Universidad de los Andes
MATE-4330

Functional Analysis

Problem Sheet 8

Hilbert spaces.

Hand in: March 19, 2010

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1. Let H be Hilbert space, $(x_n)_{n\in\mathbb{N}} \subseteq H$ and $x_0 \in H$. Then the following is equivalent:
   (a) $x_n \to x_0$.
   (b) $\|x_n\| \to \|x_0\|$ and $x_n \xrightarrow{w} x_0$.

2. Let H be a Hilbert space, $V, W \subseteq H$ closed subspaces and P_V, P_W the corresponding orthogonal projections. Show

$$V \subseteq W \quad\iff\quad P_V = P_V P_W = P_W P_V.$$

3. Let H be a Hilbert space, $V, W \subseteq H$ closed subspaces and P_V, P_W the corresponding orthogonal projections. Show that the following is equivalent:
   (a) $P_V P_W = 0$.
   (b) $V \perp W$.
   (c) $P_V + P_W$ is a orthogonal projection.

   Show that $\mathrm{R}(P_V + P_W) = V \oplus W$ if one of the equivalent conditions above hold.

4. Let H be a Hilbert space and $B : H \times H \to \mathbb{K}$ sesquilinear. On $H \times H$ consider the norm $\|(x, y)\| := \sqrt{\|x\|^2 + \|y\|^2}$.
   (a) Show that the following is equivalent:
      (i) B is continuous.
      (ii) B is partially continuous, that is, for every fixed x_0, $y \mapsto B(x_0, y)$ is continuous and for every fixed y_0, $x \mapsto B(x, y_0)$ is continuous.
      (iii) B is bounded, that is, there exists an $M \in \mathbb{R}$ such that $\|B(x, y)\| \le M\|x\|\|y\|$ for all $x, y \in H$.
   (b) If B is continuous, then there exists a $T \in L(H)$ such that

$$B(x, y) = \langle Tx\,, y\rangle, \qquad x, y \in H.$$

   (c) If in addition there exists an $m > 0$ such that $B(x, x) \ge m\|x\|^2$, $x \in H$, then T is invertible and $\|T^{-1}\| \le m^{-1}$.