Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 10

Projections; positive operators.

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1. Let H be a Hilbert space and P_1, P_2 orthogonal projections on H_0, $H_1 \subseteq H$. Then the following is equivalent.
   (i) $H_0 \subseteq H_1$,
   (ii) $\|P_0 x\| \leq \|P_1 x\|, \quad x \in H$.
   (iii) $\langle P_0 x\,, x\rangle \leq \langle P_1 x\,, x\rangle, \quad x \in H$.
   (iv) $P_0 P_1 = P_0$.

Let H be a Hilbert space. For bounded selfadjoint operators $S, T \in L(H)$ we write $T \geq 0$ if $\langle Tx\,, x\rangle \geq 0$ for all $x \in H$ and $T \leq S$ if $S - T \geq 0$. A sequence $(T_n)_{n\in\mathbb{N}} \in L(H)$ is *increasing* if and only $T_n \leq T_{n+1}$, $n \in \mathbb{N}$. A sequence $(T_n)_{n\in\mathbb{N}} \in L(H)$ is *decreasing* if and only $(-T_n)_{n\in\mathbb{N}} \in L(H)$ is increasing.

2. Let H be a Hilbert space and $(T_n)_{n\in\mathbb{N}}$ a bounded, monotonically increasing sequence of selfadjoint operators. Show that the sequence converges strongly to an selfadjoint operator.
   Hint. If S is a non-negative operator, then $s : H \times H \to H$, $s(x, y) = \langle Sx\,, y\rangle$ is a non-negative sesquilinear form.

3. Let $(P_n)_{n\in\mathbb{N}}$ be a monotonic sequence of orthogonal projections in a Hilbert space H. Then $(P_n)_{n\in\mathbb{N}}$ converges strongly to an orthogonal projection P and
   (a) $\operatorname{rg} P = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{rg} P_n}$ if $(P_n)_{n\in\mathbb{N}}$ is increasing,
   (b) $\operatorname{rg} P = \bigcap_{n\in\mathbb{N}} \operatorname{rg} P_n$ if $(P_n)_{n\in\mathbb{N}}$ is decreasing.

4. Let X and Y be Banach spaces, $Y \neq \{0\}$, and $T : X \supseteq \mathcal{D}(T) \to Y$ a densely defined linear operator. Show:
   (a) If T is closed, then for every $y \in Y$, $y \neq 0$, there exists a $\varphi \in \mathcal{D}(T')$ such that $\varphi(y) \neq 0$. In particular, $\mathcal{D}(T') \neq \{0\}$.
   (b) There exists a linear operator T such that $\mathcal{D}(T') = 0$.