Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 7

Hilbert spaces.

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1. Let X be a pre-Hilbert space, $U \subseteq H$ a dense subspace and $x_0 \in X$ such that $\langle x_0 , u \rangle = 0$ for all $u \in U$. Show that $x_0 = 0$.

2. Let $w \in C([0,1], \mathbb{R})$. For $x, y \in C([0,1])$ let
$$\langle x , y \rangle_w := \int_0^1 x(t)\overline{y(t)}w(t)\,\mathrm{d}t.$$
Find a necessary and sufficient condition for w such that $\langle \cdot , \cdot \rangle_w$ is an inner product. When is the norm induced by $\langle \cdot , \cdot \rangle_w$ equivalent to the usual L_2 norm?

3. Let $1 \leq p \leq \infty$. For $f \in L_p(\mathbb{R})$ and $s \in \mathbb{R}$ let define $T_s : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ by $(T_s f)(t) := f(t-s)$. Obviously the T_s are linear isometries.
   (a) Let $1 \leq p < \infty$. Show that $T_s \xrightarrow{s} \mathrm{id}$ for $s \to 0$. Do the T_s converge in norm?
   (b) Do the T_s converge in norm or strongly in the case $p = \infty$?

4. Show that $W^m(\Omega)$, $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces.

For problem 4:
For $\Omega \subseteq \mathbb{R}$ define the set of *test functions*
$$\mathscr{D}(\Omega) := \{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) \subseteq \Omega \ \text{ is compact}\}.$$
For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha \varphi = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} \varphi$ if the derivative exists.
Let $f \in L_2(\Omega)$. A function $g \in L_2(\Omega)$ is called the αth weak derivative of f if
$$\langle g , \varphi \rangle = (-)^{|\alpha|} \langle f , D^\alpha \varphi \rangle, \qquad \varphi \in \mathscr{D}(\Omega).$$
Note that g is uniquely defined; we denote the weak derivative g by $D^{(\alpha)} f$.
For $m \in \mathbb{N}$ we define the *Sobolev space*
$$W^m(\Omega) := \{f \in L_2(\Omega) : D^{(\alpha)} f \in L_2(\Omega), |\alpha| \leq m\}.$$
$W^m(\Omega)$ is an inner product space with
$$\langle f , g \rangle_{W^m} := \sum_{|\alpha| \leq m} \langle D^{(\alpha)} f , D^{(\alpha)} g \rangle_2.$$
Further, we define the spaces
$$H^m(\Omega) := \overline{C^m(\Omega) \cap W^m(\Omega)} \qquad \text{and} \qquad H_0^m(\Omega) := \overline{\mathscr{D}(\Omega)}$$
where the closure is taken with respect to the norm induced by $\langle \cdot , \cdot \rangle_{W^m}$.