Linear Algebra

Analysis Series

M. Winklmeier

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Contents

Chapter 4

Vector spaces and linear maps

In the following, $\mathbb{K}$ always denotes a field. You may always think of $\mathbb{K} = \mathbb{R}$, though almost everything is true also for other fields, like $\mathbb{C}$ or $\mathbb{F}_p$ where p is a prime number.

4.6 Linear maps

Definition 4.1. Let U, V be vector spaces. A function $A : U \to V$ is called a *linear map* (or *linear function* or *linear operator*) if for all $x, y \in U$ and $\lambda \in \mathbb{K}$ the following is true:

$$A(x+y) = Ax + Ay, \quad A(\lambda x) = \lambda Ax. \tag{4.1}$$

Remark 4.2. (i) Clearly, (4.1) is equivalent to

$$A(x + \lambda y) = Ax + \lambda Ay$$

for all $x, y \in U$ and $\lambda \in \mathbb{K}$.

(ii) It follows immediately from the definition that

$$A(\lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda_1 Av_1 + \cdots + \lambda_k Av_k$$

for all $v_1, \ldots, v_k \in V$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$.

(iii) The condition (4.1) says that a linear map respects the vector space structures of its domain and its target space.

Examples 4.3 (linear maps). (i) Every matrix $A \in M(m \times n)$ can be identified with a linear map $\mathbb{R}^n \to \mathbb{R}^m$.

(ii) Differentiation is a linear map, for example

 (a) $T : C^1(\mathbb{R}) \to C(\mathbb{R}),\ Tf = f'$,
 where $C^1(\mathbb{R})$ is the space of continuously differentiable functions.

 (b) $T : P_n \to P_{n-1},\ Tf = f'$.

(iii) Integration is a linear map. For example:

$$I : C([0,1]) \to C([0,1]),\ f \mapsto If \qquad \text{where } (If)(t) = \int_0^t f(s)\ \mathrm{d}s.$$

Lemma 4.4. *If A is a linear map, then $A0 = 0$.*

Proof. $0 = A0 - A0 = A(0-0) = A0$. □

Definition 4.5. Let $A : U \to V$ be a linear map.

(i) A is called *injective* (or *one-to-one*) if

$$x, y \in U,\ x \neq y \quad \Longrightarrow \quad Ax \neq Ay.$$

(ii) A is called *surjective* if for all $v \in V$ exists at least one $x \in U$ such that $Ax = v$.

(iii) A is called *bijective* if it is injective and surjective.

(iv) The *kernel of A* (or *null space of A*, *espacio nulo de A)* is

$$\ker(A) := \{x \in U : Ax = 0\}.$$

Sometimes the notations $N(A)$ or N_A instead of $\ker(A)$ are used.

(v) The *image of A* (or *range of A*, *imagen de A)* is

$$\mathrm{Im}(A) := \{v \in V : y = Ax \text{ for some } y \in U\}.$$

Sometimes the notations $\mathrm{Rg}(A)$ or $\mathrm{R}(A)$ instead of $\mathrm{Im}(A)$ are used.

Remark 4.6. (i) Observe that $\ker(A)$ is a subset of U, $\mathrm{Im}(A)$ is a subset of V. In Proposition 4.9 we will show that they are even subspaces.

(ii) It follows immediately from the definition that A is surjective if and only if $\mathrm{Im}(A) = V$.

(iii) Clearly, A is injective if and only if for all $x, y \in U$ the following is true:

$$Ax = Ay \quad \Longrightarrow \quad x = y.$$

(iv) If A is a linear injective map, then its inverse $A^{-1} : \mathrm{Im}(A) \to U$ exists and is linear too.

The following lemma is very useful.

Lemma 4.7. *A linear map A is injective if and only if* $\ker(A) = \{0\}$.

Proof. By Lemma 4.4, we always have $0 \in \ker(A)$. Assume that A is injective, then $\ker(A)$ cannot contain any other element, hence $\ker(A) = \{0\}$.
Now assume that $\ker)(A) = \{0\}$ and let $x, y \in U$ with $Ax = Ay$. By Remark 4.6 it is sufficient to show that $x = y$. By assumption, $0 = Ax - Ay = A(x - y)$, hence $x - y \in \ker(A) = \{0\}$. Therefore $x - y = 0$, which means that $x = y$. $\square$

Examples 4.8. (i) Let $A \in M(m \times n)$ with $m < n$. Then A cannot be injective.

(ii) Let $T : C^1(\mathbb{R}) \to C(\mathbb{R}), Tf = f'$ the operator of differentiation from Example 4.3. Then it is easy to see that the kernel of T consists exactly of the constant functions and the T is surjective.

Proposition 4.9. *Let $A : U \to V$ be a linear map. Then*

(i) $\ker(A)$ *is a subspace of* U.

(ii) $\operatorname{Im}(A)$ *is a subspace of* V.

Proof. (i) Let $x, y \in \ker(A)$ and $\lambda \in \mathbb{K}$. Then

$$A(x + \lambda y) = Ax + \lambda Ay = 0 + \lambda 0 = 0,$$

hence $x + \lambda y \in \ker(A)$.

(ii) Let $v, w \in \operatorname{Im}(A)$ and $\lambda \in \mathbb{K}$. Then there exist Let $x, y \in U$ such that $Ax = v$ and $Ay = y$. Then $v + \lambda w = Ax + \lambda Ay = A(x + \lambda y) \in \operatorname{Im}(A)$. hence $v + \lambda w \in \operatorname{Im}(A)$. $\square$

Since we now know that $\ker(A)$ and $\operatorname{Im}(A)$ are subspaces, the following definition makes sense.

Definition 4.10. Let $A : U \to V$ be a linear map. We define

$$\dim(\ker(A)) = \textit{nullity of } A, \qquad \dim(\operatorname{Im}(A)) = \textit{rank of } A.$$

Sometimes the notations $\nu(A) = \dim(\ker(A))$ and $\rho(A) = \dim(\operatorname{Im}(A))$ are used.

Proposition 4.11. *Let U, V be $\mathbb{K}$-vector spaces, $A : U \to V$ a linear map. Let $x_1, \dots, x_k \in U$ and set $y_1 := Ax_1, \dots, y_k := Ax_k$. Then the following is true.*

(i) *If the $x_1, \dots, x_k$ are linearly dependent, then $y_1, \dots, y_k$ are linearly dependent too.*

(ii) *If the $y_1, \dots, y_k$ are linearly independent, then $x_1, \dots, x_k$ are linearly independent too.*

(iii) *Suppose additionally that A invertible. Then $x_1, \dots, x_k$ are linearly independent if and only if $y_1, \dots, y_k$ are linearly independent.*

Remark 4.12. In general the implication "If $x_1, \dots, x_k$ are linearly independent, then $y_1, \dots, y_k$ are linearly independent." is *false*.

Proof of Proposition 4.11. (i) Assume that $x_1, \dots, x_k$ are linearly dependent. Then there exist $\lambda_1, \dots, \lambda_k \in \mathbb{K}$ such that $\lambda_1 x_1 + \dots + \lambda_k x_k = 0$ and at least one $\lambda_j \neq 0$. But then

$$\begin{aligned} 0 = A0 &= A(\lambda_1 x_1 + \dots + \lambda_k x_k) = \lambda_1 A x_1 + \dots + \lambda_k A x_k \\ &= \lambda_1 y_1 + \dots + \lambda_k y_k, \end{aligned}$$

hence $y_1, \dots, y_k$ are linearly dependent.

(ii) follows directly from (i).

(iii) follows from (i) and (ii). $\square$

Theorem 4.13. *Let U, V be finite-dimensional $\mathbb{K}$-vector spaces and let $A : U \to V$ a linear map. Moreover, let $E : U \to U$, $F : V \to V$ be linear bijective maps. Then the following is true:*

(i) $\operatorname{Im}(A) = \operatorname{Im}(AE)$, *in particular* $\dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(AE))$.

(ii) $\ker(AE) = E^{-1}(\ker(A))$ *and* $\dim(\ker(A)) = \dim(\ker(AE))$.

(iii) $\ker(A) = \ker(FA)$, *in particular* $\dim(\ker(A)) = \dim(\ker(FA))$.

(iv) $\operatorname{Im}(FA) = F(\operatorname{Im}(A))$ *and* $\dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(FA))$.

In summary we have

$$\boxed{\begin{array}{ll} \ker(FA) = \ker(A), & \ker(AE) = E^{-1}(\ker(A)), \\ \operatorname{Im}(FA) = F(\operatorname{Im}(A)), & \operatorname{Im}(AE) = \operatorname{Im}(A). \end{array}} \tag{4.2}$$

and

$$\boxed{\begin{array}{c} \dim\ker(A) = \dim\ker(FA) = \dim\ker(AE) = \dim\ker(FAE), \\ \dim\operatorname{Im}(A) = \dim\operatorname{Im}(FA) = \dim\operatorname{Im}(AE) = \dim\operatorname{Im}(FAE). \end{array}} \tag{4.3}$$

Remark 4.14. In general, $\ker(A) = \ker(AE)$ and $\ker(A) = \ker(FA)$ is false. Take for example $U = V = \mathbb{R}^2$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $E = F = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then clearly the hypotheses of the theorem are satisfied and

$$\ker(A) = \operatorname{gen}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad \operatorname{Im}(A) = \operatorname{gen}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

but

$$\ker(AE) = \operatorname{gen}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}, \qquad \operatorname{Im}(FA) = \operatorname{gen}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

Remark 4.15. The theorem is also true for infinite dimensional vector spaces, but the proofs of (ii) and (iv) must be changed a little bit.

Proof of Theorem 4.13. (i) Let $v \in V$. If $v \in \operatorname{Im}(A)$, then there exists $x \in U$ such that $Ax = v$. Set $y = E^{-1}x$. Then $v = Ax = AEE^{-1}x = AEy \in \operatorname{Im}(AE)$. On the other hand, if $v \in \operatorname{Im}(AE)$, then there exists $y \in U$ such that $AEy = v$. Set $x = E$. Then $v = AEy = Ax \in \operatorname{Im}(A)$.

(ii) To show $\ker(AE) = E^{-1}\ker(A)$ observe that

$$\ker(AE) = \{x \in U : Ex \in \ker(A)\} = \{E^{-1}u : u \in \ker(A)\} = E^{-1}(\ker(A)).$$

Now let $k = \dim(\ker(A))$ and $\ell = \dim(\ker(AE))$.

Choose a basis $u_1, \dots, u_k$ of $\ker(A)$ and set $w_1 := E^{-1}u_1, \dots, w_k := E^{-1}u_k$. Then the $u_1, \dots, u_k$ are linearly independent and for every $j = 1, \dots, k$ we have that $AEw_j = AE(E^{-1}u_j) = Au_j = 0$, hence all w_j belong to $\ker(AE)$. They are also linearly independent by Proposition 4.11, so we must have that $\ell = \dim(\ker(AE)) \geq \dim(\ker(A)) = k$.

Now choose a basis $w_1, \dots, w_k$ of $\ker(AE)$ and set $u_1 := w_1, \dots, u_k := w_k$. Then the $w_1, \dots, w_k$ are linearly independent and for every $j = 1, \dots, k$ we have that $Au_j = AEE^{-1}u_j = AEw_j = 0$, hence all u_j belong to $\ker(A)$. They are also linearly independent by Proposition 4.11, so we must have that $k = \dim(\ker(A)) \geq \dim(\ker(AE)) = \ell$.

In conclusion, we found that $\ell \geq k$ and $k \geq \ell$, so we must have $k = \ell$ as we wanted to prove.

(iii) Let $x \in U$. Then $x \in \ker(FA)$ if and only if $FAx = 0$. Since F is injective, we know that $\ker(F) = \{0\}$, hence it follows that $Ax = 0$. But this is equivalent to $x \in \ker(A)$.

(iv) To show $\operatorname{Im}(FA) = F\operatorname{Im}(A)$ observe that

$$\begin{aligned}\operatorname{Im}(FA) &= \{y \in V : y = FAx \text{ for some } x \in U\} = \{Fv : v \in \operatorname{Im}(A)\}\\ &= F(\operatorname{Im}(A)),\end{aligned}$$

Now let $k = \dim(\ker(A))$ and $\ell = \dim(\ker(AE))$. In order to prove $k = \ell$, we will show that $\ell \geq k$ and $k \geq \ell$.

Choose a basis $v_1, \dots, v_k$ of $\operatorname{Im}(FA)$ and choose $x_1, \dots, x_k \in U$ such that $v_1 = FAx_1, \dots, v_k := FAx_k$. Set $z_1 := F^{-1}v_1, \dots, z_k := F^{-1}v_k$. Then for every $j = 1, \dots, k$ we have that $z_j = F^{-1}v_j = F^{-1}FAx_j = Ax_j \in \operatorname{Im}(A)$, hence all z_j belong to $\operatorname{Im}(A)$. They are also linearly independent by Proposition 4.11 because the $v_1, \dots, v_k$ are so. Therefore we must have that $\ell = \dim(\operatorname{Im}(A)) \geq \dim(\operatorname{Im}(FA)) = k$.

Now choose a basis $z_1, \dots, z_k$ of $\mathrm{Im}(A)$ and choose $x_1, \dots, x_k \in U$ such that $z_1 = Ax_1, \dots, z_k := Ax_k$. Set $v_1 := Fz_1, \dots, v_k := Fz_k$. Then for every $j = 1, \dots, k$ we have that $v_j = Fz_j = FAx_j =\in \mathrm{Im}(FA)$, hence all v_j belong to $\mathrm{Im}(FA)$. They are also linearly independent by Proposition 4.11 because the $z_1, \dots, z_k$ are so. Therefore we must have that $k = \dim(\mathrm{Im}(FA)) \geq \dim(\mathrm{Im}(A)) = \ell$.

In conclusion, we found that $\ell \geq k$ and $k \geq \ell$, so we must have $k = \ell$ as we wanted to prove.

□

4.7 Matrices as linear maps

Let $\in M(m \times n)$. We already know that we can view A as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Hence $\ker(A)$ and $\mathrm{Im}(A)$ and the terms *injectivity* and *surjectivity* are defined.

If we view the matrix A at the same time as a linear system of equations, then we obtain the following.

Remark 4.16.

(i) $\ker(A) =$ all solutions of the homogeneous system $A\vec{x} = \vec{0}$.

(ii) A is injective
$\iff$ $\ker(A) = \{0\}$
$\iff$ the homogenous system $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

(iii) $\mathrm{Im}(A) =$ all vectors $\vec{b}$ such that the system $A\vec{x} = \vec{b}$ has a solution.

(iv) A is surjective
$\iff$ $\mathrm{Im}(A) = \mathbb{R}^m$
$\iff$ for every $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} = \vec{b}$ has at least one solution.

Definition 4.17. Let $A \in M(m \times n)$ and let $\vec{c}_1, \dots, \vec{c}_n$ be the columns of A and $\vec{r}_1, \dots, \vec{r}_m$ be the rows of A. We define

(i) $C_A := \mathrm{gen}\{\vec{c}_1, \dots, \vec{c}_m\} =:$ *column space of* A.

(ii) $R_A := \mathrm{gen}\{\vec{r}_1, \dots, \vec{r}_n\} =:$ *row space of* A,

Observe that $\vec{c}_1, \dots, \vec{c}_n \in \mathbb{R}^m$ and $\vec{r}_1, \dots, \vec{r}_m \in \mathbb{R}^n$.

It follows immediately from the definition above that

$$R_A = C_{A^t} \quad \text{and} \quad C_A = R_{A^t}. \tag{4.4}$$

Proposition 4.18. *$C_A = \mathrm{Im}(A)$, $R_A = \mathrm{Im}(A^t)$.*

Proof. Let $\vec{y} \in \mathbb{R}^m$. Then:

$$\begin{aligned}
\vec{y} \in \operatorname{Im}(A) \quad &\iff \text{ exists } \vec{x} \in \mathbb{R}^n \text{ such that } \vec{y} = A\vec{x} = (\vec{c}_1|\dots|\vec{c}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&\qquad\qquad\qquad\qquad\qquad\qquad = x_1\vec{c}_1 + \dots x_n\vec{c}_n \\
&\iff \vec{y} \in \operatorname{gen}\{\vec{c}_1, \dots, \vec{c}_n\} = C_A.
\end{aligned}$$

This shows $C_A = \operatorname{Im}(A)$. From this is follows that $R_A = C_{A^t} = \operatorname{Im}(A^t)$. □

The next theorem follows easily from the general theory in Section 4.6. We will give another proof at the end of this section.

Proposition 4.19. *Let $A \in M(m \times n)$, $E \in M(n \times n)$, $F \in M(m \times m)$ and assume that E and F are invertible. Then*

(i) $C_A = C_{AE}$.

(ii) $R_A = R_{FA}$.

Proof. (i) Note that $C_A = \operatorname{Im}(A) = \operatorname{Im}(AE) = C_{AE}$, where in the first and third equality we used Proposition 4.18, and in the second equality we used Theorem 4.13.

(ii) Recall that, if F is invertible, then F^t is invertible too. With (4.4) and what we already proved in (i), we obtain $R_{FA} = C_{(FA)^t} = C_{A^tF^t} = C_{A^t} = R_A$.

□

This theorem implies immediately the following proposition.

Proposition 4.20. *Let $A, B \in M(m \times n)$.*

(i) *If A and B are row equivalent, then*

$$\dim(\ker(A)) = \dim(\ker(B)), \quad \dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(B)),$$
$$\operatorname{Im}(A^t) = \operatorname{Im}(B^t), \quad R_A = R_B.$$

(ii) *If A and B are column equivalent, then*

$$\dim(\ker(A)) = \dim(\ker(B)), \quad \dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(B)),$$
$$\operatorname{Im}(A) = \operatorname{Im}(B), \quad C_A = C_B.$$

Proof. We will only prove (i). The claim (ii) can be proved similar (or can be deduced easily from (i) by applying (i) to the transposed matrices). If A and A are row equivalent, then there are elementary matrices $F_1, \dots, F_k \in M(m \times m)$ such that $A = F_1 \dots F_k B$. Not that all F_j are invertible. Let $F := F_1 \dots F_k$. Then F is invertible and $A = FB$. Hence all the claims in (i) follow from Theorem 4.13 and Proposition 4.19. □

The proposition above is very useful to calculate the kernel of a matrix A: Let $\widetilde{A}$ be the reduced row-echelon form of A. Then the proposition can be applied to A and $\widetilde{A}$ (for B), and we find that $\ker(A) = \ker(\widetilde{A})$. Note that determining the kernel of a matrix in reduces row-echelon form is in general very easy.
Now we will prove to technical lemmas.

Lemma 4.21. *Let $A \in M(m \times n)$. Then there exist elementary matrices $E_1, \dots, E_k \in M(n \times n)$ and $F_1, \dots, F_\ell \in M(m \times m)$ such that*

$$F_1 \cdots F_\ell A E_1 \cdots E_k = A''$$

where A'' is of the form

$$A'' = \begin{array}{cl} \begin{array}{cc} \overbrace{\qquad\qquad}^{r} & \overbrace{\qquad\quad}^{n-r} \end{array} & \\ \left(\begin{array}{c:c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 \\ \hdashline \\ 0 & 0 \end{array}\right) & \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \\1\end{matrix}}\right\} r \\ \left.\vphantom{\begin{matrix}0\\0\end{matrix}}\right\} m-r \end{matrix} \end{array} \tag{4.5}$$

Proof. Let A' be the reduced row-echelon form of A. Then there exist $F_1, \dots, F_\ell \in M(m \times m)$ such that $F_1 \cdots F_\ell A = A'$ and A' is of the form

$$A' = \begin{pmatrix} 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots \\ & & & & 1 & * & \cdots & * & 0 & * & \cdots \\ & & & & & & & & 1 & * & \cdots \\ & & & & & & & & & & \cdots \\ \\ \\ \end{pmatrix}. \tag{4.6}$$

Now clearly we can find "allowed" column transformations such that A' is transformed into the form A''. If we observe that applying row transformations is equivalent to multiply A' from the right by elementary matrices. ☐

Lemma 4.22. *Let A'' be as in (4.5). Then*

(i) $\dim(\ker(A)) = m - r =$ *number of zero rows of* A'',

(ii) $\dim(\mathrm{Im}(A)) = r =$ *number of pivots* A'',

(iii) $\dim(C_{A''}) = \dim(R_{A''}) = r$.

Proof. All assertions are clear if we note that

$$\ker(A'') = \mathrm{gen}\{e_{r+1}, \dots, e_n\}, \quad \mathrm{Im}(A'') = \mathrm{gen}\{e_1, \dots, e_r\},$$

where the e_j are the standard unit vectors (that is, their jth component is 1 and all other components are 0). ☐

Proposition 4.23. *Let $A \in M(m \times n)$ and let A' be its reduced row-echelon form. Then*

$$\dim(\mathrm{Im}(A)) = \textit{ number of pivots of } A'.$$

Proof. Let $F_1, \dots, F_\ell$, $E_1, \dots, E_k$ and A'' be as in (4.21) and set $F := F_1 \cdots F_\ell$ and $E := E_1 \cdots E_k$. It follows that $A' = FA$ and $A'' = FAE$. Clearly, the number of pivots of A' and A'' coincide. Therefore, with the help of Theorem 4.13 we obtain

$$\begin{aligned}\dim(\mathrm{Im}(A)) &= \dim(\mathrm{Im}(FAE)) \\ &= \text{ number of pivots of } A'' \\ &= \text{ number of pivots of } A'.\end{aligned}$$ □

Proposition 4.24. *Let $A \in M(m \times n)$. Then*

$$\dim(\mathrm{Im}(A)) = \dim C_A = \dim R_A.$$

That means: (rank of row space) = (rank of column space).

Proof. Since $C_A = \mathrm{Im}(A)$ by Proposition 4.18, the first equality is clear.

Now let $F_1, \dots, F_\ell$, $E_1, \dots, E_k$ and A', A'' be as in Lemma 4.21 and set $F := F_1 \cdots F_\ell$ and $E := E_1 \cdots E_k$. Then

$$\begin{aligned}\dim(R_A) &= \dim(R_{FAE}) = \dim(R_{A''}) = r = \dim(C_{A''}) = \dim(C_{FAE}) \\ &= \dim(C_A).\end{aligned}$$ □

As an immediate consequence we obtain

Theorem 4.25. *Let $A \in M(m \times n)$. Then*

$$\boxed{\dim(\ker(A)) + \dim(\mathrm{Im}(A)) = n.} \tag{4.7}$$

Proof. With the notation a above, we obtain

$$\begin{aligned}\dim(\ker(A)) &= \dim(\ker(A'')) = n - r, \\ \dim(\mathrm{Im}(A)) &= \dim(\mathrm{Im}(A'')) = r\end{aligned}$$

and the desired formula follows. □

For the calculation of a basis of $\mathrm{Im}(A)$, the following theorem is useful.

Theorem 4.26. *Let $A \in M(m \times n)$ and let A' be its reduced row-echelon form with columns $\vec{c}_1, \dots, \vec{c}_n$ and $\vec{c}_1{}', \dots, \vec{c}_n{}'$ respectively. Assume that the pivot columns of A' are the columns $j_1 < \cdots < j_k$. Then $\dim(\mathrm{Im}(A)) = k$ and a basis of $\mathrm{Im}(A)$ is given by the columns $\vec{c}_{j_1}, \dots, \vec{c}_{j_k}$ of A.*

Proof. Let E be an invertible matrix such that $A = EA'$. By assumption on the pivot columns of A', we know that $\dim(\operatorname{Im}(A')) = k$ and that a basis of $\operatorname{Im}(A')$ is given by the columns $\vec{c}_{j_1}{}', \dots, \vec{c}_{j_k}{}'$. By Theorem 4.13, it follows that $\dim(\operatorname{Im}(A)) = \dim(\operatorname{Im}(A')) = k$. Now observe that definition of E we have that $E\vec{c}_\ell{}' = \vec{c}_\ell$ for every $\ell = 1, \dots, n$ and in particular this is true for the pivot columns of A'. Moreover, since E in invertible and the vectors $\vec{c}_{j_1}{}', \dots, \vec{c}_{j_k}{}'$ are linearly independent, it follows from Theorem 4.11 that the vectors $\vec{c}_{j_1}, \dots, \vec{c}_{j_k}$ are linearly independent. Clearly they belong to $\operatorname{Im}(A)$, so we have $\operatorname{gen}\{\vec{c}_{j_1}, \dots, \vec{c}_{j_k}\} \subseteq \operatorname{Im}(A)$. Since both spaces have the same dimension, they must be equal. □

Remark 4.27. The theorem above can be used to determine a basis of a subspace given in the form $U = \operatorname{gen}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^m$ as follows: Define the matrix $A = (\vec{v}_1|\dots|\vec{v}_k)$. Then clearly $U = \operatorname{Im} A$ and we can apply Theorem 4.25 to find a basis of U.

Example 4.28. Find $\ker(A)$, $\operatorname{Im}(A)$, $\dim(\ker(A))$, $\dim(\operatorname{Im}(A))$ and R_A for

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 3 & 2 & 13 & 1 \\ 0 & 2 & 4 & -1 \\ 4 & 5 & 22 & 1 \end{pmatrix}.$$

Solution. First, let us row-reduce the matrix A:

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 3 & 2 & 13 & 1 \\ 0 & 2 & 4 & -1 \\ 4 & 5 & 22 & 1 \end{pmatrix} \xrightarrow[Q_{41}(-4)]{Q_{21}(-1)} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 2 & 4 & -1 \\ 0 & 1 & 2 & -3 \end{pmatrix}$$

$$\xrightarrow[Q_{42}(1)]{Q_{32}(2)} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -5 \end{pmatrix} \xrightarrow[Q_{43}(-1)]{S_2(-1)} \begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[Q_{12}(-1)]{S_4(1/5)} \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[Q_{24}(-2)]{Q_{14}(1)} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: A'.$$

Now it follows immediately that $\dim R_A = \dim C_A = 3$ and

$$\begin{aligned} \dim(\operatorname{Im}(A)) &= \#\text{non-zero rows of } A' = 3, \\ \dim(\ker(A)) &= 4 - \dim(\operatorname{Im}(A)) = 1 \end{aligned}$$

(or: $\dim(\operatorname{Im}(A)) = \#\text{pivot columns } A' = 3$, or: $\dim(\operatorname{Im}(A)) = \dim(R_A) = 3$ or: $\dim(\ker(A)) = \#\text{non-pivot columns } A' = 1$).

Kernel of A: We know that $\ker(A) = \ker(A')$ by Theorem 4.13 or Proposition 4.20. From the explicit form of A', it is clear that $A\vec{x} = 0$ if and only if $x_4 = 0$, x_3 arbitrary, $x_2 = -2x_3$ and $x_1 = -3x_3$. Therefore

$$\ker(A) = \ker(A') = \left\{ \begin{pmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{pmatrix} : x_3 \in \mathbb{R} \right\} = \operatorname{gen} \left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Image of A: The pivot columns of A' are the columns 1, 2 and 4. Therefore, by Theorem 4.26 a basis of $\operatorname{Im}(A)$ are the columns 1, 2 and 4 of A:

$$\operatorname{Im}(A) = \operatorname{gen} \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Example 4.29. Find a basis of $\operatorname{gen}\{p_1, p_2, p_3, p_4\} \subseteq P_3$ and its dimension for

$$p_1 = x^3 - x^2 + 2x + 2, \qquad p_2 = x^3 + 2x^2 + 8x + 13,$$
$$p_3 = 3x^3 - 6x^2 - 5, \qquad p_3 = 5x^3 + 4x^2 + 26x - 9.$$

Solution. First we identify P_3 with $\mathbb{R}^4$ by $ax^3 + bx^2 + cx + d \mathrel{\widehat{=}} (a, b, c, d)^t$. The polynomials p_1, p_2, p_3, p_4 correspond to the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 8 \\ 13 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 3 \\ -6 \\ 0 \\ -5 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 5 \\ 4 \\ 26 \\ -9 \end{pmatrix}.$$

Now we use Remark 4.27 to find a basis of $\operatorname{gen}\{v_1, v_2, v_3, v_4\}$. To this end we consider the A whose columns are the vectors $\vec{v}_1, \dots, \vec{v}_4$:

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 \\ -1 & 2 & -6 & 4 \\ 2 & 8 & 0 & 26 \\ 2 & 13 & -5 & -9 \end{pmatrix}$$

Clearly, $\operatorname{gen}\{v_1, v_2, v_3, v_4\} = \operatorname{Im}(A)$, so it suffices to find a basis of $\operatorname{Im}(A)$. Applying row transformation to A, we obtain

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 \\ -1 & 2 & -6 & 4 \\ 2 & 8 & 0 & 26 \\ 2 & 13 & -5 & -9 \end{pmatrix} \longrightarrow \quad \cdots \quad \longrightarrow \begin{pmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A'.$$

The pivot columns of A' are the first and the second column, hence by Theorem 4.26, a basis of $\operatorname{Im}(A)$ are its first and second columns, i.e. the vectors $\vec{v}_1$ and $\vec{v}_2$.
It follows that p_1, p_2 form a basis of $\operatorname{gen}\{p_1, p_2, p_3, p_4\} \subseteq P_3$ and consequently $\dim(\operatorname{gen}\{p_1, p_2, p_3, p_4\}) = 2$.

Remark 4.30. Let us use the abbreviation $\pi = \operatorname{gen}\{p_1, p_2, p_3, p_4\}$. The calculation above actually shows that any two vectors of p_1, p_2, p_3, p_4 form a basis of π. To see this, observe that clearly any two of them are linearly independent, hence the dimension of their generated space is 2. On the other hand, this generated space is a subspace of π which has the same dimension. Therefore they must be equal.

Remark 4.31. If we wanted to complete p_1, p_2 to a basis of P_3, we have (at least) the two following options:

(i) Find two linearly independent vectors which are orthogonal to p_1 an p_2. This leads to a homogenous system of two equations for four unknowns, namely

$$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 &= 0, \\ x_1 + 2x_2 - 6x_3 + 4x_4 &= 0 \end{aligned}$$

or, in matrix notation, $P\vec{x} = 0$ where P is the 2×4 matrix whose rows are p_1 and p_2. Since clearly $\operatorname{Im}(P) \subseteq \mathbb{R}^2$, it follows that $\dim(\operatorname{Im}(P)) \le 2$ and therefore $\dim(\ker(P)) \ge 4 - 2 = 2$.

(ii) Another way to find $q_3, q_4 \in P_3$ such that p_1, p_2, q_3, q_4 forms a basis of P_3 is to use reduction process that was employed to find A'. Assume that E is an invertible matrix such that $A = EA'$. Such an E can be found by keeping track of the row operations that transform A into A'. Let e_j be the standard unit vectors of $\mathbb{R}^4$. Then we already know that $\vec{v}_1 = Ee_1$ and $\vec{v}_2 = Ee_2$. If we set $\vec{w}_3 = Ee_3$ and $\vec{w}_4 = Ee_4$, then $\vec{v}_1, \vec{v}_2, \vec{w}_3, \vec{w}_4$ form a basis of $\mathbb{R}^4$. This is because $e_1, \dots, e_4$ are linearly independent and E in injective. Hence $Ee_1, \dots, Ee_4$ are linearly independent too (by Proposition 4.11).

Sometimes useful is the following theorem.

Theorem 4.32. *Let $A \in M(m \times n)$. Then* $\ker(A) = (R_A)^\perp$.

Proof. Let $\vec{r}_1, \dots, \vec{r}_n$ be the rows of A. Since $R_A = \operatorname{gen}\{\vec{r}_1, \dots, \vec{r}_n\}$, it suffices to show that $\vec{x} \in \ker(A)$ if and only if $\vec{x} \perp \vec{r}_j$ for all $j = 1, \dots, m$.
By definition $\vec{x} \in \ker(A)$ if and only if

$$\vec{0} = A\vec{x} = \left(\begin{array}{c} \vec{r}_1 \\ \hline \vdots \\ \hline \vec{r}_m \end{array}\right) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \langle \vec{r}_1 \,, \vec{x} \rangle \\ \vdots \\ \langle \vec{r}_m \,, \vec{x} \rangle \end{pmatrix}$$

This is the case if and only if $\langle \vec{r}_j \,, \vec{x} \rangle$ for all $j = 1, \dots, m$, that is, if and only if $\vec{x} \perp \vec{\vec{r}}_j$ for all $j = 1, \dots, m$. ($\langle \cdot \,, \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$.) $\square$

Alternative proof of Theorem 4.32. Observe that $R_A = C_{A^t} = \operatorname{Im}(A^t)$. So we have to show that $\ker(A) = (\operatorname{Im}(A^t))^\perp$. Recall that $\langle Ax\,, y\rangle = \langle x\,, A^t y\rangle$. Therefore

$$\begin{aligned} x \in \ker(A) &\iff Ax = 0 \iff Ax \perp \mathbb{R}^m \\ &\iff \langle Ax\,, y\rangle = 0 \text{ for all } y \in \mathbb{R}^m \\ &\iff \langle x\,, A^t y\rangle = 0 \text{ for all } y \in \mathbb{R}^m \iff x \in (\operatorname{Im}(A))^t. \qquad \square \end{aligned}$$

Finally we want to give an alternative (coordinate free!) proof of Theorem 4.25 ...