Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 9

Orthogonality.

Hand in: March 26, 2010

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1. Let H be a Hilbert space, $Y \subseteq H$ a subspace and $\varphi_0 \in Y'$. Show that there exists exactly one extension $\varphi \in H'$ of φ_0 with $\|\varphi_0\| = \|\varphi\|$.

2. Let X be a normed space, $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$. Then the following is equivalent:
   (a) $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally to x.
   (b) For every $\varepsilon > 0$ there exists a finite subset $A \subseteq \mathbb{N}$ such that for every finite set B with $A \subseteq B \subseteq \mathbb{N}$
   $$\Big\| \sum_{b\in B} x_b - x \Big\| < \varepsilon.$$

3. Let H be a Hilbert space. For a linear operator $P : H \to H$ the following is equivalent:
   (a) P is an orthogonal projection.
   (b) $P^2 = P$ and $\langle Px\,, y\rangle = \langle x\,, Py\rangle$.

4. **Haar functions.** Let $\psi = \chi_{[0,\,1/2)} - \chi_{[1/2,\,1)}$. For $n, k \in \mathbb{Z}$ define
   $$\psi_{n,k} : \mathbb{R} \to \mathbb{R}, \qquad \psi_{n,k}(t) = 2^{k/2}\,\psi(2^k t - n).$$
   For $k \in \mathbb{N}_0$ and $n \in \{1, 2, \ldots, 2^k - 1\}$ let
   $$h_{2^k+n} : [0,1] \to \mathbb{R}, \qquad \begin{cases} h_{2^k+n}(t) = \psi_{k,n}(t), & \text{for } t \in [0,1), \\ h_{2^k+n}(1) = \lim\limits_{t\to 1^-} \psi_{k,n}(t). \end{cases}$$
   and $h_0(t) = 1,\ t \in [0,1]$.
   (a) $(h_j)_{j\in\mathbb{N}_0}$ is a orthonormal system in $L_2[0,1]$ and $(\psi_{n,k})_{n,k\in\mathbb{Z}}$ is a orthonormal system in $L_2(\mathbb{R})$.
   (b) $T : L_2[0,1] \to L_2[0,1],\ Tf = \sum_{j=0}^{2^k-1} \langle f\,, h_j\rangle h_j$ is a orthonormal projection on the subspace
   $$U = \{f \in L_2[0,1] : f \text{ const. in intervals } [r2^{-k}, (r+1)2^{-k}) \text{ with } r \in \mathbb{N}_0\}.$$
   (c) For $f \in C[0,1]$, the series $\sum_{j=0}^{\infty} \langle f\,, h_j\rangle h_j$ converges uniformely to f.
   (d) $(h_j)_{j\in\mathbb{N}_0}$ is an orthonormal basis of $L_2[0,1]$.
   (e) $(\psi_{k,n})_{k,n\in\mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{R})$.