2x+4y = 5$ $\lightning$

$\Rightarrow$ The system has no solution.

(C) •
$$\left.\begin{aligned} (1)\ & x+2y = 1\\ (2)\ & 3x+6y = 3\end{aligned}\right\} \quad \begin{aligned} &\Rightarrow x = 1-2y\\ &\text{in } (2)\quad 3-6y+6y = 3 \Rightarrow 3 = 3.\end{aligned}$$

$\Rightarrow$ every pair (x,y) which satisfies (1), automatically satisfies (2)!

$\Rightarrow$ The system has infinitely many solutions.

Geometric interpretation: Suppose that $(A,B) \neq (0,0)$

All x, y which satisfy $Ax + By = U$ lie on a line with slope $-A/B$ if $B \neq 0$, and parallel to y-axis if $B = 0$.

$\leadsto$ all x, y which satisfy simultaneously

$$\left| \begin{array}{l} Ax + By = U \\ Cx + Dy = V \end{array} \right|$$

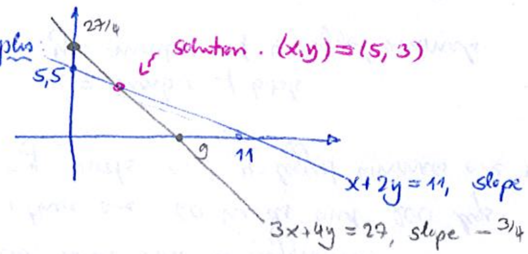
lie on the intersection of two lines.

3 possible cases:

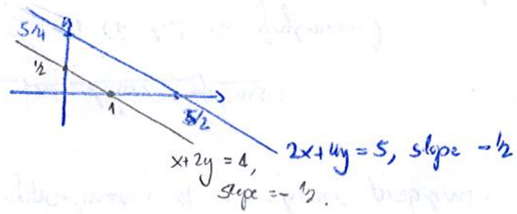
- The lines are not parallel $\Rightarrow$ exactly one intersection
- The lines are parallel, but not equal: $\Rightarrow$ no intersection
- The lines are parallel and equal $\Rightarrow$ infinitely many intersections.

In the examples:

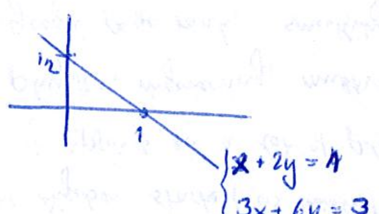
(A)



(B)



(C)



Remark. If $A = B = 0$ then:

Case 1 $Ax + By = 0$ describes the plane $\mathbb{R}^2$

Case 2 $Ax + By = U \neq 0$ has no solution.

Now general case:

$$\begin{array}{ll} ① & Ax + By = U \\ ② & Cx + Dy = V \end{array} \qquad (*)$$

- $A \neq 0 \neq C$. $\leadsto$ $\begin{array}{l} C \cdot ① \\ A \cdot ② \end{array} \left| \begin{array}{ll} ACx + BCy = CU & ①' \\ ACx + ADy = AV. & ②' \end{array} \right.$

$②' - ①' \Leftrightarrow \left\{ \begin{array}{l} ACx + BCy = CU \\ (AD - BC)\, y = AV - CU. \end{array} \right.$

Case 1. $AD - BC \neq 0 \Rightarrow y = \dfrac{AV - CU}{AD - BC}$ and $x = \dfrac{U}{A} - \dfrac{B}{A} y = \dfrac{UD - BV}{AD - BC}$

$\Longrightarrow$ exactly one solution.

Case 2 $AD - BC = 0 \Longrightarrow$ No solution if $AV - CU \neq 0$

inf. many solutions if $AV - CU = 0$

(Choose y arbitrary and $x = \frac{1}{A}(U - By)$).

- $A = 0, C \neq 0$ $\leadsto \left| \begin{array}{l} By = U \\ Cx + Dy = V \end{array} \right|$

Case 1 $B \neq 0 \Rightarrow y = U/B$, $x = \frac{1}{C}(V - D\frac{U}{B})$ $\leadsto$ exactly one solution

Note: in this case: $AD - BC = -BC \neq 0$

Case 2 $B = 0 \Longrightarrow \left\{ \begin{array}{l} \text{no solution if } U \neq 0 \\ \text{inf. many solutions if } U = 0 \end{array} \right|$ in this case: $AD - BC = 0$.

- $A \neq 0, C = 0$: similar
- $A = C = 0$: $\begin{array}{l} By = U \\ Dy = V \end{array} \Longrightarrow$ the original system has no or infinitely many solutions.

Note. $AB - CD = 0$.

Conclusion:

$AD - BC \neq 0 \Leftrightarrow$ exactly one solution
$AD - BC = 0 \Leftrightarrow$ the system has either no solution or infinitely many solutions!

$AB - CD$ se llama el determinante del sistema $(*)$

1. $\mathbb{R}^2$ and $\mathbb{R}^3$.

1.1. Vectors in $\mathbb{R}^2$.

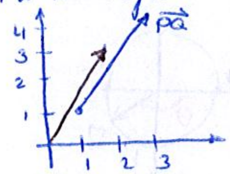
Intuition. A vector in $\mathbb{R}^2$ is "an arrow",
that is: "something that has length and direction"
= "a directed line segment".

Each two arrows with same length and same direction are called equivalent.

The set of all equivalent "arrows" is called a vector.
Each arrow in such a set is called a representant (representación del vector).

Special representant: Arrow whose initial point is the origin.

Example: $P, Q \in \mathbb{R}^2$, $P(1,1)$, $Q(3,4)$.



$\Rightarrow$ Every vector can be identified with a point in $\mathbb{R}^2$
Every point P in $\mathbb{R}^2$ can be identified with the vector $\vec{OP}$.

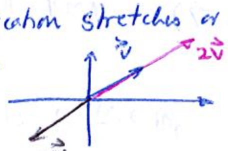
Notation $P(a,b) \rightsquigarrow \vec{v} = \vec{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$ or $\vec{v} = (a,b)$.

Examples of vectors: Forces; velocities, ...

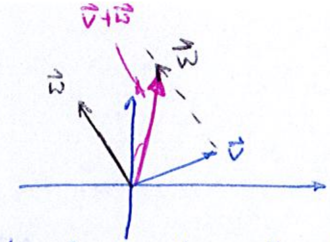
Vector space properties of $\mathbb{R}^2$:

- Sum of vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$: $\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \in \mathbb{R}^2$
- Multiplication by $\lambda \in \mathbb{R}$: $\lambda \vec{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$.

Graphically: • Multiplication stretches or shrinks; always: $\vec{v}$ and $\lambda\vec{v}$ are parallel.



- Sum of two vectors:



Examples: • Body with two forces acting on it:

• Superposition of velocities: Moving body on a moving train
$\vec{v}_{total} = \vec{v}_{train} + \vec{v}_{body\ on\ train}$.

Vector space properties Let $\vec{u}, \vec{v} \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

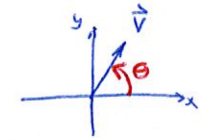
- Associativity: $(\vec{u}+\vec{v})+\vec{w} = \vec{u} + (\vec{v}+\vec{w})$
- Commutativity: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- Identity element: $\exists \vec{0}$ tq. $\forall \vec{v} \in V \quad 0 + \vec{v} = \vec{v}$
- Inverse element: $\forall \vec{v} \in V \ \exists \hat{\vec{v}} \in V$ st. $\vec{v} + \hat{\vec{v}} = \vec{0}$
- Mult by id: $1 \cdot \vec{v} = \vec{v} \ (\vec{v} \in V)$
- Compatibility: $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$
- Distributivity: $(\lambda+\mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$
  $\lambda(\vec{v}+\vec{u}) = \lambda\vec{v} + \lambda\vec{u}$.

Easy to show: $0\vec{v} = \vec{0}$; $-\vec{v} = -1 \cdot \vec{v}$.

Clearly, a vector is known, if we know its length and its angle with the positive x-axis.

Definition. $\vec{v} \in \mathbb{R}^2$; $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$.

$\Rightarrow \|\vec{v}\| := \sqrt{a^2+b^2}$ = length/longitud/norm of $\vec{v}$



How to find θ: Always $\tan\theta = b/a$ if $a \neq 0$

$$\Rightarrow \boxed{\theta = \begin{cases} \arctan(b/a), & a > 0 \\ \pi - \arctan(b/a), & a < 0 \\ \pi/2 & a = 0,\ b > 0 \\ -\pi/2 & a = 0,\ b < 0 \end{cases}} \rightsquigarrow \text{gives } \theta \in (-\pi, \pi]$$