Universidad de los Andes
MATE-4330

Functional Analysis

Problem Sheet 12

Spectrum; approximate eigenvalues. Hand in: April 23, 2010

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1. Find the point spectrum, continuous spectrum and the residual spectrum of the left shift and the right shift:

$$R : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}), \quad R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots),$$
$$L : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}), \quad L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

2. Let X be a Banach space and $S, T \in L(X)$. Show $\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}$.

   Hint. Show: if $\mathrm{id} - ST$ is invertible, then $\mathrm{id} + T(\mathrm{id} - ST)^{-1}S$ is the inverse of $\mathrm{id} - TS$.

3. Let X be a Banach space and $T \in L(X)$. $\lambda \in \mathbb{C}$ is called *approximate eigenvalue* if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} (T - \lambda)x_n = 0$.

   (a) Every approximate eigenvalue belongs to $\sigma(T)$.

   (b) Every boundary point of $\sigma(T) \subseteq \mathbb{C}$ is an approximate eigenvalue of T.

   (c) If X is a Hilbert space and if T is selfadjoint, then every $\lambda \in \sigma(T)$ is an approximate eigenvalue of T.

4. Let X be a complex Banach space. For $T \in L(X)$ and $\lambda \in \mathbb{C}$ let

$$\mathcal{A}_\lambda(T) := \{x \in X : x \in \ker(T - \lambda)^n \text{ for some } n \in \mathbb{N}\}$$

   be the algebraic eigenspace of T in $\lambda \in \sigma_p(T)$.

   (a) Let $\lambda_0 \in \sigma_p(T)$, $x_0 \in \mathcal{A}_{\lambda_0}(T) \setminus \{0\}$ and $\mathcal{D} := \{\lambda \in \mathbb{C} : |\lambda| \geq \|T\|\}$. Define

$$f : \mathcal{D} \to X, \qquad f(\lambda) := (\lambda - T)^{-1}x_0,$$
$$g : \mathbb{C} \setminus \{\lambda_0\} \to X, \qquad g(\lambda) := \frac{1}{\lambda - \lambda_0} \sum_{n=0}^{\infty} \left(\frac{\lambda_0 - T}{\lambda_0 - \lambda} \right)^n x_0.$$

   Show that g is a holomorphic extension of f.

   (b) Let $\lambda_1, \dots, \lambda_n$ pairwise distinct eigenvalues of T and $\mathcal{A}_{\lambda_j}(T)$ the corresponding algebraic eigenspaces. Choose $x_j \in \mathcal{A}_{\lambda_j}(T)$, $x_j \neq 0$, $j = 1, \dots, n$. Then the vectors $x_1, \dots, x_n$ are linearly independent.