Universidad de los Andes
MATE-4330

Functional Analysis

Problem Sheet 13

Compact operators.

Hand in: April 30, 2010

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1. Let $X = \ell_2(\mathbb{N})$.

   (a) Let $\lambda_0 \in \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ such that $\lim\limits_{n\to\infty} \lambda_n = \lambda_0$ and define
   $$T : X \to X, \quad Tx = (\lambda_n x_n)_{n\in\mathbb{N}} \quad \text{for } x = (x_n)_{n\in\mathbb{N}}.$$
   Find $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$.

   (b) Show that for every compact set $K \subseteq \mathbb{C}$ there exists an operator $T \in L(X)$ such that $\sigma(T) = K$.

2. Let X be an infinite dimensional Banach space and $K \in L(X)$ a compact operator, $K \neq 0$. Show that there exists a non-trivial closed subspace U of X such that $B(U) \subseteq U$ for every $B \in L(X)$ which commutes with K (that is: $BK = KB$).

3. Let X be a reflexive Banach space, Y a Banach space and $T \in L(X, Y)$. Show that the following is equivalent:

   (a) T is compact.

   (b) $w\text{-}\lim\limits_{n\to\infty} x_n = 0$ implies $\lim\limits_{n\to\infty} Tx_n = 0$ for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$.

4. Let $\tau : [0, 1] \to [0, 1]$ be continuous and
   $$A : C[0, 1] \to C[0, 1], \qquad Af := f \circ \tau.$$
   For which τ is A compact?